AN ECONOMIC INDEX OF RISKINESS

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Abstract

Define the *riskiness* of a gamble as the reciprocal of the absolute risk aversion (ARA) of an individual with constant ARA who is indifferent between taking and not taking that gamble. We characterize this index by axioms, chief among them a "duality" axiom which, roughly speaking, asserts that less risk-averse individuals accept riskier gambles. The index is positively homogeneous, continuous, and subadditive, respects first and second order stochastic dominance, and for normally distributed gambles, is half of variance/mean. Examples are calculated, additional properties derived, and the index is compared with others.

JEL Codes: C00, C43, D00, D80, D81, E44, G00.
Keywords: Riskiness; risk aversion; expected utility; decision making under uncertainty; portfolio choice; Sharpe ratio; value at risk; coherent measures of risk.

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1 Introduction

On March 21, 2004, an article on the front page of the New York Times presented a picture of allegedly questionable practices in some state-run pension funds. Among the allegations were that these funds often make unduly risky investments, recommended by consultants who are interested parties. The concept of "risky investment" is commonplace in financial discussions, and seems to have clear conceptual content. But when one thinks about it carefully and tries to pin it down, it is elusive. Can one measure riskiness *objectively*—independently of the person or entity taking the risk?

Conceptually, whether or not a person takes a gamble depends on two distinct considerations (Diamond and Stiglitz 1974):

(i) the attributes of the gamble, and in particular, how *risky* it is; and
(ii) the attributes of the person, and in particular, how *averse* he is to risk.

The classic contributions of Arrow (1965, 1971) and Pratt (1964) address item (ii) by defining *risk aversion,* which is a personal, subjective concept, depending on the utility function of the individual in question. But they do not define *riskiness;* they do not address item (i). It is like speaking about subjective time perception ("this movie was too long") without having an objective measure of time ("three hours"), or about heat or cold aversion ("it's too cold in here") without an objective measure of temperature ("20° F").

This paper addresses item (i); it develops an index of riskiness of gambles. The concept is based on that of risk aversion: We think of riskiness as a kind of "dual" to risk aversion—specifically, as that aspect of a gamble to which a risk-averter is averse. So on the whole, we expect individuals who are less risk averse to take riskier gambles. As Machina and Rothschild (1987) put it, "risk is what risk-averters hate."

Unlike some other riskiness indices that have been proposed in disciplines such as finance, statistics, and psychology (Section 7), ours is based on economic, decision-theoretic ideas, such as the duality principle roughly enunciated above, and respect for first- and second-order stochastic dominance (Section 4.3). Clearly, riskiness is related to dispersion, so a good riskiness measure should be monotonic with respect to (w.r.t.) second-order stochastic dominance. Less well understood, perhaps, is that riskiness should also relate to location, and thus be monotonic w.r.t. first-order stochastic dominance; in particular, that a gamble that is *sure* to yield more than another should be considered less risky. Both stochastic dominance criteria are uncontroversial, and one advantage of our index is that it completes the partial ordering on gambles that they induce.

The plan of the paper is as follows: Section 2 discusses the purpose and potential uses of the proposed index—what it is we are seeking. Section 3 is devoted to the basic axiomatic definition of the index, and its numerical characterization. Section 4 relates our index to Arrow-Pratt risk aversion. Specifically, it carefully discusses our basic axiom, "Duality," in its own right as well as in relation to Arrow-Pratt (Section 4.1); and it characterizes our index in terms of Arrow-Pratt constant absolute risk-aversion—CARA—as outlined in the abstract (Section 4.2). Section 5 sets forth some desirable properties of the index (in addition to the axioms) such as continuity, respect for stochastic dominance, subadditivity, its dimension (dollars), its behavior for normal gambles, for independent gambles, for "diluted" gambles, and for compound gambles, its ordinal characterization, its interpretation in terms of CRRA—constant relative risk aversion, and the relatively greater weight it puts on losses vis-a-vis gains. Section 6 discusses some numerical examples, meant to give an intuitive feel for the index. Section 7 reviews the literature, and Section 8 is devoted to proofs (throughout, assertions that are not proved on the spot and are not immediate are proved there). Section 9 concludes.

2 The Concept and its Uses

As remarked above, the concept of "riskiness" is ubiquitous in financial discussions. Investors are told that one investment may hold an opportunity for high returns but be "risky," whereas another may be "safer" but yield lower returns. Mutual funds are characterized as "safe" or "venture capital" or "blue chip" or "volatile;" bonds are rated AAA, AA, etc.; and so on. We repeatedly hear that an investment that is appropriate for one investor may be "too risky" for another. Or that a pension fund makes "unduly risky" investments.

Here we propose to *quantify* riskiness—describe it with numbers, rather than adjectives or letter "ratings." The main purpose of such a quantification is the same as that of the adjectives and the letter ratings—to help investors and other decision makers make their decisions. For example, the investments of pension funds could be required not to exceed a stated level of riskiness. Or an investor, on being told the riskiness index of an investment, could say "well, that's too risky for me," or "that's a little risky but I'll go for it," or "hey, that sounds just right for me." Or an advisor could say, if you're living on a pension you should not accept gambles that exceed such and such a riskiness, but if you're young and have plenty of opportunities, you could up that by so-and-so much.

From this viewpoint it is clear that if the gamble g is *sure* to yield more than h, it cannot be considered riskier. We are considering risk-averse decision makers—those for whom risks are undesirable—who, "all other things being equal," prefer less risky alternatives.

But riskiness and desirability are not opposites; a less risky gamble is not *always* more desirable. That depends on the decision maker, and on other parameters in addition to riskiness, such as the mean, maximum loss, opportunities for gain, and so on; indeed, on the whole distribution. Desirability is subjective, depending on the decision maker; one may prefer gamble g to gamble h, while another prefers h to g. Riskiness, on the other hand, is objective; it is the same for all individuals. Given two gambles, a more risk-averse individual may well prefer the less risky gamble, while a less risk-averse individual may find that the opportunities afforded by the riskier gamble outweigh the risk involved.

Like any index or summary statistic—the Gini index of inequality, parameters of distributions (mean, median, variance, ...), the Shapley value of a game, market indices (Dow Jones, S&P 500, ...), cost-of-living indices, difficulty ratings of rock climbs (3, 4, 5.1-5.13; I, II, II) and ski runs (green, blue, red, black), and so on—the riskiness index summarizes a complex, high-dimensional object by a single number. Needless to say, no index captures all the relevant aspects of the situation being summarized. But once accepted, it takes on a life of its own; its "consumers" internalize its content through repeated use.

In addition to these practical uses, a riskiness index could also be a useful research tool. For example, Rabin (2000) asserts that most people would reject a gamble yielding +\$105 or −\$100 with half-half probabilities. While this sounds plausible on its face, it is difficult to verify empirically (as opposed to "experimentally"), since such gambles are not readily available in the real world. What one *can* ask is, do people accept gambles with a "similar" level of riskiness? Once one has a measure of riskiness, one can approach that question by looking at real-life gambles; e.g., insurance contracts.[1]

Early attempts to quantify riskiness were based on mean and variance only (see Machina and Rothschild 1987). Defending this approach, Tobin (1969, p.14) wrote that its critics "owe us more than demonstrations that it rests on restrictive assumptions. They need to show us how a more general and less vulnerable approach will yield the kind of comparative-static results that economists are interested in." That is what our index aims to do.

3 Axiomatic Characterization

In this paper, a *utility function* is a von Neumann-Morgenstern utility function for money, strictly monotonic, strictly concave,[2] twice continuously differentiable, and defined over the entire real line. A *gamble* g is a random variable with real values[3]—interpreted as dollar amounts—some of which are negative, and that has positive expectation.

