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Research Article

An Efficient Analytical Approach for the Periodicity of Nano/Microelectromechanical Systems' Oscillators

Naveed Anjum,[1] **Jamshaid Ul Rahman**,[2] **Ji-Huan He**,[3,4,5] **Md. Nur Alam**,[6] **and Muhammad Suleman**[7]

[1]*Department of Mathematics, Government College University, Faisalabad 38000, Pakistan*
[2]*Abdus Salam School of Mathematical Sciences, GC University, Lahore 54600, Pakistan*
[3]*National Engineering Laboratory for Modern Silk, College of Textile and Engineering, Soochow University, Suzhou 215003, China*
[4]*School of Science, Xi'an University of Architecture and Technology, Xi'an 710055, China*
[5]*School of Mathematics and Information Science, Henan Polytechnic University, Jiaozuo 454003, China*
[6]*Department of Mathematics, Pabna University of Science and Technology, Pabna 6600, Bangladesh*
[7]*Department of Mathematics, Comsat University, Islamabad 45550, Pakistan*

Correspondence should be addressed to Md. Nur Alam; nuralam.pstu23@gmail.com

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Periodic behavior analysis of nano/microelectromechanical systems (N/MEMS) is an essential field owing to their many promising applications in microinstruments. The interesting and unique properties of these systems, particularly, small size, batch fabrication, low power consumption, and high reliability, have fascinated researchers and industries to implement these structures for the production of different microdevices. The dynamic oscillatory behavior of N/MEMS is very intricate due to the various types of nonlinearities present in these structures. The foremost objective of this study is to explore the periodicity of oscillatory problems from N/MEMS. The variational iteration method (VIM), which has been considered as an effective approach for nonlinear oscillators, is coupled with the Laplace transform to obtain the approximate analytic solution of these nonlinear vibratory systems with fewer computations. This coupling of VIM and Laplace transform not only helps in the identification of the Lagrange multiplier without getting into the details of the cryptic theory of variations, but also finds the frequency-amplitude relationship and the analytic approximate solution of N/MEMS. A generalized vibratory equation for N/MEMS is followed by three examples as special cases of this generalized equation are given to elucidate the effectivity of the coupling. The solution obtained from the Laplace-based VIM not only exhibits good agreement with observations numerically but also higher accuracy yields when compared to other established techniques in the open literature.

1. Introduction

A few decades have passed since the revelation and advancement of nano/microelectromechanical systems (N/MEMS). This innovation now has touched a level of maturity that, nowadays, several N/MEMS devices are being utilized in our daily life, ranging from pressure sensors and accelerometers in cars, radiofrequency switches, micromirrors in electronics devices such as Plasma TVs, microphones in the telecommunication industry, and inertia sensors in video games [1–6]. Conversely, with this developing demand on the N/MEMS innovation come incredible challenges. Dynamic analysis is one of them and has experienced rapid development [7]. The oscillators from N/MEMS have rich dynamics, and there are many phenomena involved in the dynamic analysis of a N/MEMS such as pull-in instability, phase diagram, and hysteresis. However, the focus of this manuscript is on the periodic solution property of N/MEMS. The exact solutions of N/MEMS, which are hard to find, play a vital role in examining the properties and behavior of these systems. Thus, researchers are interested in finding at least analytic solutions because they have more detail which helps with better insight into these systems.

In the past decades, several techniques have been proposed to get the approximate analytic solution of N/MEMS problems such as the homotopy perturbation method (HPM), higher-order HPM [8], Taylor series [9], energy balance technique [10], spreading residual harmonic balance method [11], higher-order Hamiltonian method [12], Adomian decomposition method (ADM) [13], Li-He modified HPM [14], modified ADM [15], variational approach [16], Galerkin decomposition method [17], and so on. It is also noted that, besides these methods, there are various analytical techniques for getting the approximate solution to the nonlinear equations, for example, the He-Laplace method [18], global residual harmonic balance method [19], integral transform-based methods [20–22], max-min approach [23], frequency-amplitude formulation method [24], Hamiltonian approach [25], and others [26–29]. Moreover, there have been several review articles that have appeared on the analytical methods for oscillatory problems during the past decade [30–32].

The variational iteration method (VIM) [33] is one of the most powerful techniques among the aforementioned methods, capable to solve linear and nonlinear, ordinary and partial differential equations [34–41] analytically and leading to truthful results. It was first proposed in 1998 [33] and has been extensively discussed, including its extensions and modifications [22, 41]. The main theme of the method involves the construction of a suitable iterative formula with a Lagrange multiplier that is optimally determined with the help of the variational theory. As there is no need to linearize or treat the nonlinear terms, therefore, authors [22] recently recommend that Laplace transform a simpler method to evaluate the multiplier, rendering the approach available to researchers facing different nonlinear problems. Additionally, it is noticed that nonlinear oscillators benefit greatly from this modification.

In this study, we construct a generalized nonlinear vibrational problem for N/MEMS, which under various conditions, reduces to different physical systems such as electrostatic force-based N/MEMS, the dynamic behavior of the microbeams induced by van der Waals attractions, the periodicity of the multiwalled carbon nanotubes under the effect of an electric field, etc. A Laplace-based VIM (LVIM) is employed to obtain a general solution of these microsystems and hence to obtain the deflection (y) and the oscillator's frequency (Ω) for different scenarios as the particular cases of the generalized problem. We match the findings of LVIM to those yielded numerically using the fourth-order Runge–Kutta method (RK4) and other established methods to endorse its usefulness for N/MEMS.

2. Formalism

Nonlinear oscillators often hold the following equation as follows:

$$\ddot{y}(t) + f(y) = 0, \tag{1}$$

$$\begin{aligned} y(0) &= B, \\ \dot{y}(0) &= 0. \end{aligned} \tag{2}$$

Equation (1) can be written as follows:

$$\ddot{y} + \Omega^2 y + g(y) = 0, \tag{3}$$

where $g(y) = f(y) - \Omega^2 y$.

Recently, authors [22] proposed a simple way of identifying the Lagrange multiplier for the equation (1) which is based on the Laplace transform. Let us revisit the general methodology.

