DE GRUYTER Open Mathematics 2020; 18: 738–748

Research Article

Hijaz Ahmad, Tufail A. Khan, and Shao-Wen Yao*

An efficient approach for the numerical solution of fifth-order KdV equations

https://doi.org/10.1515/math-2020-0036
received November 15, 2019; accepted May 22, 2020

Abstract: The main aim of this article is to use a new and simple algorithm namely the modified variational iteration algorithm-II (MVIA-II) to obtain numerical solutions of different types of fifth-order Korteweg-de Vries (KdV) equations. In order to assess the precision, stability and accuracy of the solutions, five test problems are offered for different types of fifth-order KdV equations. Numerical results are compared with the Adomian decomposition method, Laplace decomposition method, modified Adomian decomposition method and the homotopy perturbation transform method, which reveals that the MVIA-II exceptionally productive, computationally attractive and has more accuracy than the others.

Keywords: Korteweg-de Vries equation, variational iteration algorithm-II, fifth-order KdV equation, Caudrey-Dodd-Gibbon equation, Lax equation, Kawahara equation, Sawada-Kotera equation

MSC 2010: 35Q53

1 Introduction

Nonlinear partial differential equations (PDEs) have turned into a helpful tool for delineating a large number of physical problems that arise in many fields of mathematics and science, including fluid dynamics, chemical physics, hydrodynamics, fluid mechanics, heat and mass transfer, solid-state physics, chemical kinematics, plasma physics, etc. [1–12]. The Korteweg-de Vries (KdV) equation is a nonlinear PDE and assumes a significant role with numerous applications, for example, in illustration of magneto-acoustic waves, ion-acoustic waves, nonlinear LC circuit's waves and shallow water waves in plasmas. This equation has experienced a few extensions and modifications leading to numerous types of KdV equations showing up in three, five, seven or more order forms deferential equations. However, some notable types of them are specifically noteworthy in physical phenomena. These types of the KdV equation of fifth order are as follows:

1. The Sawada-Kotera (SK) equation [13]:

$$\frac{\partial u}{\partial t} + 45u^2\frac{\partial u}{\partial x} + 15\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} + 15u\frac{\partial^3 u}{\partial x^3} + \frac{\partial^5 u}{\partial x^5} = 0. \tag{1}$$

2. The Caudrey-Dodd-Gibbon (C-D-G) equation [14]:

$$\frac{\partial u}{\partial t} + 180u^2\frac{\partial u}{\partial x} + 30\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} + 30u\frac{\partial^3 u}{\partial x^3} + \frac{\partial^5 u}{\partial x^5} = 0. \tag{2}$$

* **Corresponding author: Shao-Wen Yao,** School of Mathematics and Information Science, Henan Polytechnic University, Jiaozuo 454000, China, e-mail: yaoshaowen@hpu.edu.cn
Hijaz Ahmad: Department of Basic Sciences, University of Engineering and Technology, Peshawar 25000, Pakistan, e-mail: hijaz555@gmail.com
Tufail A. Khan: Department of Basic Sciences, University of Engineering and Technology, Peshawar 25000, Pakistan, e-mail: tufailmarwat@uetpeshawar.edu.pk

3. The Lax equation [15,16]:

$$\frac{\partial u}{\partial t} + 30u^2\frac{\partial u}{\partial x} + 30\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} + 10u\frac{\partial^3 u}{\partial x^3} + \frac{\partial^5 u}{\partial x^5} = 0. \tag{3}$$

4. The fifth-order KdV equation [17]:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - u\frac{\partial^3 u}{\partial x^3} + \frac{\partial^5 u}{\partial x^5} = 0. \tag{4}$$

5. The Kawahara equation [18]:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} - \frac{\partial^5 u}{\partial x^5} = 0. \tag{5}$$

The fifth-order KdV equation, also called generalized KdV, is the indispensable model for numerous physical phenomena in quantum mechanics and nonlinear optics. Its general solution is not known while the exact solution for the particular case of solitary waves of the KdV equation of fifth order is reported in [19]. To solve this model numerically, several methods can be found in the literature, such as Wazwaz [16] has employed the tanh and sine-cosine methods, Djidjeli et al. [20] used finite difference schemes, Chun [21] utilized the Exp-function method, Abbasbandy and Zakaria [22] employed the homotopy analysis method for soliton solution, Goswami et al. [18] applied homotopy perturbation transform method (HPTM), Kaya [23,24] used the decomposition method as well as Adomian decomposition method, Rafei and Ganji [25] provided a comparative study of homotopy perturbation method and Adomian decomposition method, Lei et al. used the homogeneous balance method [26], Helal and Mehanna [27] provided a comparative investigation of the Adomian decomposition method and Crank-Nicholas method, Handibag and Karande [17] have developed the Laplace decomposition method and Bakodah [28] has used the modified Adomian decomposition method.