Say that an agent with utility function u *accepts* a gamble g *at* wealth w if $\mathrm{E}u(w+g) > u(w)$, where E stands for "expectation;" that is, if he prefers taking the gamble at w to refusing it. Otherwise, he *rejects* it. Call agent i *uniformly no less risk-averse than* agent j (written $i \trianglerighteq j$) if whenever i accepts a gamble at some wealth, j accepts that gamble at any wealth. Call i *uniformly*[4] *more* risk-averse than j (written $i \triangleright j$) if $i \trianglerighteq j$ and $j \ntrianglerighteq i$.

Define an *index* as a positive real-valued function on gambles (to be thought of as measuring riskiness). Given an index Q, say that "gamble g is riskier than gamble h" if $Q(g) > Q(h)$. We consider two axioms for Q, the first of which

[1] Rejecting insurance is like accepting a gamble. Since insurance usually has negative expectation for the purchaser, rejecting it has positive expectation.

[2] Strict monotonicity means that the individual likes money; strict concavity, that he is risk-averse—prefers the expected value of a gamble over the gamble itself.

[3] For simplicity, we assume for now that it takes finitely many values, each with positive probability. This assumption will be relaxed in the sequel.

[4] See Section 4.1 for a discussion of this terminology.

posits a kind of "duality" between riskiness and risk aversion; roughly, that less risk-averse agents accept riskier gambles. The axioms are as follows:

DUALITY:[5] If $i \rhd j$, i accepts g at w, and $Q(g) > Q(h)$, then j accepts h at w.

In words, duality says that if the more risk-averse of two agents accepts the riskier of two gambles, then a fortiori the less risk-averse agent accepts the less risky gamble.

POSITIVE HOMOGENEITY: $Q(tg) = tQ(g)$ for all positive numbers t.

Positive Homogeneity embodies the *cardinal* nature of riskiness. If g is a gamble, it makes sense to say that $2g$ is "twice as" risky as g, not just "more" risky. Similarly, tg is t times as risky as g. Our main result is now as follows:

THEOREM A: *For each gamble g, there is a unique positive number $R(g)$ with*
(3.1) $\mathrm{E}e^{-g/R(g)} = 1$.
The index R thus defined satisfies Duality and Positive Homogeneity; and, any index satisfying these two axioms is a positive multiple of R.

We call $R(g)$ the *riskiness* of g. Both axioms are essential: omitting either admits indices that are not positive multiples of R. But Duality is by far the more central: Together only with certain weak conditions of continuity and monotonicity—but *not* Positive Homogeneity—it already implies that the index is ordinally equivalent to R (Section 5.9).

4 Relation with Arrow-Pratt

4.1 Risk Aversion and Duality

To understand the concept of uniform comparitive risk aversion (Section 3) that underlies our treatment, recall first that Arrow (1965, 1971) and Pratt (1964) define the coefficient of *absolute risk aversion* (ARA) of an agent i with utility function u_i and wealth w as $\rho_i(w) := \rho(w, u_i) := -u_i''(w)/u_i'(w)$. Now, call i *no less risk-averse* than j if at any given wealth, j accepts any gamble that i accepts.[6] Then

(4.1.1) *i is no less risk averse than j if and only if $\rho_i(w) \geq \rho_j(w)$ for all w.*

Our concept of $i \unrhd j$—that i is *uniformly* no less risk-averse than j—is much stronger. It says that if i accepts a gamble at some wealth, j also accepts it—not only at that given wealth, but at *any* wealth. Parallel to (4.1.1), we then have

(4.1.2) *i is* uniformly *no less risk averse than j if and only if $\rho_i(w_i) \geq \rho_j(w_j)$ for all w_i and w_j* (i.e., $\min_w \rho_i(w) \geq \max_w \rho_j(w)$).

[5] Throughout, the universal quantifier applies to variables that are not explicitly quantified otherwise. For example, the duality axiom should be understood as being prefaced by: "For all gambles g, h, agents i, j, and wealth w,".

[6] Closely related—in view of (4.1.1)—is the concept of Diamond and Stiglitz (1974, p.346), who call i *more* risk-averse than j if $\rho_i(w) > \rho_j(w)$ for all w. But this has no straightforward equivalent in terms of finite gambles.

Arrow-Pratt risk-aversion is a "local" concept, in that it concerns i's attitude towards infinitesimally small gambles at a specified wealth only; in contrast, our two concepts of comparitive risk-aversion are "global," in two senses: (i) they apply to gambles of arbitrary finite size, and (ii) the gambles may be taken at any wealth. Thus our concepts seem more direct, straightforward, and natural; no limiting process is involved—one deals directly with real gambles. On the other hand, we get only partial orders, whereas Arrow and Pratt define a numerical index (and so a total order). The three concepts are related by (4.1.1) and (4.1.2).

For one agent to be uniformly more risk-averse than another—$i \rhd j$—is a very strong requirement. It is precisely this strength that makes the duality axiom highly acceptable: Since this strong requirement appears in the hypothesis of the axiom, the axiom as a whole calls for very little, and what it does call for is eminently reasonable.

4.2 CARA

An agent i is said to have *constant absolute risk aversion* (CARA) if his ARA is a constant α that does not depend on his wealth. In that case, i is called a CARA agent, and his utility u a CARA utility, both with *parameter* α. There is an essentially[7] unique CARA utility with parameter α, given by $u(w) = -e^{-\alpha w}$. While defined in terms of the local concept of risk aversion, CARA may in fact be characterized (or equivalently, defined) in global terms, as follows:

(4.2.1) *An agent i has CARA if and only if for any gamble g and any two wealth levels, i either accepts g at both levels, or rejects g at both levels.*

In words, whether or not i accepts a gamble g depends only on g, not on the wealth level. CARA utility functions thus constitute a kind of medium or context in which gambles may be evaluated "on their own," without reference to wealth; in particular, one can speak of CARA agents "accepting" or "rejecting" a gamble, without specifying the wealth. This kind of "wealth-free environment" is, of course, precisely what we want when seeking an objective riskiness measure. We then have

(4.2.2) *If a CARA agent accepts a gamble, then any CARA agent with a smaller parameter also accepts the gamble. Equivalently, if a CARA agent rejects a gamble, then any CARA agent with a larger parameter also rejects the gamble.*

From (4.2.2) it follows that for each gamble g, there is precisely one "cut-off" value of the parameter, such that g is accepted by CARA agents with smaller parameter, and rejected by CARA agents with larger parameter. The larger the parameter, the more risk-averse the agent; so the duality principle—that less risk-averse agents accept riskier gambles—indicates that this cut-off might be a good *inverse* measure of riskiness. And indeed, we have

[7] Up to an additive and a positive multiplicative constant.

Theorem B: *The riskiness $R(g)$ of a gamble g is the reciprocal of the number α such that a CARA person with parameter α is indifferent between taking and not taking the gamble.*

Proof: Follows from (3.1) and the form of CARA utilities.

Note that Theorem B goes a little beyond Theorem A in characterizing riskiness; it actually fixes the index numerically, not just within a positive constant. Note, too, that while Theorem B might not unreasonably have served as a *definition* of riskiness, it is, in fact, *not* a definition; it is a *theorem*—a proven consequence of our axioms (Section 3).

5 Some Properties of Riskiness

5.1 The Parameters of Riskiness

The riskiness of a gamble depends on the gamble only—indeed, on its **distribution** *only*—and not on any other parameters, such as the utility function of the decision maker or his wealth.