Initially, the correction functional for equation (3) is established as follows:

$$y_{k+1}(t) = y_k(t) + \int_0^t \lambda(t-\psi)\left[\ddot{y}_k(\psi) + \Omega^2 y_k(\psi) + \tilde{g}_k(\psi)\right]\mathrm{d}\psi, \quad k = 0, 1, 2, \ldots\ldots, \tag{4}$$

where λ is the multiplier, y_k depicts the kth solution, and $\tilde{g}_k$ is a restricted variation i.e., $\delta\tilde{g}_k = 0$. The integration in equation (4) is ultimately a convolution; therefore, we can employ the Laplace transform easily. Utilizing the properties of the Laplace transform, and then through restricted variation, the Lagrange multiplier is identified as follows:

$$\lambda(t) = -\frac{1}{\Omega}\sin \Omega t. \tag{5}$$

Finally, the correction functional will get the following form:

$$L\left[y_{k+1}(t)\right] = L\left[y_k(t)\right] - \frac{1}{\Omega}L[\sin \Omega t]L\left[\ddot{y}_k(t) + \Omega^2 y_k(t) + \tilde{g}_k(y)\right]. \tag{6}$$

A detail derivation about the aforementioned method of solution can be seen in Ref. [21].

3. Applications

This section is devoted to a general vibratory system for N/MEMS to explain the theory described in formalism, followed by three well-known N/MEMS as the special cases of this general problem.

Consider the motion of microstructures represented with a nonlinear ordinary differential equation characterized by the general form of a group of oscillators from N/MEMS [10–12, 14, 39, 42–44] used in nanoscience and nanotechnology.

$$\begin{aligned} &\left(\alpha_0 + \alpha_1 y + \alpha_2 y^2 + \alpha_3 y^3 + \alpha_4 y^4\right)\ddot{y} + \alpha_5 + \alpha_6 y \\ &\quad + \alpha_7 y^2 + \alpha_8 y^3 + \alpha_9 y^4 + \alpha_{10} y^5 + \alpha_{11} y^6 + \alpha_{12} y^7 = 0, \end{aligned} \tag{7}$$

where $\alpha_0, \alpha_1, \ldots, \alpha_{12}$ are constants found in result of transforming a multivariable differential equation to an ordinary differential equation using Galerkin approach. Dividing equation (7) by α_0 yields

$$\left(1 + d_1 y + d_2 y^2 + d_3 y^3 + d_4 y^4\right)\ddot{y} + d_5 + d_6 y + d_7 y^2 + d_8 y^3 + d_9 y^4 + d_{10} y^5 + d_{11} y^6 + d_{12} y^7 = 0, \tag{8}$$

where $d_j = \alpha_j/\alpha_0$ for $j = 1, 2, \cdots, 12$. Let us rewrite equation (8) as follows:

$$\ddot{y} + \Omega^2 y + g(y) = 0, \tag{9}$$

where

$$g(y) = \left(d_1 y + d_2 y^2 + d_3 y^3 + d_4 y^4\right)\ddot{y} + d_5 + \left(d_6 - \Omega^2\right) y + d_7 y^2 + d_8 y^3 + d_9 y^4 + d_{10} y^5 + d_{11} y^6 + d_{12} y^7. \tag{10}$$

The iterative formula, equation (6), can be expressed as follows:

$$L[y_{k+1}(t)] = L[y_k] - \frac{1}{\Omega} L[\sin \Omega t] L\begin{bmatrix} \left(1 + d_1 y_k + d_2 y_k^2 + d_3 y_k^3 + d_4 y_k^4\right)\ddot{y} + d_5 + d_6 y_k \\ + d_7 y_k^2 + d_8 y_k^3 + d_9 y_k^4 + d_{10} y_k^5 + d_{11} y_k^6 + d_{12} y_k^7 \end{bmatrix}. \tag{11}$$

Assuming the initial solution

$$y_0(t) = B \cos \Omega t. \tag{12}$$

After simple calculations, we have

$$L[y_1(t)] = L[B \cos \Omega t] - \frac{1}{\Omega} L[\sin \Omega t] L\begin{bmatrix} \Gamma_0 + \Gamma_1 \cos \Omega t + \Gamma_2 \cos 2\,\Omega t + \Gamma_3 \cos 3\,\Omega t + \\ \Gamma_4 \cos 4\,\Omega t + \Gamma_5 \cos 5\,\Omega t + \Gamma_6 \cos 6\,\Omega t + \Gamma_7 \cos 7\,\Omega t \end{bmatrix}. \tag{13}$$

The following relation helps in solving the above-mentioned equation:

$$L^{-1}\left(L[\sin \Omega t] L[\cos \kappa \Omega t]\right) = \begin{cases} \dfrac{1}{2} t \sin \Omega t, & \kappa = 1 \\ \dfrac{\cos \Omega t - \cos \kappa \Omega t}{\Omega\left(\kappa^2 - 1\right)}, & \kappa \neq 1 \end{cases},$$

$$\begin{aligned} y_1 = {} & B \cos \Omega t + \frac{\Gamma_0}{\Omega^2}(\cos \Omega t - 1) - \frac{\Gamma_1}{2\Omega} t \sin \Omega t - \frac{\Gamma_2}{3\Omega^2}(\cos \Omega t - \cos 2\,\Omega t) \\ & - \frac{\Gamma_3}{8\Omega^2}(\cos \Omega t - \cos 3\,\Omega t) - \frac{\Gamma_4}{15\Omega^2}(\cos \Omega t - \cos 4\,\Omega t) - \frac{\Gamma_5}{24\Omega^2}(\cos \Omega t - \cos 5\,\Omega t) \\ & - \frac{\Gamma_6}{35\Omega^2}(\cos \Omega t - \cos 6\,\Omega t) - \frac{\Gamma_7}{48\Omega^2}(\cos \Omega t - \cos 7\,\Omega t), \end{aligned} \tag{14}$$

where the expression of coefficients $\Gamma_0, \Gamma_1, \ldots, \Gamma_7$ can be depicted as follows:

$$\Gamma_0 = -B\Omega^2\left(\frac{d_1B}{2}+\frac{3d_3B^3}{8}\right)+d_5+\frac{d_7B^2}{2}+\frac{3d_9B^4}{8}+\frac{5d_{11}B^6}{16},$$
$$\Gamma_1 = -B\Omega^2\left(1+\frac{3d_2B^2}{4}+\frac{5d_4B^4}{8}\right)+Bd_6+\frac{3d_8B^3}{4}+\frac{5d_{10}B^5}{8}+\frac{35d_{12}B^7}{64},$$
$$\Gamma_2 = -B\Omega^2\left(\frac{d_1B}{2}+\frac{d_3B^3}{2}\right)+\frac{d_7B^2}{2}+\frac{d_9B^4}{2}+\frac{15d_{11}B^6}{32},$$
$$\Gamma_3 = -B\Omega^2\left(\frac{d_2B^2}{4}+\frac{5d_4B^4}{16}\right)+\frac{d_8B^3}{4}+\frac{5d_{10}B^5}{16}+\frac{21d_{12}B^7}{64},$$
$$\Gamma_4 = -B\Omega^2\left(\frac{d_3B^3}{8}\right)+\frac{d_9B^4}{8}+\frac{3d_{11}B^6}{16},$$
$$\Gamma_5 = -B\Omega^2\left(\frac{d_4B^4}{16}\right)+\frac{d_{10}B^5}{16}+\frac{7d_{12}B^7}{64},$$
$$\Gamma_6 = \frac{d_{11}B^6}{32},$$
$$\Gamma_7 = \frac{d_{12}B^7}{64}. \tag{15}$$

To ensure the periodicity requires that the coefficient of $t \sin \omega t$ equal to zero, thus

$$\frac{\Gamma_1}{2\Omega} = 0, \tag{16}$$

or

$$-B\Omega^2\left(1+\frac{3d_2B^2}{4}+\frac{5d_4B^4}{8}\right)+Bd_6 +\frac{3d_8B^3}{4}+\frac{5d_{10}B^5}{8}+\frac{35d_{12}B^7}{64}=0, \tag{17}$$

yields

$$\Omega = \sqrt{\frac{64d_6+48d_8B^2+40d_{10}B^4+35d_{12}B^6}{64+48d_2B^2+40d_4B^4}}, \tag{18}$$

and thus the first-order approximation for the analytic solution of the equation (7) is as follows:

$$y_{VIM} = a_0+(a_1+B)\cos\Omega t+a_2\cos 2\,\Omega t+a_3\cos 3\,\Omega t + a_4\cos 4\,\Omega t+a_5\cos 5\,\Omega t+a_6\cos 6\,\Omega t+a_7\cos 7\,\Omega t, \tag{19}$$

where

$$a_0 = -\frac{\Gamma_0}{\Omega^2},$$
$$a_1 = \frac{1}{\Omega^2}\left(\Gamma_0-\frac{\Gamma_2}{3}-\frac{\Gamma_3}{8}-\frac{\Gamma_4}{15}-\frac{\Gamma_5}{24}-\frac{\Gamma_6}{35}-\frac{\Gamma_7}{48}\right),$$
$$a_2 = \frac{\Gamma_2}{3\Omega^2},$$
$$a_3 = \frac{\Gamma_3}{8\Omega^2},$$
$$a_4 = \frac{\Gamma_4}{15\Omega^2},$$
$$a_5 = \frac{\Gamma_5}{24\Omega^2},$$
$$a_6 = \frac{\Gamma_6}{35\Omega^2},$$
$$a_7 = \frac{\Gamma_7}{48\Omega^2}. \tag{20}$$

We shall now examine the several physically relevant N/MEMS cases considering various sets of parameter values in equation (7).

3.1. CASE I: Motion of Electrically Excited Microbeam. Consider the motion of an electrically actuated model of a microbeam [10, 12].

$$\left(b_0 + b_1 y^2 + b_2 y^4\right)\ddot{y} + b_3 y + b_4 y^3 + b_5 y^5 + b_6 y^7 = 0, \quad (21)$$

where the expression of coefficients $b_0, b_1, \ldots, b_7$ are as follows:

$$\begin{aligned}
b_0 &= \int_0^1 \xi^2 d\eta, \\
b_1 &= -2\int_0^1 \xi^4 d\eta, \\
b_2 &= \int_0^1 \xi^6 d\eta, \\
b_3 &= \int_0^1 \left(\xi\xi'''' - N\xi\xi'' - V^2\xi^2\right) d\eta, \\
b_4 &= \int_0^1 \left(-2\xi^3\xi'''' + 2N\xi^3\xi'' - \beta\xi\xi'' \int_0^1 \xi'^2 d\eta\right) d\eta, \\
b_5 &= \int_0^1 \left(\xi^5\xi'''' - N\xi^5\xi'' + 2\beta\xi^3\xi'' \int_0^1 \xi'^2 d\eta\right) d\eta, \\
b_6 &= -\int_0^1 \left(\beta\xi^5\xi'' \int_0^1 \xi'^2 d\eta\right) d\eta,
\end{aligned} \quad (22)$$

where $\xi(\eta) = 16\eta^2(1-\eta)^2$ is the trail function. Equation (21) may be achieved from the generalized equation (7) by choosing the parameters $\alpha_1 = \alpha_3 = \alpha_5 = \alpha_7 = \alpha_9 = \alpha_{11} = 0$, $\alpha_0 = b_0$, $\alpha_2 = b_1$, $\alpha_4 = b_2, \alpha_6 = b_3$, $c_8 = b_4$, $c_{10} = b_5$ and $c_{12} = b_6$.