The main objective of this article is to employ modified variational iteration algorithm-II (MVIA-II) for the numerical treatment of nonlinear PDEs in physical sciences and engineering via the KdV equation of fifth order. The article is prepared as follows. In Section 2, MVIA-II is described. In Section 3, five different forms of KdV equations have been investigated to show the applicability and precision of the modified algorithm, and in Section 4, conclusions are given.

2 Implementation of MVIA-II

In this section, we will describe MVIA-II for finding the numerical solutions of fifth-order KdV equations. Suppose the generalized form of the KdV equation of fifth order:

$$\frac{\partial u}{\partial t} + \alpha u\frac{\partial^3 u}{\partial x^3} + \beta\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} + \gamma u^2\frac{\partial u}{\partial x} + \frac{\partial^5 u}{\partial x^5} = 0, \tag{6}$$

where γ, β and α are nonzero arbitrary constants. Approximate solution $u_{k+1}(x)$ of Eq. (6) for given initial condition $u_0(x)$ can be acquired as follows:

$$\begin{aligned} u_{k+1}(x,t,h) &= u_0(x,t,h) + h\int_0^t \lambda(\eta)\Bigg[\frac{\partial u_k(x,\eta,h)}{\partial(\eta)} + \alpha \widetilde{u_k(x,\eta,h)\frac{\partial^3 u_k(x,\eta,h)}{\partial(x)^3}} \\ &\quad + \beta\widetilde{\frac{\partial u_k(x,\eta,h)}{\partial(x)}\frac{\partial^2 u_k(x,\eta,h)}{\partial(x)^2}} + \gamma\widetilde{u_k^2(x,\eta,h)\frac{\partial u_k(x,\eta,h)}{\partial(x)}} + \widetilde{\frac{\partial^5 u_k(x,\eta,h)}{\partial(x)^5}}\Bigg]d\eta, \end{aligned} \tag{7}$$

where λ and h are two unknown parameters, the first one is known as the Lagrange multiplier [29], while the second one is an auxiliary parameter, which was used to accelerate the convergence in different methods [30–37], but we are the first to use it in VIA-II. The Lagrange multiplier can be found by taking δ on both sides of the recurrence relation (7) w.r.t. the variable $u_k(x)$,

$$\begin{aligned}\delta u_{k+1}(x,t,h) &= \delta u_0(x,t,h) + h\delta\int_0^t \lambda(\eta)\left[\frac{\partial u_k(x,\eta,h)}{\partial(\eta)} + \alpha\widetilde{u_k(x,\eta,h)\frac{\partial^3 u_k(x,\eta,h)}{\partial(x)^3}}\right. \\ &\left. + \beta\widetilde{\frac{\partial u_k(x,\eta,h)}{\partial(x)}\frac{\partial^2 u_k(x,\eta,h)}{\partial(x)^2}} + \widetilde{\gamma u_k^2(x,\eta,h)\frac{\partial u_k(x,\eta,h)}{\partial(x)}} + \widetilde{\frac{\partial^5 u_k(x,\eta,h)}{\partial(x)^5}}\right]\mathrm{d}\eta,\end{aligned} \tag{8}$$

where $\widetilde{u_k(x,\eta,h)}$ is a restricted term, i.e., $\delta\widetilde{u_k(x,\eta,h)} = 0$. The significant value of $\lambda(\eta)$ can be identified by making use of the variation theory [38–40]. While h is an auxiliary term which is utilized to ensure convergence of numerically obtained solution ideally by limiting norm 2 of residual error over the space of the given system. The ideal decision of this h improves the precision and proficiency of the algorithm. For the optimal solution of the auxiliary parameter, we define a residual function for the approximated solution:

$$\begin{aligned}r_p(x,t,h) &= \frac{\partial u_p(x,t,h)}{\partial(t)} + \alpha u_p(x,t,h)\frac{\partial^3 u_p(x,t,h)}{\partial(x)^3} + \beta\frac{\partial u_p(x,t,h)}{\partial(x)}\frac{\partial^2 u_p(x,t,h)}{\partial(x)^2} \\ &+ \gamma u_p^2(x,t,h)\frac{\partial u_p(x,t,h)}{\partial(x)} + \frac{\partial^5 u_p(x,t,h)}{\partial(x)^5},\end{aligned} \tag{9}$$

where p is the number of approximation. The square of function (9) for the pth-order approximation with respect to parameter h for $(x,t)\,\epsilon\,[a,b]\times[a,b]$ is

$$\left(\frac{1}{(b+1)^2}\sum_{i=a}^{b}\sum_{j=a}^{b}(r_p(i,j,h))^2\right)^{1/2}. \tag{10}$$

The lowest value of function (10) occurs at the point which should be selected and that will be the value of auxiliary parameter h. In small domains where the standard VIA-II gives high-order accuracy, this technology gives the value of h as 1. Putting values of both the parameters in the recurrence relation, we get an iterative formula:

$$\begin{aligned}u_{k+1}(x,t,h) &= u_0(x,t,h) + h\int_0^t \lambda(\eta)\left[\alpha u_k(x,\eta,h)\frac{\partial^3 u_k(x,\eta,h)}{\partial(x)^3} + \beta\frac{\partial u_k(x,\eta,h)}{\partial(x)}\frac{\partial^2 u_k(x,\eta,h)}{\partial(x)^2}\right. \\ &\left. + \gamma u_k^2(x,\eta,h)\frac{\partial u_k(x,\eta,h)}{\partial(x)} + \frac{\partial^5 u_k(x,\eta,h)}{\partial(x)^5}\right]\mathrm{d}\eta.\end{aligned} \tag{11}$$

Introducing with a proper initial guess, one can get the different approximations by utilizing the iterative structure (11). This gives an exact solution $u(x)$, when

$$u(x) = \lim_{k\to\infty} u_k(x). \tag{12}$$

This algorithm is named as MVIA-II. We utilize this modified algorithm for the solution of the different types of KdV equations of fifth order, which is able to provide numerical results for nonlinear and linear problems, in a direct way very accurately.

3 Numerical examples

In this section, MVIA-II is employed for the numerical treatment of the SK equation, C-D-G equation, a fifth-order KdV equation, Lax equation and Kawahara equation. Numerical and graphical results gained from the modified algorithm are very encouraging, empowering, noteworthy and significant. Illustrated test problems revealed the effectiveness and power of the suggested algorithm.

3.1 Test Problem 1

The SK equation (1) having the following initial condition

$$u(x, 0) = 2k^2 \operatorname{sech}^2[k(x - c)], \tag{13}$$

and the exact solution of SK equation (1) with condition (13) was given by [28]

$$u(x, t) = 2k^2 \operatorname{sech}^2[k(x - 16k^4 t - c)]. \tag{14}$$

The numerical solutions for Test Problem 3.1 using MVIA-II are reported in Table 1 of SK equation (1). To show the efficiency and applicability of the proposed algorithm in comparison with adomian decomposition method (ADM) [24] and modified ADM [28], the absolute errors are reported in Tables 1 and 2 for various values of t and x and $k = 0.01$ and $c = 0.0$. A full agreement between the results of MVI-II and exact solution is observed, which confirms the validity of the proposed algorithm. In comparison with [24,28] results, one can ensure that the results of MVIA-II are more efficient and reliable. Results are also shown through illustrations in Figure 1 at different times t and for different x, while the behaviour of approximate and exact solutions can be seen in Figure 2.

