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An Early Mathematical Presentation of Consumer's Surplus

Yeung-Nan Shieh

San Jose State University, yeung-nan.shieh@sjsu.edu

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An early mathematical presentation of consumer's surplus

Yeung−Nan Shieh
Department of Economics, San Jose State University

Abstract

This paper shows the first presentation of consumer's surplus as a definite integral is in Launhardt's 1885 masterpiece, Mathematische Begrundung Der Volkswirtschaftslehre. In chapter 32, Launhardt applied integral calculus to derive the consumer's surplus of a decrease in the freight rate for each consumer at a market point and for all consumers in the whole market area. Launhardt's analysis is typically reproduced in the modern literature on non−spatial and spatial economics without any acknowledgement of Launhardt. Launhardt's name deserves to be mentioned alongside with Dupuit and Marhall as an early anticipator of many key elements in the theory of consumer's surplus.

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1. Introduction

It has long been accepted that the concept of consumer's surplus in terms of the "Marshallian triangle" originated with Dupuit (1844) and was popularized by Marshall (1890, 1920). However, the contribution of Wilhelm Launhardt, a pioneer of mathematical economics, has been ignored. This paper attempts to show that the first presentation of consumer's surplus as a definite integral is in chapter 32 of Launhardt's masterpiece, *Mathematical Principles of Economics* (1885, 1993). In fact, Launhardt discovered the definite-integral formulation of consumer's surplus in the course of investigating the welfare-maximizing freight rate for a monopolistic railway company. His analysis has become the standard method of illustrating consumer's surplus and is typically reproduced in the modern literature on non-spatial and spatial economics without any acknowledgement of Launhardt.

2. Launhardt's Contribution to Consumer's Surplus

Launhardt's contribution to consumer's surplus was couched in the von Thunen circular space that has the following characteristics. A monopolistic firm is located at a point on an unbounded plain where consumers are evenly distributed. All consumers are identical and each has the same linear demand function. The railway business has monopoly power and carries the goods from the firm to all consumers. The freight rate per unit of distance per unit of quantity is set by the monopolistic railway.

Under these conditions, Launhardt assessed the effect of a reduction of the freight rate on what he called "the savings on the price of goods [that] benefits the consumer"; i.e., the consumer's surplus. (1885, p. 202; 1993, p. 181) He specified the demand function for each consumer at distance r from the firm as:

$$q = b(a - p - tr) \tag{1}$$

where q = quantity demanded, p = mill price, t = constant freight rate, r = the distance from the firm, and p + tr = the full-price paid by the consumer at distance r, i.e., the delivered price. It is assumed that $a > 0$, $b > 0$.[1] Launhardt also pointed out that "[a]t a given distance r demand commences when the rate of freight is reduced to $t' = (a - p)/r$." (1885, p. 201; 1993, p. 180).

If the freight rate falls by dt, the quantity demanded of the goods at distance r will increase by

$$dq = brdt \tag{2}$$

The price for an increase in the amount shipped would be $(p + tr)dq = (p + tr)brdt$ as the consumer pays p + tr for a unit of the goods.

Next, Launhardt considered the case where "…the freight rate is reduced from t to t_1 the goods price will change from $p + tr$ to $p + t_1r$" (1885, p. 202; 1993, p. 180). Multiply each side of (2) by $(t - t_1)r$, one obtains

[1] The notation used by Launhardt is rather awkward for the modern reader and potentially misleading. We use the convention notation in economic textbooks to replace Launhardt's.

$$(t - t_1)rdq = (t - t_1)br^2dt \tag{3}$$

Clearly, for an increase in the amount shipped by dq, the consumer saves $(t - t_1)rdq$ or $(t - t_1)br^2dt$.

With this in mind, the integral, as Launhardt pointed out, "[f]rom the freight rate t' = (a – p)/r…, for which the dispatch of goods commences to a destination at a distance of r [down to]… the freight rate t_1" (1885, p. 202; 1993, pp. 180-81) would be

$$cs(r) = br^2\int_{t_1}^{t'} (t - t_1)dt = (1/2)br^2(t' - t_1)^2 \tag{4}$$

Substituting t' = (a – p)/r into (4), one obtains

$$cs(r) = (1/2)b(a - p - t_1r)^2 \tag{5}$$

Equation (5) is the equivalent of Launhardt's result (1985, p. 202; 1993, p. 181). This is the well-known formula for consumer's surplus with linear demand curve. It indicates that the area of the consumer's surplus triangle is (1/2) (base x height). In the modern economic literature, this result is typically reproduced by economists who are apparently unfamiliar with Launhardt's work. For example, see McKenna and Rees (1992, pp.75-77), Shy (1995, pp. 52-53), Takayama (1987), Varian (2003. chapter 14).