Let us rewrite equation (21) as

$$\left(1 + m_1 y^2 + m_2 y^4\right)\ddot{y} + m_3 y + m_4 y^3 + m_5 y^5 + m_6 y^7 = 0, \quad (23)$$

where the coefficients $m_j = b_j/b_0\,(j = 1, 2, \ldots, 6)$. The frequency-amplitude relationship can be attained using equation (18) by substituting aforementioned parameters as follows:

$$\Omega_{VIM} = \sqrt{\frac{64m_3 + 48m_4B^2 + 40m_5B^4 + 35m_6B^6}{64 + 48m_1B^2 + 40m_2B^4}}, \quad (24)$$

which differs from the frequency calculated by the energy balance method (EBM) [10], which is as follows:

$$\Omega_{EBM} = \sqrt{\frac{4b_3 + 3b_4B^2 + 7b_5B^4/3 + 15b_6B^6/8}{4b_0 + 2b_1B^2 + b_2B^4}}. \quad (25)$$

The approximate solution of equation (21) by using equation (19) is as follows:

$$\begin{aligned}
y_{VIM} &= \left[B - (\Lambda_1 + \Lambda_2 + \Lambda_3)\right]\cos\Omega t + \Lambda_1 \cos 3\,\Omega t \\
&\quad + \Lambda_2 \cos 5\,\Omega t + \Lambda_3 \cos 7\,\Omega t,
\end{aligned} \quad (26)$$

where

$$\begin{aligned}
\Lambda_1 &= \frac{1}{8\Omega^2}\left[-A\Omega^2\left(\frac{m_1B^2}{4} + \frac{5m_2B^4}{16}\right) + \frac{m_4B^3}{4} + \frac{5m_5B^5}{16} + \frac{21m_6B^7}{64}\right], \\
\Lambda_2 &= \frac{1}{24\Omega^2}\left[-A\Omega^2\left(\frac{m_2B^4}{16}\right) + \frac{m_5B^5}{16} + \frac{7m_6B^7}{64}\right], \\
\Lambda_3 &= \frac{1}{48\Omega^2}\left[\frac{m_6B^7}{64}\right].
\end{aligned} \quad (27)$$

And, the approximate analytic result by the EBM is

$$y_{EBM} = B\cos\left(\sqrt{\frac{4b_3 + 3b_4B^2 + 7b_5B^4/3 + 15b_6B^6/8}{4b_0 + 2b_1B^2 + b_2B^4}}\,t\right). \quad (28)$$

We depict the deflection of microbeam y obtained from LVIM (solid red lines) (equation (26)) with time t for four sets of parameter values (B, N, V, β) in the left side column of Figure 1 with the same yield by EBM (solid black lines) (equation (28)) and also obtained numerically by utilizing RK4 (solid blue line). This evaluation validates that the findings from the LVIM and those attained by RK4 match remarkably well.

We also graph errors in the deflection of microbeams with respect to their values evaluated using RK4. Red circles and black stars with dashed lines represent the errors of the LVIM $(y_{RK4} - y_{LVIM})$ and EBM $(y_{RK4} - y_{EBM})$, respectively, map errors against time for the similar values of parameters in the right side column of Figure 1. All panels in the right side column confirm that the accuracy of the solution obtained by LVIM is much improved in comparison to the solution obtained by means of EBM because the margin of error is less in the case of LVIM. Moreover, it is notable that the error in EBM is increasing with an increase in amplitude, but the error in LVIM is insignificant.

The effectiveness of the LVIM for the nonlinear analytic frequency and the approximate solution can be seen in Figure 2 which represents the influence of different