3.2 Test Problem 2

The C-D-G equation (22) having the initial condition:

$$u(x, 0) = \frac{k^2 e^{kx}}{(1 + e^{kx})^2}, \tag{15}$$

and the exact solution of equation (22) with condition (15) was given by [28]

$$u(x, t) = \frac{k^2 e^{k(x - k^4 t)}}{(1 + e^{k(x - k^4 t)})^2}. \tag{16}$$

Table 1: Numerical results of SK equation (1) in terms of absolute error using MVIA-II for Test Problem 3.1

x/t	$t = 0.2$		$t = 0.4$		$t = 5.0$	
	MVIA-II	**[28]**	**MVIA-II**	**[28]**	**MVIA-II**	**[28]**
2	0	1.54499×10^{-18}	2.71050×10^{-20}	4.52654×10^{-18}	2.71050×10^{-20}	6.6435×10^{-15}
4	2.71050×10^{-20}	5.36680×10^{-18}	5.42101×10^{-20}	1.12757×10^{-17}	8.13151×10^{-20}	1.32874×10^{-14}
6	2.71050×10^{-20}	1.13841×10^{-17}	2.71050×10^{-20}	2.02746×10^{-17}	8.13151×10^{-20}	1.99336×10^{-14}
8	2.71050×10^{-20}	1.97054×10^{-17}	5.42101×10^{-20}	3.15232×10^{-17}	5.42101×10^{-20}	2.65820×10^{-14}
10	5.42101×10^{-20}	3.02492×10^{-17}	5.42101×10^{-20}	4.50486×10^{-17}	5.42101×10^{-20}	3.32327×10^{-14}

Table 2: Comparison of numerical solutions ADM of SK equation (1) for Test Problem 3.1

x/t	$t = 0.2$		$t = 0.4$		$t = 0.5$	
	MVIA-II	**[24]**	**MVIA-II**	**[24]**	**MVIA-II**	**[24]**
0.1	0	9.59980×10^{-16}	2.71050×10^{-20}	1.91996×10^{-15}	8.13151×10^{-20}	6.6435×10^{-15}
0.2	2.71050×10^{-20}	1.91996×10^{-15}	8.13151×10^{-20}	3.83989×10^{-14}	5.42101×10^{-20}	1.32874×10^{-14}
0.3	2.71050×10^{-20}	2.87980×10^{-15}	2.71050×10^{-20}	5.75966×10^{-14}	5.42101×10^{-20}	1.99336×10^{-14}
0.4	2.71050×10^{-20}	3.83973×10^{-15}	2.71050×10^{-20}	7.67932×10^{-14}	5.42101×10^{-20}	2.65820×10^{-14}
0.5	2.71050×10^{-20}	4.79941×10^{-15}	5.42101×10^{-20}	9.59871×10^{-14}	5.42101×10^{-20}	3.32327×10^{-14}

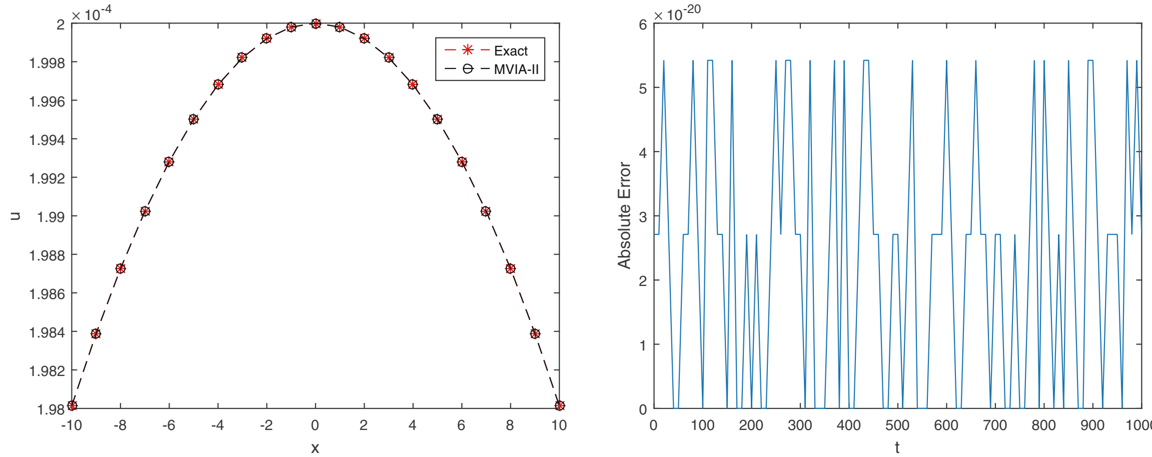

Figure 1: The exact and numerical solutions of Test Problem 3.1 for $t = 10$ and $k = 0.01$.