5.2 Dimension

Riskiness is measured in dollars. For an "operational" interpretation of the dollar amount, see Section 5.10.

5.3 Monotonicity w.r.t. Stochastic Dominance

The most uncontroversial, widely accepted notions of riskiness are provided by the concept of *stochastic dominance* (Hadar and Russell (1969), Hanoch and Levy (1969), Rothschild and Stiglitz (1970)). Say that a gamble g *first-order dominates* (FOD) g_* if $g \geq g_*$ for sure, and $g > g_*$ with positive probability; and g *second-order dominates* (SOD) g_* if g_* may be obtained from g by "mean-preserving spreads"—by replacing some of g's values with random variables whose mean is that value. Say that g *stochastically* dominates g_* (in either sense) if there is a gamble distributed like g that dominates g_* (in that sense).

An index Q is called *first- (second-) order monotonic* if $Q(g) < Q(g_*)$ whenever g F(S)OD g_*. First- and second-order dominance constitute partial orders. One would certainly expect any reasonable notion of riskiness to extend these partial orders—i.e., to be both first- and second-order monotonic. And indeed, *the riskiness index R is monotonic in both senses.*

5.4 Continuity

Call an index Q *continuous* if $Q(g_n) \to Q(g)$ whenever the g_n are uniformly bounded and converge to g in probability.[8] With this definition, *the riskiness index R is continuous;* in words, when two gambles are likely to be close,

[8] I.e., for every $\varepsilon > 0$, there is an N such that $\text{Prob}\{|g_n - g| > \varepsilon\} < \varepsilon$ for all $n > N$.

their riskinesses are close. Therefore, it is also continuous in weaker senses; e.g., $R(g_n) \to R(g)$ whenever the g_n converge to g uniformly.[9]

5.5 Diluted Gambles

If g is a gamble, p a number strictly between 0 and 1, and g^p a compound gamble that yields g with probability p and 0 with probability $1-p$, then

$R(g^p) = R(g)$.

Though at first this may sound counterintuitive, on closer examination it is very reasonable; indeed, *any* expected utility maximizer—risk averse or not—accepts g^p if and only if he accepts g.

5.6 Compound Gambles

If two gambles g and h have the same riskiness r, then a compound gamble yielding g with probability p and h with probability $1-p$ also has riskiness r.

More generally,

(5.6.1) *the riskiness of a compound of two gambles lies between their riskinesses.*

5.7 Normal Gambles

If the gamble g has a normal distribution,[10] *then*

$R(g) = \mathrm{Var}g/2\mathrm{E}g$,

where Var stands for "variance." Indeed, set $\mathrm{Var}g =: \sigma^2$ and $\mathrm{E}g =: \mu$. The density of g's distribution is $e^{-(x-\mu)^2/2\sigma^2}/\sigma\sqrt{2\pi}$, so

$$\mathrm{E}\, e^{-g/(\sigma^2/2\mu)} = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(x-\mu)^2/2\sigma^2} e^{-x/(\sigma^2/2\mu)}dx$$
$$= \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-[(x^2-2\mu x+\mu^2)+(4\mu x)]/2\sigma^2}dx$$
$$= \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(x+\mu)^2)/2\sigma^2}dx = 1.$$

So (3.1) holds with $R(g) := \sigma^2/2\mu$, so that is indeed the riskiness of g.

5.8 Sums of Gambles

If g and h are independent identically distributed (i.i.d.) *gambles with riskiness r, then $g+h$ also has riskiness r.* Indeed, the hypothesis yields $\mathrm{E}e^{-g/r} = \mathrm{E}e^{-h/r} = 1$. Since g and h are independent, so are $e^{-g/r}$ and $e^{-h/r}$, so $1 = \mathrm{E}e^{-g/r}\ \mathrm{E}e^{-h/r} = \mathrm{E}(e^{-g/r}e^{-h/r}) = \mathrm{E}(e^{-(g+h)/r})$, so $R(g+h) = r$.

It follows that the sum of n i.i.d. gambles has the *same* riskiness as each one separately. This contrasts with the expectation—and the variance—of such a sum, which is n times the corresponding quantity for a single gamble. In

[9] I.e., for every $\varepsilon > 0$, there is an N such that $\sup|g_n - g| < \varepsilon$ for all $n > N$. In words, when two gambles are always close, their riskinesses are close.

[10] As defined in Section 3, a gamble has only finitely many values; so strictly speaking, its distribution cannot be normal. We therefore redefine a "gamble" as a random variable g (Borel-measurable function on a probability space) for which $\mathrm{E}e^{-\alpha g}$ exists for all positive α.

the case of riskiness, one might say that location and dispersion considerations, which act in opposite directions, cancel each other out, and the result is that the riskiness stays the same.

More generally,

(5.8.1) *if g and h are independent, then the riskiness of $g+h$ lies between those of g and h.*

An interesting consequence is that a person—or entity like a pension fund—that does not want its portfolio to exceed a certain level of riskiness need only see to it that each of the independent investments it makes does not exceed that level.

Even without independence, we still have *subadditivity:*[11]

(5.8.2) $R(g+h) \leq R(g) + R(h)$

for *any* gambles g and h. Moreover, equality in (5.8.2) obtains when g is a positive multiple of h (that follows from homogeneity), and *only* then. We thus get a spectrum of circumstances, which is most transparent when the two gambles are identically distributed, and so have the same riskiness r: When the gambles are "totally" positively correlated—i.e., equal—the risks reinforce each other, and the sum has riskiness precisely $2r$. When they are independent, the risks neither reinforce nor hinder each other, and the sum has the same riskiness r as each of the gambles separately. When they are "totally" negatively correlated, the risk is minimal—but need not vanish.

5.9 Ordinality

If we are looking only for an ordinal index—i.e., wish to define "riskier," without saying *how much* riskier—then we can replace the homogeneity axiom by conditions of monotonicity and continuity.

An index Q for which $Q(g) > Q(h)$ if and only if $R(g) > R(h)$ is called *ordinally equivalent* to R. We have already seen that the riskiness index R satisfies the duality axiom (Theorem A), is continuous (Section 5.4), and is both first- and second-order monotonic (Section 5.3). In the opposite direction, *any continuous and first-order monotonic index that satisfies the duality axiom is ordinally equivalent to R.* Moreover, *continuity, monotonicity and duality are essential for this result;* without any one of them, it fails.

5.10 An "Operational" Interpretation in Terms of CRRA

An agent i is said to have *constant relative risk aversion* (CRRA) if his ARA is inversely proportional to his wealth w; i.e., if $w\rho_i(w)$ is a constant, called the CRRA *parameter.* CRRA is an expression of the idea that wealthier people are less risk-averse. Here wealth is assumed positive, so unlike in the rest of this paper, we here discuss utility functions defined on the positive reals only.

Like with CARA, there is an essentially unique CRRA utility with a given parameter. For parameter 1, it is the classic logarithmic utility, $\log w$, where

[11] We thank Sergiu Hart for this observation, and for its proof.

"log" denotes the natural logarithm (i.e., to base e), originally proposed by Daniel Bernoulli (1731).[12] In terms of this utility, one can lend operational meaning to the riskiness $R(g)$ of a gamble g as follows:

(5.10.1) *An agent with logarithmic utility and initial wealth w accepts a gamble g if $w + \min g > R(g)$, and rejects it if $w + \max g < R(g)$.*

That is, g is accepted if taking the gamble necessarily results in a wealth greater than $R(g)$, and rejected if it necessarily results in a wealth smaller than $R(g)$. Thus when the range of g is small compared to its riskiness, the riskiness represents an approximate cut-off; the gamble is accepted if the initial wealth is considerably greater than the riskiness, and is rejected if it is considerably less.