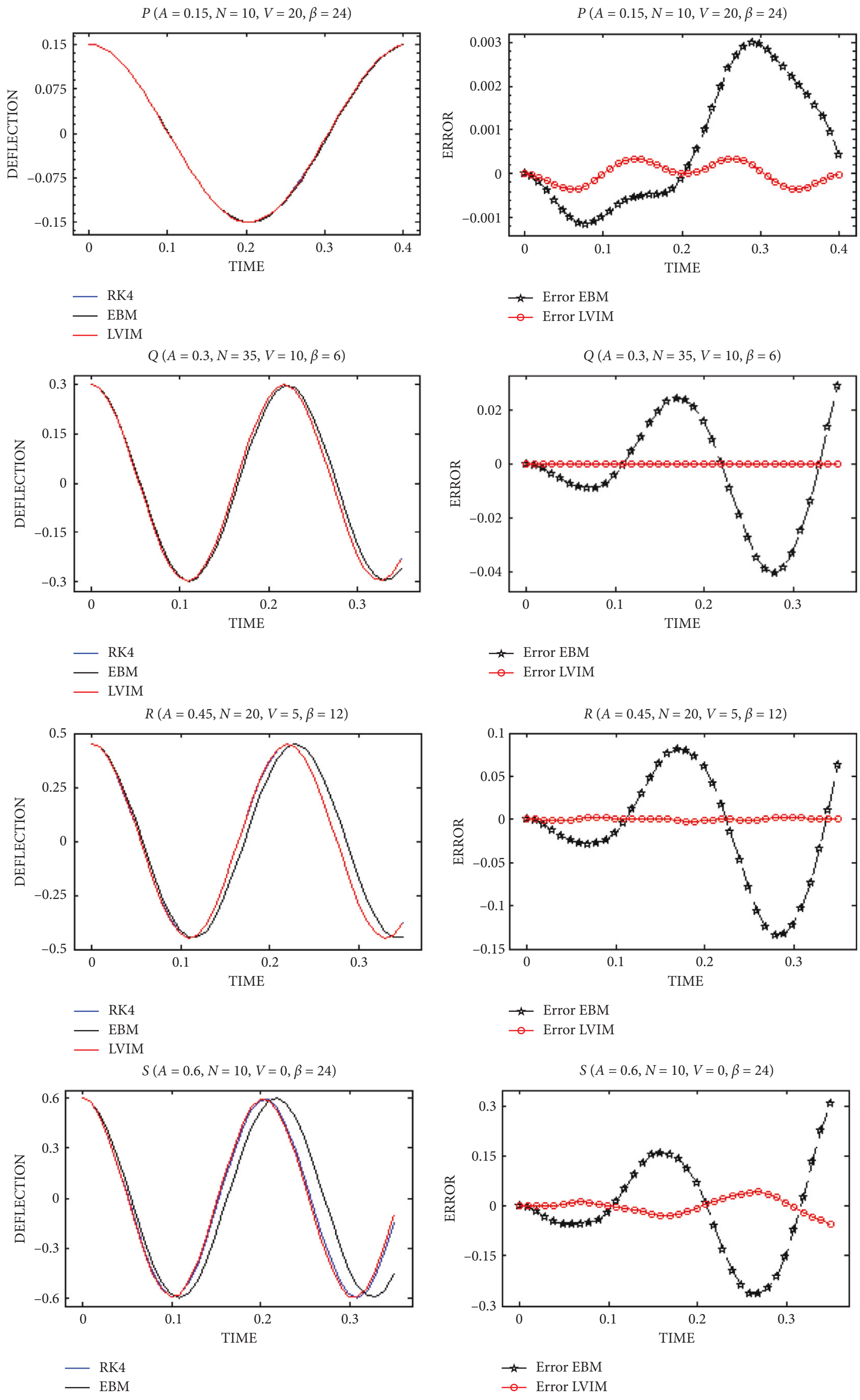

Figure 1: Comparison of results obtained by LVIM and EBM with RK4 findings for electrically excited microbeam.

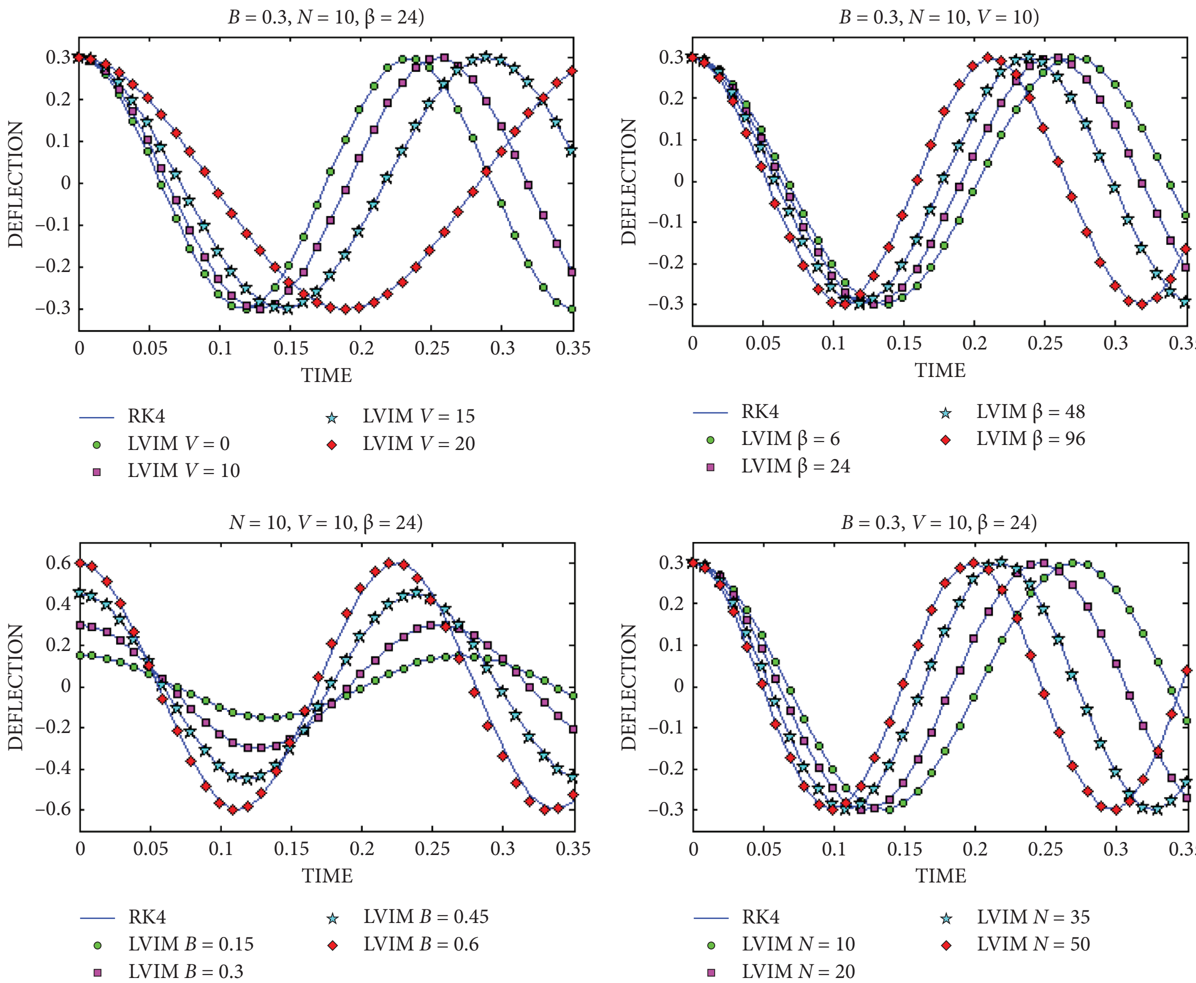

FIGURE 2: Influence of the parameters on the deflection of electrically excited microbeam.

parameters on the deflection of microbeams. To this end, one of the mentioned parameters is supposed to change while the three other ones remain constant. The graphs in this figure demonstrate the accuracy of the solution obtained in equation (26) because the observations attained by the LVIM are in good agreement with those achieved numerically using RK4.