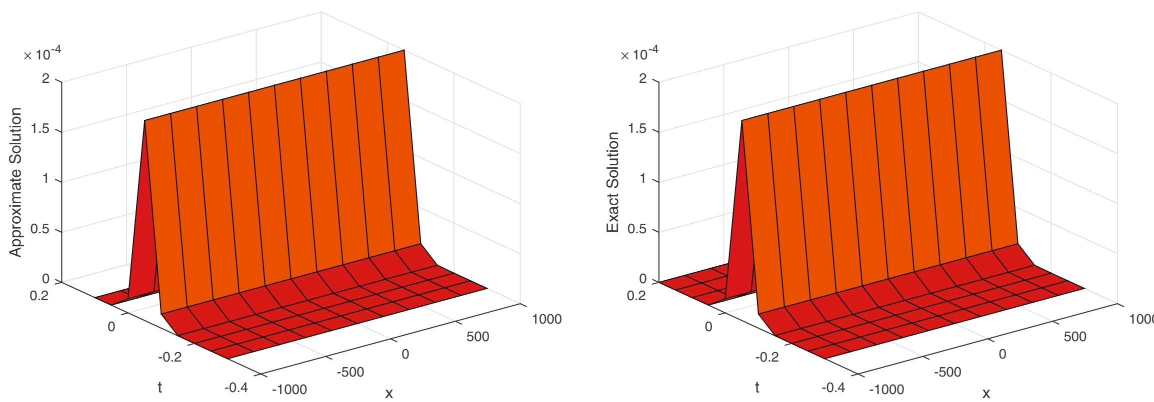

Figure 2: Approximate (MVIA-II) (left) and exact (right) solutions of Test Problem 3.1 for $k = 0.01$ in space-time graph form.

The obtained numerical results from the MVIA-II for Test Problem 3.2 are reported in Table 3. Results by MVIA-II show improvement over previous ones reported by [28]. Comparison of approximate and exact solutions is also shown through curves in Figure 3 for $k = 0.01$ and time $t = 10$. While Figure 4 shows the behaviour of the approximate and exact solution, which indicates that the proposed algorithm is a powerful mathematical tool for getting an accurate numerical solution of the C-D-G equation. The results of the proposed algorithm (MVIA-II) are compared with the results of the modified Adomian decomposition method [28]. It is well defined from the numerical and graphical results that the performance of the proposed algorithm is more precise.

3.3 Test Problem 3

The Lax equation (3) having the following initial condition

$$u(x, 0) = 2k^2[2 - 3\tanh^2 k(x - c)], \tag{17}$$

and the exact solution of Lax's equation with condition (17) was given by [28]

$$u(x, t) = 2k^2[2 - 3\tanh^2 k(x - 56k^4 t - c)]. \tag{18}$$

From the numerical results of Test Problem 3.3 using MVIA-II reported in Table 4, we conclude that the obtained results are in quite agreement with the exact and with the results given in [28]. Figure 5 shows the comparison of approximate and exact solutions of Test Problem 3.3 for $t = 10$ and $k = 0.01$.

Table 3: Comparison of numerical solutions of C–D–G equation (22) for Test Problem 3.2

x/t	$t = 0.4$		$t = 0.8$		$t = 5.0$	
	MVIA-II	[**28**]	**MVIA-II**	[**28**]	**MVIA-II**	[**28**]
2	6.77626×10^{-21}	3.11957×10^{-21}	6.77626×10^{-21}	1.7893×10^{-20}	1.01643×10^{-20}	2.48025×10^{-18}
4	3.38813×10^{-21}	2.70138×10^{-21}	6.77626×10^{-21}	2.01336×10^{-20}	1.35525×10^{-20}	5.70681×10^{-18}
6	3.38813×10^{-21}	5.50257×10^{-21}	0	3.91461×10^{-20}	1.01643×10^{-20}	8.93996×10^{-18}
8	6.77626×10^{-21}	4.74685×10^{-21}	6.77626×10^{-21}	5.79897×10^{-20}	6.77626×10^{-21}	1.21797×10^{-17}
10	3.38813×10^{-21}	7.21049×10^{-21}	6.77626×10^{-21}	5.63358×10^{-20}	3.38813×10^{-21}	1.54092×10^{-17}