Launhardt's consumer's surplus can also be shown graphically. In Figure 1, the linear demand curve intersects the vertical full-price line at (p + t'r) = a, and the horizontal quantity line at ab. If the freight rate is at t_1, the shaded triangle area is what we called the "Marahallian consumer's surplus triangle".

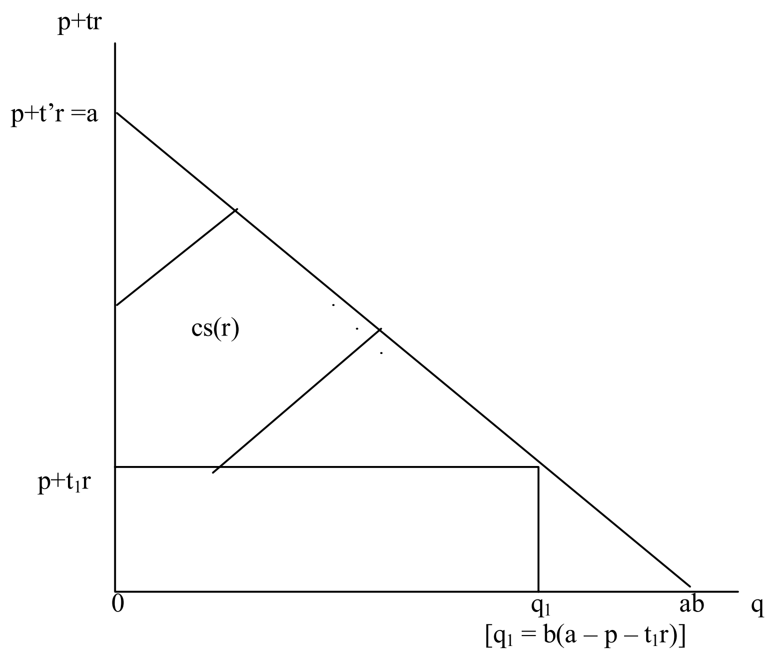

Figure 1. Consumer's Surplus at market point r.

After finding the consumer's surplus for each consumer located at distance r from the firm, i.e., cs(r) in (5), Launhardt assumed that per surface unit has one consumer and showed that the aggregate consumer's surplus in the circular market area with radius $R^* = (a - p)/t_1$ takes the form of:

$$CS = 2\pi\int_0^{R^*} cs(r)rdr = 2\pi\int_0^{R^*} (1/2)b(a - p - t_1r)^2rdr = (\pi b/12t_1^2)(a - p)^4 \tag{6}$$

Equation (6) is Launhardt's equation (144), (1885, p. 203; 1993, p.181). This result is quite similar to the aggregate consumer's surplus obtained in the theory of spatial pricing policies. For example, see Holahan (1975, p. 500), Beckmann (1999, p. 187), Greenhut, Norman and Hung (1987, pp. 119-120), Ohta (1988, pp. 194-204). It has become the standard analytical device in the modern literature on location theory and spatial economics. However, nobody has mentioned Launhardt's name in this connection.

3. Conclusions

This paper has shown Launhardt as a major, and one of the earliest contributors to the theory of consumer's surplus. It is not surprising that Launhardt's work has not been widely appreciated because of inaccessibility of source material on Launhardt and his analysis of consumer's surplus is in the spatial setting. However, the available evidence (as developed above) indicates that Launhardt's analysis, in 1885, already contained a sophisticated and original presentation of consumer's surplus in terms of the "Marshallian triangle". Modern writers, who are apparently unfamiliar with Launhardt's work, always use his analytical devices.

Considering the perceptiveness of Launhardt's analysis, it is clear that Launhardt's name deserves to be mentioned alongside with Dupuit and Marshall as an early anticipator of many key elements in the non-spatial and spatial consumer's surplus analyses.

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