3.2. CASE II: Motion of Nanobeams Actuated by Van der Waals Attractions. Consider the motion of an N/MEMS of nanobeams induced by the Van der Waals attractions [11, 39]. Intermolecular interactions or van der Waals force have been used instead of electrostatic force in this microstructure for actuation. The mathematical model can be represented as follows:

$$\begin{aligned}\left(h_0 + h_1 y + h_2 y^2 + h_3 y^3\right)\ddot{y} + h_4 + h_5 y + h_6 y^2 \\ + h_7 y^3 + h_8 y^4 + h_9 y^5 + h_{10} y^6 = 0,\end{aligned} \quad (29)$$

where the coefficients $h_0, h_1, \ldots, h_{10}$ can be written as follows and a detailed derivation of equation (29) and the physical understanding of each coefficient are available in Refs. [11, 39].

$$\begin{aligned}
h_0 &= \int_0^1 \xi^2 d\eta,\\
h_1 &= -3\int_0^1 \xi^3 d\eta,\\
h_2 &= 3\int_0^1 \xi^4 d\eta,\\
h_3 &= -\int_0^1 \xi^5 d\eta,\\
h_4 &= -\lambda\int_0^1 \xi\, d\eta,\\
h_5 &= \int_0^1 (\xi\xi'''' - N\xi\xi'')d\eta,\\
h_6 &= \int_0^1 \left(-3\xi^2\xi'''' + 3N\xi^2\xi''\right)d\eta,\\
h_7 &= \int_0^1 \left(3\xi^3\xi'''' - 3N\xi^3\xi''\right)d\eta - \beta\int_0^1 \xi\xi'' d\eta \int_0^1 \xi'^2 d\eta,\\
h_8 &= \int_0^1 \left(-\xi^4\xi'''' + N\xi^4\xi''\right)d\eta + 3\beta\int_0^1 \xi^2\xi'' d\eta \int_0^1 \xi'^2 d\eta,\\
h_9 &= -3\beta\int_0^1 \xi^3\xi'' d\eta \int_0^1 \xi'^2 d\eta,\\
h_{10} &= \beta\int_0^1 \xi^4\xi'' d\eta \int_0^1 \xi'^2 d\eta,
\end{aligned} \quad (30)$$

where $\xi(\eta) = \sin \pi\eta$ is the trail function. Equation (29) is solved for different trail function by means of the spreading residue harmonic balance method (SRHBM) [11], which can be attained by putting the parameters $\alpha_0 = h_0,\ \alpha_1 = h_1,\ \alpha_2 = h_2,\ \alpha_3 = h_3,\ \alpha_4 = 0, \alpha_5 = h_4,\ \alpha_6 = h_5, \alpha_7 = h_6,\ \alpha_8 = h_7, \alpha_9 = h_8,\ \alpha_{10} = h_9,\ \alpha_{11} = h_{10}$, and $\alpha_{12} = 0$ in the general form of equation (7).

Equation (29) can rewrite in the following form:

$$\begin{aligned}&\left(1 + k_1 y + k_2 y^2 + k_3 y^3\right)\ddot{y} + k_4 + k_5 y + k_6 y^2 + k_7 y^3\\&\quad + k_8 y^4 + k_9 y^5 + k_{10} y^6 = 0,\end{aligned} \tag{31}$$

where the coefficients $k_n = h_n/h_0\,(n = 1, 2, \ldots, 10)$. The nonlinear frequency of this oscillatory system using LVIM can be gained by placing the abovementioned parameters in equation (18) and can be written as follows:

$$\Omega = \sqrt{\frac{8k_5 + 6k_7 B^2 + 5k_9 B^4}{8 + 6k_2 B^2}}, \tag{32}$$

which is similar to the frequency of order first calculated by SRHBM [11]. The approximate analytic solution of equation (29) is obtained by LVIM from equation (19) as follows:

$$\begin{aligned}y_{LVIM} &= e_0 + (e_1 + B)\cos \Omega t + e_2 \cos 2\ \Omega t + e_3 \cos 3\ \Omega t\\&\quad + e_4 \cos 4\ \Omega t + e_5 \cos 5\ \Omega t + e_6 \cos 6\ \Omega t,\end{aligned} \tag{33}$$

where

$$\begin{aligned}e_0 &= -\frac{\Upsilon_0}{\Omega^2},\\ e_1 &= \frac{1}{\Omega^2}\left(\Upsilon_0 - \frac{\Upsilon_2}{3} - \frac{\Upsilon_3}{8} - \frac{\Upsilon_4}{15} - \frac{\Upsilon_5}{24} - \frac{\Upsilon_6}{35}\right),\\ e_2 &= \frac{\Upsilon_2}{3\Omega^2},\\ e_3 &= \frac{\Upsilon_3}{8\Omega^2},\\ e_4 &= \frac{\Upsilon_4}{15\Omega^2},\\ e_5 &= \frac{\Upsilon_5}{24\Omega^2},\\ e_6 &= \frac{\Upsilon_6}{35\Omega^2},\end{aligned} \tag{34}$$

where the expression of coefficients $\Upsilon_0, \Upsilon_1, \ldots, \Upsilon_2$ can be identified as follows:

$$\begin{aligned}\Upsilon_0 &= -B\Omega^2\left(\frac{k_1 B}{2} + \frac{3k_3 B^3}{8}\right) + k_4 + \frac{k_6 B^2}{2} + \frac{3k_8 B^4}{8} + \frac{5k_{10} B^6}{16},\\ \Upsilon_1 &= -B\Omega^2\left(1 + \frac{3k_2 B^2}{4}\right) + Bk_5 + \frac{3k_7 B^3}{4} + \frac{5k_9 B^5}{8},\\ \Upsilon_2 &= -B\Omega^2\left(\frac{k_1 B}{2} + \frac{k_3 B^3}{2}\right) + \frac{k_6 B^2}{2} + \frac{k_8 B^4}{2} + \frac{15k_{10} B^6}{32},\\ \Upsilon_3 &= -B\Omega^2\left(\frac{k_2 B^2}{4}\right) + \frac{k_7 B^3}{4} + \frac{5k_9 B^5}{16},\\ \Upsilon_4 &= -B\Omega^2\left(\frac{k_3 B^3}{8}\right) + \frac{k_8 B^4}{8} + \frac{3k_{10} B^6}{16},\\ \Upsilon_5 &= \frac{k_9 B^5}{16},\\ \Upsilon_6 &= \frac{k_{10} B^6}{32}.\end{aligned} \tag{35}$$

Figure 3 represents the deflection obtained analytically from a numerical solution for the vibration of nanobeams excited by Van der Waals attraction. We have also plotted the variation of error for the above-mentioned system in the corresponding bottom panels. From the top panel of Figure 3, it is seen that the approximate results were achieved numerically using RK4 (blue line), the SRHBM [11] (black line), and those obtained by the LVIM (red line) equation (33) for two sets of parameters (B, N, β, λ), which reveals the accuracy of the findings achieved by the present application of LVIM. Errors of the SRHBM are symbolized with black stars with solid lines, while errors of LVIM are denoted with red circles against time for the same parameter values in the bottom panel confirming the supremacy of LVIM over SRHBM. Furthermore, in the top panel, a significant difference between the solutions of RK4 and SRHBM can be observed on the trough part of the wave, but the LVIM solution matches extremely well in that part. The same is observed in the error graph shown in the bottom panel. These facts authenticate the great potential of the LVIM for solving nonlinear problems over SRHBM.