(5.10.1) is an immediate consequence of the following proposition, which is of interest in its own right, and does not assume CRRA:

(5.10.2) *If $\rho_i(x) < 1/R(g)$ for all x between $w + \min g$ and $w + \max g$, then i accepts g at w; if $\rho_i(x) > 1/R(g)$ for all such x, then i rejects g at w.*

5.11 Extending the Domain

So far, riskiness is defined on the domain of "gambles:" random variables g with some negative values and $\mathrm{E}g > 0$. On this domain, the range of the riskiness ranges is the positive reals; i.e., strictly between 0 and ∞. Outside of this domain, the basic relation that determines riskiness—Equation (3.1)—has no solution. The domain may be extended by defining $R(g) := 0$ when there are no negative values, and $R(g) := \infty$ when $\mathrm{E}g \leq 0$. Intuitively, this makes good sense: When there are no negative values, there is no risk, and when $\mathrm{E}g \leq 0$, no risk-averse agent will accept g. When g vanishes identically, we have a "singular point," where the riskiness remains undefined.

With these definitions, the properties of the index continue to apply. Thus it still respects first- and second-order stochastic dominance, though now only weakly.[13] It is also "continuous," under the usual meaning of "$\to \infty$." The other properties also apply, *mutatis mutandis.*

5.12 Emphasis on Losses

As we shall see in Section 6, the riskiness index R is much more sensitive to the loss side of a gamble than to its gain side. Technically, that is because the exponential on the right side of (3.1) has a positive exponent if and only if the value of g is negative. Conceptually, too, the idea of "risk" is usually associated with possible losses rather than with gains; one speaks more of risking losses than of risking smaller gains.

Many of the indices discussed in the literature (see Sections 7.3 - 7.5) also emphasize loss. But there, the emphasis is built in; the *definitions* explicitly put more weight on the loss side. With the index R, the definition as such does

[12] Alternatively characterized by marginal utility being inversely proportional to wealth.

[13] When $g\mathrm{FOD}g_*$ or $g\mathrm{SOD}g_*$, one can now conclude only the weak inequality $R(g) \leq R(g_*)$.

not distinguish between losses and gains, and indeed there is no sharp division between them; the distinction emerges naturally from the analysis.

6 Some Numerical Examples

6.1 A Benchmark

A gamble that results in a loss of l with probability $1/e$, and a "very large" gain with the remaining probability, has riskiness l. Formally, if $g_{M,l}$ yields $-l$ and M with probabilities $1/e$ and $1-(1/e)$ respectively, then $\lim_{M\to\infty} R(g_{M,l}) = l$.

By Positive Homogeneity, one may think of this as "calibrating" the unit of riskiness: Any gamble with riskiness \$1 is "as risky" as one in which the possible loss is \$1 and the possible gain is "very large," where the loss probability is $1/e$—the probability of "no success" in a Poisson distribution with mean 1.

6.2 Some Half-Half Gambles

We have just seen that the riskiness of a gamble yielding a loss of 1 with probability $1/e$, and a large gain with the remaining probability, is close to 1. If the probabilities are half-half, the riskiness goes up to $1/\log 2 \approx 1.44$. If the gain decreases to 3 (so the expectation decreases from ∞ to 1), the riskiness goes up again, but not by much—only to 1.64. If the gain decreases to 1.1—so the expectation is only 0.05—the riskiness jumps to 11.01. As the gain approaches 1—i.e., the expectation approaches 0—the riskiness approaches ∞. The riskiness of a half-half gamble yielding -\$100 or \$105 (Rabin 2000) is $\$2,100$.

6.3 Insurance

To buy insurance is to reject a gamble. For example, suppose you insure a risk of losing $\$20,000$ with probability 0.001 for a premium of \$100—like when buying loss damage waiver in a car rental. Thus you end up with $-\$100$ for sure. If you decline the insurance, you are faced with a gamble that yields $-\$20,000$ with probability .001, and 0 with probability 0.999. If we normalize[14] so that rejecting the gamble is worth 0, then the gamble yields $-\$19,900$ with probability .001, and \$100 with probability 0.999. The riskiness of this gamble is $\$7,491$.

6.4 Riskiness, Desirability, and Acceptance

A riskier gamble need not be less desirable, even when both gambles have the same mean. For example, let g be a 1/2—1/2 gamble yielding -3 or 5, and let h be a 7/8—1/8 gamble yielding -1 or 15. The respective riskinesses of g

[14] You cannot "stay where you are;" you must either pay the premium, which means moving to your current wealth w less \$100, or decline the insurance, which means moving to $w - \$100$ plus the gamble g described in this sentence. That is like choosing between g and \$0, from what your vantage point would be if your current wealth were $w - \$100$.

and h—both of which have mean 1—are 7.7 and 9.2, but a CARA agent i with sufficiently high parameter α will prefer the riskier gamble h; he will essentially disregard the gains in both gambles, and will prefer a loss that though more likely, is smaller in magnitude. Indeed, i's utilities for g and h are, respectively, $-\frac{1}{2}e^{3\alpha}(1+o(1))$ and $-\frac{7}{8}e^{\alpha}(1+o(1))$; for sufficiently high α, the second is higher than the first.

Moreover, there are even agents who accept the riskier gamble h and reject the less risky one g. For example, that is so at wealth 0 for an agent with utility function $u(x) := \min(2x, x)$. To be sure, the function u is not twice continuously differentiable; but it can easily be modified so that it will be, without substantially affecting the example.

On the other hand, such an agent cannot be CARA. Indeed, as we have seen (4.2.2), if a CARA agent rejects a gamble, then he rejects any riskier gamble.

7 The Literature

This section reviews other indices, and compares them to ours. A prominent feature of many is that they are not monotonic w.r.t. first-order dominance; indeed, they may rate a gamble g riskier than h even though h is *sure* to yield more than g. The review is not exhaustive; we content ourselves with discussing some of the indices, and briefly mentioning some others.

7.1 Measures of Dispersion

Pure measures of dispersion like standard deviation, variance, mean absolute deviation ($\mathrm{E}|g-\mathrm{E}g|$), and interquartile range[15] have been suggested as indices of riskiness; see the survey of Machina and Rothschild (1987). These indices measure *only* dispersion, taking little account of the gamble's actual values. Thus if g and $g+c$ are gambles, where c is a positive constant, then any of these indices rate $g+c$ precisely as risky as g, in spite of it's being *sure* to yield *more* than g. An even stranger index (op. cit.) is entropy,[16] which totally disregards the values of the gamble, taking into account only their probabilities; thus a gamble with three equally probable (but different) values has entropy $\log_2 3$, no matter what its values are. It seems obvious that such measures of dispersion cannot embody the economic, decision-making notion of riskiness set forth in Section 3. As Hanoch and Levy (1970, p.344) put it, "The identification of riskiness with variance, or with any other single measure of dispersion, is clearly unsound. There are many obvious cases where more dispersion is desirable, if it is accompanied by an upward shift in the locations of the distribution, or by an increasing positive asymmetry."

[15] The difference between the first and third quartiles of the gamble's distribution. So, if g yields -\$100, -\$1, \$2, and \$1000 with probability 1/4 each, then the interquartile range is \$3.

[16] $-\quad{}_k\, p_k \log_2 p_k$, where the p_k range over the probabilities of the gamble's different values.