Figure 4 demonstrates the effect of change in midpoint deflection of the nanobeams due to variation in parameter values of the model. The LVIM results and those attained by RK4 are almost similar which indicates that LVIM can correctly predict the oscillatory behavior of these microstructures.

3.3. CASE III: Motion of Multiwalled Carbon Nanotubes. Consider the equation of motion of multiwalled carbon nanotubes that includes both the electrostatic and Van der Waals attraction forces for actuating [42].

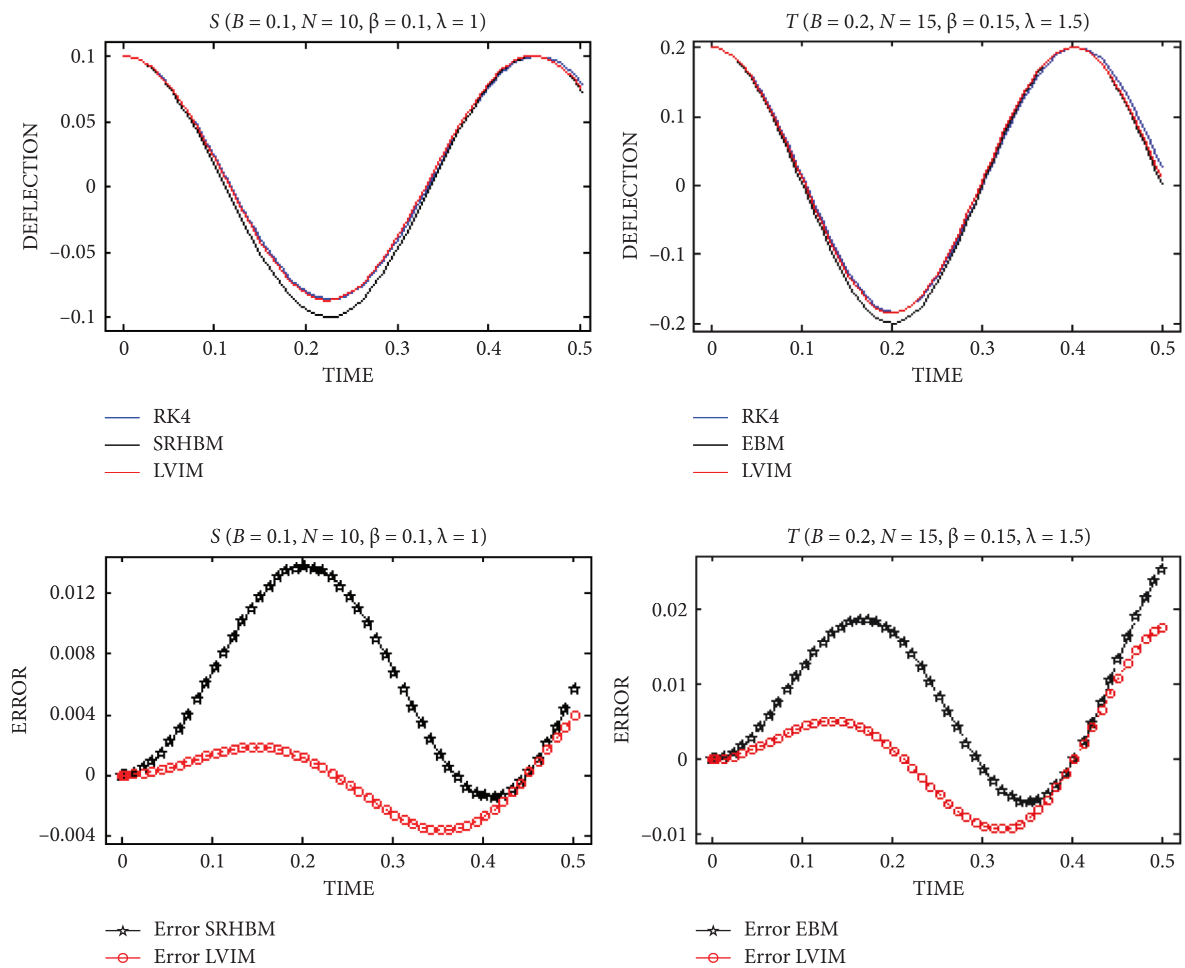

FIGURE 3: Comparison of results obtained by LVIM and SRHBM with RK4 findings for the nanobeams actuated by Van der Waals force.

$$\ddot{y} + \ell_0 + \ell_1 y + \ell_2 y^2 + \ell_3 y^3 + \ell_4 y^4 = 0, \tag{36}$$

where the coefficients $\ell_0, \ell_1, \ldots, \ell_4$ and detailed derivation of the model equation can be found in [42]. This vibratory model can be achieved by substituting $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_{10} = \alpha_{11} = \alpha_{12} = 0$, $\alpha_0 = 1$, $\alpha_5 = \ell_0$, $\alpha_6 = \ell_1$, $\alpha_7 = \ell_2$, $\alpha_8 = \ell_3$, and $\alpha_9 = \ell_4$ in the generalized equation (7). The LVIM frequency may obtained from equation (18) as follows:

$$\Omega = \sqrt{\ell_1 + \frac{3}{4}\ell_3 B^2}. \tag{37}$$

The LVIM solution of equation (36) is as follows:

$$\begin{aligned} y_{HPLTM} = {} & \Psi_0 + [\Psi_1 + B]\cos \Omega t + \Psi_2 \cos 2\,\Omega t \\ & + \Psi_3 \cos 3\,\Omega t + \Psi_4 \cos 4\,\Omega t, \end{aligned} \tag{38}$$