7.2 Standard Deviation/Mean

Standard deviation/mean is related to the *Sharpe Ratio,* a measure of "risk-adjusted returns" frequently used to evaluate portfolio selection; see, e.g., Bodie, Kane and Marcus (2002) and Welch (2005). Specifically, any portfolio is associated with a gamble g; the Sharpe ratio of the portfolio is defined[17] as μ/σ, where μ is the mean of g, and σ its standard deviation. Portfolios with a smaller Sharpe ratio are considered riskier, so σ/μ—the reciprocal of the Sharpe ratio—might be considered an index of riskiness of the portfolio.

This index violates M-FOD. Indeed, let g be a gamble yielding -1 with probability 0.02 and 1 with probability 0.98, and h a gamble that yields -1 with probability 0.02, yields 1 with probability 0.49, and yields 2 with probability 0.49. Then g has $\mu = 0.96$ and $\sigma = 0.28$, so $\sigma/\mu = 7/24 \approx 0.29$. For h, the numbers are $\mu = 1.45$ and $\sigma = 7\sqrt{3}/20$, so $\sigma/\mu = 7\sqrt{3}/29 \approx 0.42$. Thus h is rated *more* risky than g, though hFODg. Moreover, when ε is positive but small, $h + \varepsilon$ is *sure* to yield more than g, but is nevertheless rated riskier.

A final remark, regarding normal gambles, is of interest. As we said, the Sharpe ratio is viewed as a measure of risk-adjusted returns. If one takes the ratio of the mean μ to the riskiness index—which in this case $= \sigma^2/\mu$, by Section 5.7—the result is $2\mu^2/\sigma^2$, which is ordinally equivalent to the Sharpe ratio. Thus, the Sharpe ratio ranks normal gambles by their riskiness-adjusted expected returns. Matters are different for non-normal gambles.

7.3 Value at Risk

Another index used extensively by banks and finance professionals in portfolio risk management is *value at risk* (VaR). This depends on a parameter called a *confidence level.* At a 95% confidence level, the VaR of a gamble g is the absolute value of its fifth percentile, when that is non-positive, and 0 otherwise. In words, it is the greatest possible loss, ignoring losses with probability less than 5%. Thus a gamble yielding -\$1,000,000, -\$1, and \$100,000 with respective probabilities of 0.04, 0.02, and 0.94 has a 95% VaR of \$1, and so does the gamble yielding -\$1 and \$100,000 with 0.06 and 0.94 probabilities.

This index depends on a parameter—the confidence level—whose "appropriate" value is not clear. Also, it ignores completely the gain side of the gamble; in particular, it violates M-FOD. And even on the loss side, it concentrates only on that loss that "hits" the confidence level.

7.4 "Coherent" Measures of Risk

Artzner, Delbaen, Eber, and Heath (1999) call an index Q *coherent* if it satisfies five axioms: (i) Positive Homogeneity, (ii) Subadditivity, (iii) Weak First-Order Monotonicity, (iv) "Relevance," and (v) "Translation invariance." Axiom (i) is as in our Section 3; (ii) is our (5.8.2) (except that with us it follows from the axioms, whereas they assume it). Their (iii) says that if $g \geq h$ identically, then

[17]The standard definition looks more complicated, but boils down to this.

$Q(g) \geq Q(h)$, which for us follows from first-order monotonicity (Section 5.3). Thus our index obeys their first three axioms. Their (iv) concerns "gambles" with no positive values, which we exclude.[18] Their Axiom (v) says that if c is a constant, then $Q(g+c) = Q(g) - c$, which is not the case for our index.

Like ours, their indices measure risk in dollars. But their five axioms are very far from determining the index. Indeed, for *any* family of probability measures μ on the underlying probability space, the supremum of $\mathrm{E}_\mu(-g)$ over the family is a coherent index. One example is $|\min g|$, which violates first and second order monotonicity and also continuity; but there are very many others. All these indices violate our duality axiom.

7.5 Additional Indices

Brachinger (2002) and Brachinger and Weber (1997) are good surveys of the psychological literature. Like VaR and the "coherent" measures, these measures of *perceived* risk take the form of families rather than proposing a single index. The studies include Coombs (1969), Pollatsek and Tversky (1970), Fishburn (1977, 1982, 1984), Luce (1980), Sarin (1987), Luce and Weber (1988), and Jia, Dyer and Butler (1999).

Of all these, Sarin's measure $S(g) := \mathrm{E}e^{-g}$ is the closest to our index R. This is monotonic w.r.t. FOD, so it must violate duality. Indeed, let g be the gamble that assigns probability 0.01 to a loss of 1 and probability 0.99 to a gain of 2. Then $S(2g) = 0.09 < 0.16 = S(g)$. In contrast, $R(2g) = 2R(g) > R(g)$. To see that S violates duality, set $\alpha := 1/R(g)$. By (3.1), a CARA agent i with parameter $\frac{5}{6}\alpha$ accepts g, while a CARA agent j with parameter $\frac{2}{3}\alpha$—who is less risk-averse than i—rejects $2g$, which is rated less risky than g by S. So S violates Duality. It also violates Positive Homogeneity.

8 Proofs

8.1 Preliminaries

In this section, agents i and j have utility functions u_i and u_j, and Arrow-Pratt coefficients ρ_i and ρ_j of absolute risk aversion. Since utilities may be modified by additive and positive multiplicative constants, we may—and do—assume throughout the following that

(1) $u_i(0) = u_j(0) = 0$ and $u_i'(0) = u_j'(0) = 1$.

LEMMA 2: For some $\delta > 0$, suppose that $\rho_i(w) > \rho_j(w)$ at each w with $|w| < \delta$. Then $u_i(w) < u_j(w)$ whenever $|w| < \delta$ and $w \neq 0$.

PROOF: Let $|y| < \delta$. If $y > 0$, then by (1),

$$\log u_i'(y) = \log u_i'(y) - \log u_i'(0) = \int_0^y (\log u_i'(z))' dz = \int_0^y (u_i''(z)/u_i'(z)) dz$$
$$= \int_0^y -\rho_i(z) dz < \int_0^y -\rho_j(z) dz = \log u_j'(y).$$

[18] Unless the domain is extended (Section 5.11), when the riskiness is $+\infty$.

If $y < 0$, the reasoning is similar, but the inequality is reversed, because then $\int_0^y = -\int_0^{|y|}$. Thus $\log u_i'(y) \lesseqgtr \log u_j'(y)$ when $y \gtreqless 0$, so also $u_i'(y) \lesseqgtr u_j'(y)$ when $y \gtreqless 0$.

So if $w > 0$, then by (1), $u_i(w) = \int_0^w u_i'(y)dy < \int_0^w u_j'(y)dy = u_j(w)$, and if $w < 0$, then $u_i(w) = -\int_0^{|w|} u_i'(y)dy < -\int_0^{|w|} u_j'(y)dy = u_j(w)$, q.e.d.

COROLLARY 3: If $\rho_i(w) \leq \rho_j(w)$ for all w, then $u_i(w) \geq u_j(w)$ for all w.

PROOF: Similar to that of Lemma 2, with i and j interchanged, strict inequalities replaced by weak inequalities, and the restriction to $|w| < \delta$ eliminated.

LEMMA 4: If $\rho_i(w_i) > \rho_j(w_j)$, then there is a gamble g that j accepts at w_j and i rejects at w_i.

PROOF: W.l.o.g.[19] $w_i = w_j = 0$, so $\rho_i(0) > \rho_j(0)$. Since u_i and u_j are twice continuously differentiable, it follows that there is a $\delta > 0$ such that $\rho_i(w) > \rho_j(w)$ at each w with $|w| < \delta$. So by Lemma 2,

(5) $u_i(w) < u_j(w)$ whenever $|w| < \delta$ and $w \neq 0$.