where

$$\begin{aligned} \Psi_0 &= -\frac{1}{\Omega^2}\left[\ell_0 + \frac{\ell_2 B^2}{2} + \frac{3\ell_4 B^4}{8}\right], \\ \Psi_1 &= \frac{1}{\Omega^2}\left[\frac{\ell_2 B^2}{3} - \frac{\ell_3 B^3}{32} + \frac{\ell_4 B^4}{5}\right], \\ \Psi_2 &= \frac{1}{3\Omega^2}\left[\frac{\ell_2 B^2}{2} + \frac{\ell_4 B^4}{2}\right], \\ \Psi_3 &= \frac{1}{8\Omega^2}\left[\frac{\ell_3 B^3}{4}\right], \end{aligned} \tag{39}$$

and

$$\Psi_4 = \frac{1}{15\Omega^2}\left[\frac{\ell_4 B^4}{8}\right]. \tag{40}$$

Both frequency and solution are similar to those given by [42] gained by means of the iteration perturbation method and also the same as those given by [43, 44] employing the parameter expansion method.

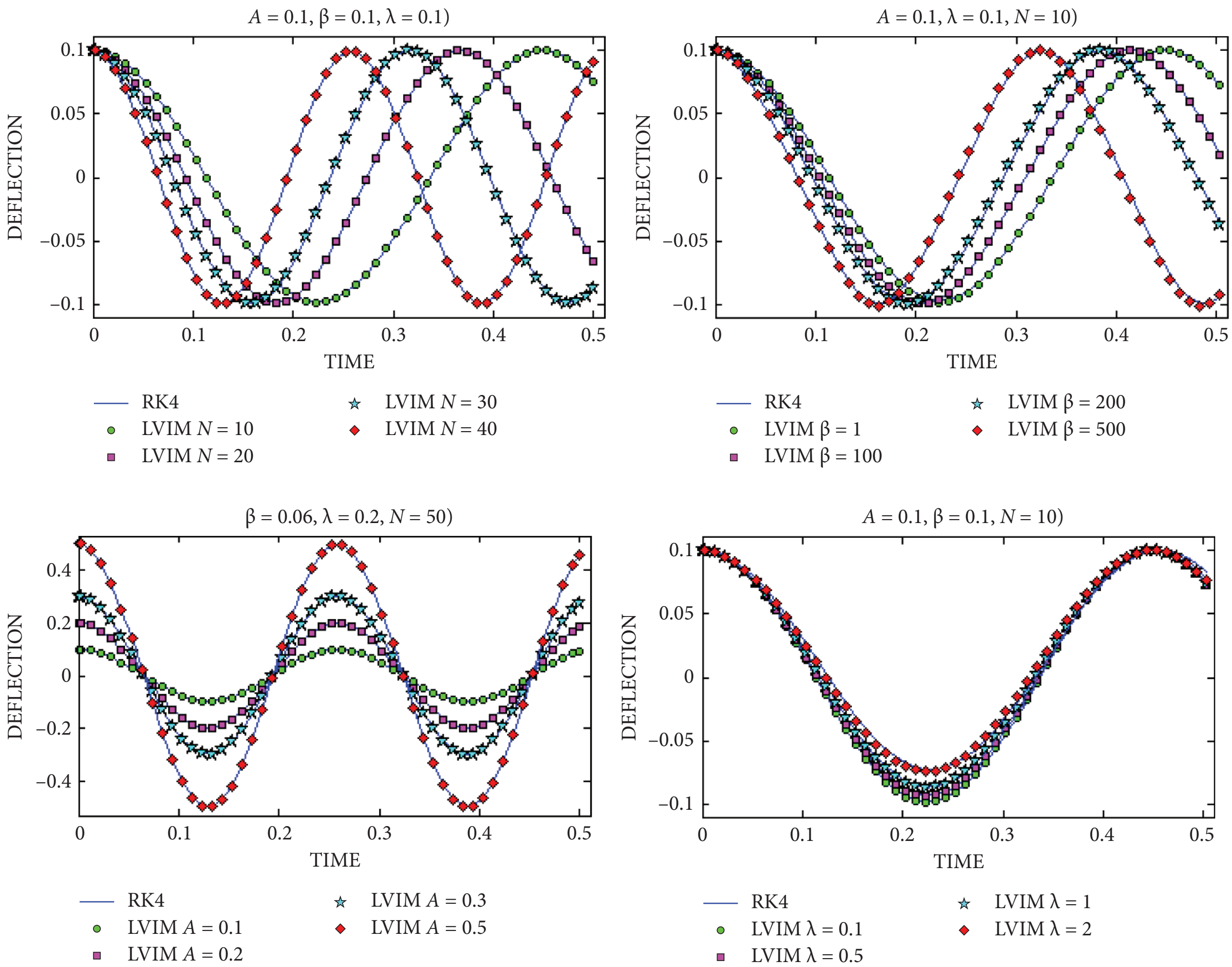

Figure 4: Influence of parameters on deflection of nanobeams actuated by Van der Waals force.

4. Conclusion

A vibratory equation for N/MEMS in its general form is reduced to ordinary differential equations with nonlinearities such as the motion of a microbeam actuated electrically, the vibration of nanobeams under the effect of Van der Waals attractions, and the motion of multiwalled carbon nanotubes. The variational iteration method (VIM) coupled with the Laplace transformation is utilized to achieve the nonlinear analytic frequency and approximate solution of the generalized model of N/MEMS and its relevant systems with great success. We considered some novel variational iteration formulas where the Lagrange multiplier is identified with the help of the Laplace transform which eliminates the elusive theory of variations. The merit of the proposed method is its simplicity (no need to do any integration) and capability to solve nonlinear models with high accuracy. Comparative results of the Laplace-based variational iteration method, energy balance process, spreading residual harmonic balance technique, and Runge–Kutta scheme were given to show the efficacy of the suggested strategy. It is concluded that the obtained results for the generalized model enable us to examine numerous nonlinear physical N/MEMS easily in a similar way.

Data Availability

No datasets were generated or analyzed during the current study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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