Choose ε with $0 < \varepsilon < \delta/2$. For $0 \leq x \leq \varepsilon$, and $k = i, j$, set $f_k(x) := \frac{1}{2}u_k(-\varepsilon + x) + \frac{1}{2}u_k(\varepsilon + x)$. By (5),

(6) $f_i(x) < f_j(x)$ for all x.

By (6), concavity, and (1), $f_i(0) < f_j(0) \leq u_j(0) = 0$. By monotonicity of the utilities, $f_i(\varepsilon) = \frac{1}{2}u_i(2\varepsilon) > \frac{1}{2}u_i(0) = 0$. So $f_i(y) = 0$ for some y between 0 and ε, since f_i is continuous. So by (6), $f_j(y) > 0$. So if $\eta > 0$ is sufficiently small, then $f_j(y-\eta) > 0 > f_i(y-\eta)$. So if g is the half-half gamble yielding $-\varepsilon + y - \eta$ or $\varepsilon + y - \eta$, then $\mathrm{E}u_j(g) = f_j(y-\eta) > 0 > f_i(y-\eta) = \mathrm{E}u_i(g)$. So j accepts g whereas i rejects it, q.e.d.

8.2 Proof[20] of (4.1.1)

"Only if": Assume i no less risk-averse than j; we must show

(7) $\rho_i(w) \geq \rho_j(w)$ for all wealth levels w.

If not, then there is a w with $\rho_i(w) < \rho_j(w)$. So by Lemma 4, there is a gamble that i accepts at w and j rejects at w, contradicting i being less risk-averse than j. So (7) is proved.

"If": Assume (7); we must show that for each wealth level w and gamble g, if i accepts g at w, then j accepts g at w. W.l.o.g. $w = 0$, so we must show that

(8) if i accepts g at 0, then j accepts g at 0.

From (1), (7), and Corollary 3 (with i and j reversed), we conclude $u_j(w) \geq u_i(w)$ for each w. So $\mathrm{E}u_j(g) \geq \mathrm{E}u_i(g)$, which yields (8), q.e.d.

[19] "Without loss of generality." For arbitrary w_i and w_j, define $u_i^*(x) := (u_i(x + w_i) - u_i(w_i))/u_i'(w_i)$ and u_j^* similarly, and apply the current reasoning to u_i^* and u_j^*.

[20] (4.1.1) and (4.1.2) are needed in the proof of Theorem A, so we prove them first.

8.3 Proof of (4.1.2)

(4.1.2) follows from (4.1.1) by shifting the independent variable on one of the utilities to make $w_i = w_j$.

8.4 Proof of Theorem A

For $\alpha > 0$, let $u_\alpha(x) = (1 - e^{-\alpha x})/\alpha$; this is a CARA utility function with parameter α. The functions u_α satisfy (1), so by Lemma 2 (with δ arbitrarily large), their graphs are "nested;" that is,

(9) if $\alpha > \beta$, then $u_\alpha(x) < u_\beta(x)$ for all $x \neq 0$.

To see that there is a unique $R(g) > 0$ satisfying (3.1), set $f(\alpha) := \mathrm{E}e^{-\alpha g} - 1$, and note that f is convex, $f(0) = 0$, $f'(0) = -\mathrm{E}g < 0$, and $f(M) > 0$ for M sufficiently large. So there is a unique $\gamma > 0$ with $f(\gamma) = 0$, and we set $R(g) := 1/\gamma$.

To see that R satisfies the duality axiom, let i, j, g, h, w, be as in the hypothesis of that axiom; w.l.o.g. $w = 0$. Set $\gamma := 1/R(g)$, $\eta := 1/R(h)$, $\alpha_i := \inf \rho_i$, $\alpha_j := \sup \rho_j$. Thus

(10) $\mathrm{E}u_\gamma(g) = (1 - \mathrm{E}e^{-\gamma g})/\gamma = 0$ and $\mathrm{E}u_\eta(h) = (1 - \mathrm{E}e^{-\eta h})/\eta = 0$.

By hypothesis, $R(g) > R(h)$, so $\eta > \gamma$. By Corollary 3,

(11) $u_i(x) \leq u_{\alpha_i}(x)$ and $u_{\alpha_j}(x) \leq u_j(x)$ for all x.

Now assume $\mathrm{E}u_i(g) > 0$; we must prove that $\mathrm{E}u_j(h) > 0$. From $\mathrm{E}u_i(g) > 0$ and (11) it follows that $\mathrm{E}u_{\alpha_i}(g) > 0$. So by (10), $\mathrm{E}u_\gamma(g) = 0 < \mathrm{E}u_{\alpha_i}(g)$. So by (9), $\gamma > \alpha_i$. By (4.1.2), $\alpha_i \geq \alpha_j$, so $\eta > \gamma$ yields $\alpha_j < \eta$. Then (10), (9) and (11) yield $0 = \mathrm{E}u_\eta(h) < \mathrm{E}u_{\alpha_j}(h) < \mathrm{E}u_j(h)$, so indeed, R satisfies the duality axiom. That R is positively homogeneous is immediate, so indeed, R satisfies the axioms.

In the opposite direction, let Q be an index that satisfies the axioms. We first show that

(12) Q is ordinally equivalent to R.

If this is not true, then there must exist g and h that are ordered differently by Q and R. This means that either the respective orderings are reversed, i.e.,

(13) $Q(g) > Q(h)$ and $R(g) < R(h)$,

or that equality holds for exactly one of the two indices; i.e.,

(14) $Q(g) > Q(h)$ and $R(g) = R(h)$

or

(15) $Q(g) = Q(h)$ and $R(g) > R(h)$.

If either (14) or (15), then by homogeneity, replacing g by $(1-\varepsilon)g$ for sufficiently small positive ε leads to reversed inequalities. So w.l.o.g. we may assume (13).

Now let $\gamma := 1/R(g)$, $\eta := 1/R(h)$; then (10) holds. By (13), $\gamma > \eta$. Choose μ and ν so that $\gamma > \mu > \nu > \eta$. Then $u_\gamma(x) < u_\mu(x) < u_\nu(x) < u_\eta(x)$ for all $x \neq 0$. So by (10), $\mathrm{E}u_\mu(g) > \mathrm{E}u_\gamma(g) = 0$ and $\mathrm{E}u_\nu(h) < \mathrm{E}u_\eta(h) = 0$. So if i and j have utility functions u_μ and u_ν respectively, then i accepts g and j rejects

h. But from $\mu > \nu$ and (4.1.2), it follows that $i \trianglerighteq j$, contradicting the duality axiom for Q. So (12) is proved.

To see that Q is a positive multiple of R, let g_0 be an arbitrary but fixed gamble, and set $\lambda := Q(g_0)/R(g_0)$. If g is any gamble, and $t := Q(g)/Q(g_0)$, then $Q(tg_0) = tQ(g_0) = Q(g)$, so $tR(g_0) = R(tg_0) = R(g)$ by the ordinal equivalence between Q and R, so $R(g)/R(g_0) = t = Q(g)/Q(g_0)$, so $Q(g)/R(g) = Q(g_0)/R(g_0) = \lambda$, so $Q(g) = \lambda R(g)$. This completes the proof of Theorem A.

Needless to say, both duality and positive homogeneity are essential to Theorem A. Indeed, the mean Eg is positively homogeneous, but violates duality, and the index $[R(g)]$, where $[x]$ denotes the integer part of x, satisfies duality, but is not positively homogeneous. Neither Eg nor $[R(g)]$ are even ordinally equivalent to R.

8.5 Proof of (4.2.1)

"Only if:" All CARA utility functions have the form $-e^{-\alpha x}$. Thus i accepts g at wealth w if and only if $-\mathrm{E}e^{-\alpha(g+w)} > -e^{-\alpha w}$, i.e., if and only if $\mathrm{E}e^{-\alpha g} < 1$; and this condition does not depend[21] on w.

"If:" Suppose i's Arrow-Pratt index of absolute risk aversion is not constant, say $\rho(w) > \rho(w_*)$. Consider a gamble yielding $\pm\delta$ with probabilities p and $1-p$ respectively, and let $p_\delta(w)$ be that p for which i is indifferent at w between taking and not taking the gamble. Then[22] $\rho(w) = \lim_{\delta\to 0}(p_\delta(w) - \frac{1}{2})/\delta$; i.e., noting that even-money $\frac{1}{2}-\frac{1}{2}$ bets are always rejected by risk-averse utility maximizers, the Arrow-Pratt index is the probability *premium* over $\frac{1}{2}$, per dollar, that is needed for i to be indifferent between taking and not taking a small even-money gamble. So, if δ is sufficiently small, $q - \frac{1}{2}$ lies half-way between $\rho(w)$ and $\rho(w_*)$, and g is an even money gamble yielding $\pm\delta$ with probabilities q and $1-q$ respectively, then i accepts g at w_* and rejects it at w; this proves the contrapositive of "if," and so "if" itself.

8.6 Proof of (4.2.2)

Let g_1 be a gamble, g_2 a riskier gamble. For $\alpha \geq 0$ and $i = 1, 2$, set $f_i(\alpha) := \mathrm{E}e^{-\alpha g_i} - 1$. We saw (near the start of the proof of Theorem A) that $f_i(0) = 0$, $f_i(\alpha) < 0$ when $0 < \alpha < 1/R(g_i)$, $f_i(1/R(g_i)) = 0$, and $f(\alpha) > 0$ when $\alpha > 1/R(g_i)$. So a CARA agent with parameter α accepts g_i if and only if $f_i(\alpha) < 0$; i.e., if and only if $\alpha \in (0, 1/R(g_i))$. Since $1/R(g_2) < 1/R(g_1)$, it follows that if the agent rejects g_1, then he also rejects g_2, as was to be proved. This proves the second sentence, and so the whole assertion.

[21] Pratt (1964, p.130) makes a similar argument for preferences between gambles.

[22] E.g., see Aumann and Kurz (1977), Section 6; but there may well be earlier sources.

8.7 Proof of the Claims in Section 5.3

For $\alpha \geq 0$, set $f(\alpha) := \mathrm{E}e^{-\alpha g}$, $f_*(\alpha) := \mathrm{E}e^{-\alpha g_*}$. If g FOD g_*, then $f(\alpha) < f_*(\alpha)$ whenever $\alpha > 0$. From this and the proof that (3.1) has a unique positive root,[23] it follows that the unique positive root of $f_* = 1$ is smaller than that of $f = 1$, so $R(g_*) > R(g)$, as asserted.

If g SOD g_*, then, too, $f(\alpha) < f_*(\alpha)$, because of the strict convexity of $e^{-\alpha x}$ as a function of x. The remainder of the proof is as before.

8.8 Proof of the Claim in Section 5.4

For $\alpha \geq 0$, set $f(\alpha) := \mathrm{E}e^{-\alpha g}$, $f_n(\alpha) := \mathrm{E}e^{-\alpha g_n}$; denote the unique positive root of $f = 1$ by γ, of $f_n = 1$ by γ_n. We have $f_n \to f$, uniformly in any finite interval. Now $f(\gamma/2) < 1$ and $f(2\gamma) > 1$. So for n sufficiently large, $f_n(\gamma/2) < 1$ and $f_n(2\gamma) > 1$, so $\gamma/2 < \gamma_n < 2\gamma$. Suppose that the γ_n have a limit point $\gamma_* \neq \gamma$; arguing by contradiction, we may assume w.l.o.g. that it is the limit. For any $\varepsilon > 0$, we have $|f_n(\gamma_n) - f(\gamma_n)| < \varepsilon$ for n sufficiently large, because of the uniform convergence. Also $|f(\gamma_n) - f(\gamma_*)| < \varepsilon$, because of the continuity of f. So $|f_n(\gamma_n) - f(\gamma_*)| < 2\varepsilon$. So $\lim f_n(\gamma_n) = f(\gamma_*) \neq 1$, contradicting $f_n(\gamma_n) = 1$; q.e.d.

8.9 Proof of (5.6.1)

Denote by $g^p \oplus h^{1-p}$ the compound gamble that yields g with probability p and h with probability $1-p$. By Theorem A, the riskiness $R(g^p \oplus h^{1-p})$ is the reciprocal of the unique positive root of $f = 1$, where $f(\alpha) := \mathrm{E}e^{-\alpha(g^p \oplus h^{1-p})} = p\mathrm{E}e^{-\alpha g} + (1-p)\mathrm{E}e^{-\alpha h}$. So if $f(\alpha) = 1$, then it cannot be that both $\mathrm{E}e^{-\alpha g}$ and $\mathrm{E}e^{-\alpha h}$ are > 1, and it cannot be that both $\mathrm{E}e^{-\alpha g}$ and $\mathrm{E}e^{-\alpha h}$ are < 1. So $\mathrm{E}e^{-\alpha g} \leq 1$ and $\mathrm{E}e^{-\alpha h} \geq 1$, say. So $1/R(g^p \oplus h^{1-p}) = \alpha \leq 1/R(g)$ and similarly $1/R(g^p \oplus h^{1-p}) = \alpha \geq 1/R(h)$. Thus $R(g) \leq R(g^p \oplus h^{1-p}) \leq R(h)$, as asserted.

8.10 Proof of (5.8.1)

By Theorem A, the riskiness $R(g+h)$ is the reciprocal of the unique positive root of $f = 1$, where $f(\alpha) := \mathrm{E}e^{-\alpha(g+h)}$. Because g and h are independent, $f(\alpha) = \mathrm{E}e^{-\alpha g}e^{-\alpha h} = \mathrm{E}e^{-\alpha g}\mathrm{E}e^{-\alpha h}$. So if $f(\alpha) = 1$, then it cannot be that both $\mathrm{E}e^{-\alpha g}$ and $\mathrm{E}e^{-\alpha h}$ are > 1, and it cannot be that both $\mathrm{E}e^{-\alpha g}$ and $\mathrm{E}e^{-\alpha h}$ are < 1. So $\mathrm{E}e^{-\alpha g} \leq 1$ and $\mathrm{E}e^{-\alpha h} \geq 1$, say. So $1/R(g+h) = \alpha \leq 1/R(g)$ and similarly $1/R(g+h) = \alpha \geq 1/R(h)$. Thus $R(g) \leq R(g+h) \leq R(h)$, as asserted.

8.11 Proof of (5.8.2)

Set $r := R(g)$, $r' := R(h)$, and $\lambda := r/(r+r') \in (0,1)$. Then $(g+h)/(r+r') =$

[23] Near the beginning of the proof of Theorem A.

$\lambda(g/r) + (1-\lambda)(h/r')$, so from (3.1) and the convexity of the exponential, we get $\mathrm{E}e^{-(g+h)/(r+r')} \le \lambda \mathrm{E}e^{-g/r} + (1-\lambda)\mathrm{E}e^{-h/r'} = 1$, so $r + r' \le R(g+h)$ (see the second paragraph in the proof of Theorem A), as asserted.

8.12 Proof of the Ordinal Characterization in Section 5.9

The proof of ordinal equivalence follows that of (12) above. If either (14) or (15) holds, and Q is first-order monotonic, then replacing g by $g - \varepsilon$ for sufficiently small positive ε leads to reversed inequalities; this follows from first-order monotonicity and continuity. The remainder of the proof of (12) is as above.

To see that first-order monotonicity is essential, define

$$Q(g) := \begin{cases} R(g), & \text{when } 0 < R(g) \le 1, \\ 1, & \text{when } 1 \le R(g) \le 2, \\ R(g) - 1, & \text{when } 2 \le R(g). \end{cases}$$

Thus Q collapses the interval $[1, 2]$ in the range of R to a single point. It may be seen that it is continuous and satisfies the duality axiom, but is not first-order monotonic; and there are g and h (in the "collapsed" region) satisfying (15), so Q is not ordinally equivalent to R.

To see that continuity is essential, let A be a non-empty proper subset of the set $R^{-1}(1)$ of all gambles with riskiness 1. Define

$$Q(g) := \begin{cases} R(g), & \text{when } R(g) < 1 \text{ or } g \in A, \\ R(g) + 1, & \text{when } R(g) > 1 \text{ or } g \in R^{-1}(1) \backslash A. \end{cases}$$

One may think of Q as resulting from R by "tearing" along the "seam" $R(g) = 1$, with the seam itself going partly to the upper fragment and partly to the lower fragment. It may be seen that Q is first-order monotonic and satisfies the duality axiom, but is not continuous; and there are g and h (on the "seam") satisfying (15), so Q is not ordinally equivalent to R.

Finally, as already argued at the end of Section 7, Sarin's index $S(g)$ is continuous and first-order monotonic, but violates duality.

8.13 Proof of (5.10.2)

To prove the first sentence, let u_i be i's utility, and define a utility u_j as follows: when x is between $w + \min g$ and $w + \max g$, define $u_j(x) := u_i(x)$; when $x \le w + \min g$, define $u_j(x)$ to equal a CARA utility with parameter $\rho_i(w + \min g)$ and $u_j(w+\min g) = u_i(w+\min g)$ and $u_j'(w+\min g) = u_i'(w+\min g)$; when $x \ge w + \max g$, define $u_j(x)$ to equal a CARA utility with parameter $\rho_i(w + \max g)$ and $u_j(w + \max g) = u_i(w + \max g)$ and $u_j'(w + \max g) = u_i'(w + \max g)$. Let u_k be a CARA utility with parameter $(1/R(g)) - \varepsilon$. Then

(16) $\min_x \rho_k(x) > \max_x \rho_j(x)$

for positive ε sufficiently small. By Theorem B (Section 4.2), a CARA person with parameter $1/R(g)$ is indifferent between taking and not taking g. Therefore k, who is less risk-averse, accepts g. So by (16) and (4.1.2), j also accepts g. But between the minimum and maximum of $w + g$, the utilities of i and j are the same. So i accepts g at w. This proves the first sentence of (5.10.2); the proof of the second sentence is similar.

9 Conclusion

We have defined a numerical index of the *riskiness* of a gamble with stated dollar outcomes and stated probabilities. It is denominated in dollars, monotonic w.r.t. first and second order stochastic dominance, continuous in about any sense one wishes, positively homogeneous, and satisfies a duality condition that says, roughly, that agents who are more risk-averse are less likely to accept gambles that are riskier. Moreover, it is the *only* index satisfying these conditions.

REFERENCES

Arrow, K. J. (1965), *Aspects of the Theory of Risk-Bearing,* Helsinki: Yrjö Jahnssonin Säätiö.

—(1971), *Essays in the Theory of Risk Bearing,* Chicago: Markham Publishing Company.

Artzner, P., F. Delbaen, J.-M. Eber, and D. Heath (1999), "Coherent Measures of Risk," *Mathematical Finance* 9, 203-228.

Aumann, R. J. and M. Kurz (1977), "Power and Taxes," *Econometrica* 45, 1137-1161.

Bernoulli, D. (1731), "Letter on July 4 to Nicolas Bernoulli;" published (1738), "Specimen Theoriae Novae de Mensura Sortis," in *Commentarii Academiae Scientiarum Imperialis Petropolitanae* 5; reprinted in translation, (1954), "Exposition of a New Theory on the Measurement of Risk," *Econometrica* 22, 23-36.

Bodie, Z., A. Kane and A. J. Marcus (2002), *Investments,* 5th edition, New York: McGraw Hill.

Brachinger, H. W. (2002), "Measurement of Risk," in Derigs, U. (Ed.), *Optimization and Operations Research,* pp. 1119-1137, Encyclopedia of Life Support Systems (EOLSS), UNESCO, EOLSS Publishers Co Ltd.

Brachinger, H. W. and M. Weber (1997), "Risk as a Primitive: a Survey of Measures of Perceived Risk," *Operations Research Spectrum* 19, 235-250.

Coombs, C. H. (1969), "Portfolio Theory: A Theory of Risky Decision Making," in G. T. Guilbaud (ed.), *La Decision,* Paris: Centre National de la Recherche Scientifique.

Diamond, P. A. and J. Stiglitz (1974), "Increases in Risk and Risk Aversion," *Journal of Economic Theory* 8, 337-360.

Fishburn, P. C. (1977), "Mean-Risk Analysis with Risk Associated with Below Target Returns," *American Economic Review* 67, 116-126.

—(1982), "Foundations of Risk Measurement (II): Effects of Gains on Risk," *Journal of Mathematical Psychology* 25, 226-242.

—(1984), "Foundations of Risk Measurement (I): Risk as Probable Loss," *Management Science* 30, 396-406.

Hadar, J. and W.R. Russell (1969), "Rules for Ordering Uncertain Prospects," *American Economic Review* 59, 25-34.

Hanoch, G. and H. Levy (1969), "The Efficiency Analysis of Choices Involving Risk," *Review of Economic Studies* 36, 335-346.

—(1970), "Relative Effectiveness of Efficiency Criteria for Portfolio Selection," *Journal of Financial Quantitative Analysis* 5, 63-76.

Jia, J., J. Dyer and J. Butler (1999), "Measures of Perceived Risk," *Management Science* 45, 519-532.

Luce, R. D. (1980), "Several Possible Measures of Risk," *Theory and Decision* 12, 217-228. Correction: (1981), *Theory and Decision* 13, 381.

Luce, R. D. and E. U. Weber (1988), "An Axiomatic Theory of Conjoint Expected Risk," *Journal of Mathematical Psychology* 30, 188-205.

Machina, M. and M. Rothschild (1987), "Risk," in J. Eatwell et al (eds.), *The New Palgrave Dictionary of Economics,* London: McMillan.

Palacios-Huerta, I., R. Serrano and O. Volij (2004), "Rejecting Small Gambles Under Expected Utility," Working Paper, April.

Pollatsek, A. and A. Tversky (1970), "A Theory of Risk," *Journal of Mathematical Psychology* 7, 540-553.

Pratt, J. (1964), "Risk Aversion in the Small and in the Large," *Econometrica* 32, 122-136.

Rabin, M. (2000), "Risk Aversion and Expected Utility: A Calibration Theorem," *Econometrica* 68, 1281-1292.

Rothschild, M. and J. E. Stiglitz (1970), "Increasing Risk: I. A Definition," *Journal of Economic Theory* 2, 225-243.

Sarin, R. K. (1987), "Some Extensions of Luce's Measures of Risk," *Theory and Decision* 22, 125-141.

Tobin, J. (1969), "Comment on Borch and Feldstein," *Review of Economic Studies* 36, 13-14.

Welch, I. (2005), *A First Course in Corporate Finance,* available at http://welch.econ.brown.edu/book/

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