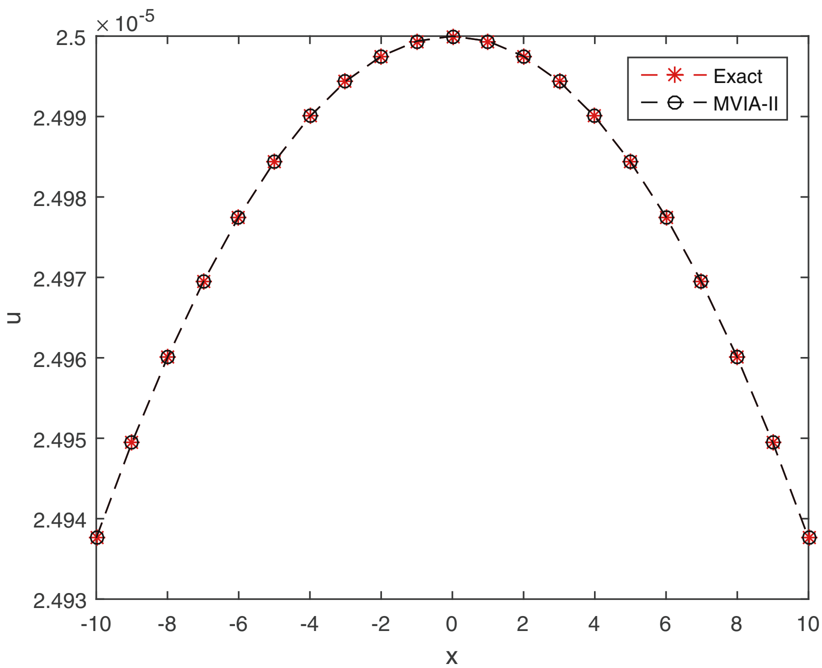

Figure 3: The exact and numerical solutions' comparison of Test Problem 3.2 for $k = 0.01$ and time $t = 10$.

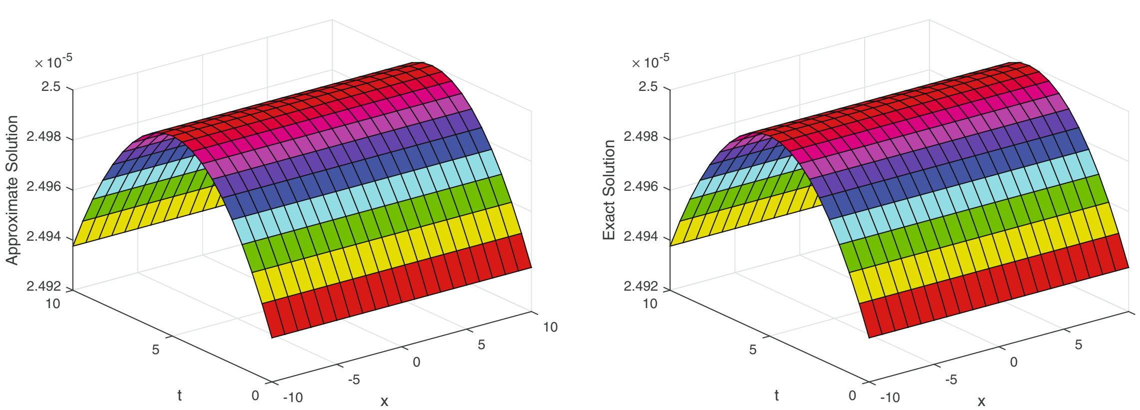

Figure 4: Approximate (MVIA-II) (left) and exact (right) solutions of Test Problem 3.2 for $k = 0.01$.

Table 4: Comparison of numerical solutions of Lax equation (3) for Test Problem 3.3

x/t	$t = 0.2$		$t = 0.8$		$t = 5.0$	
	MVIA-II	**[28]**	**MVIA-II**	**[28]**	**MVIA-II**	**[28]**
2	5.75976×10^{-14}	5.76197×10^{-14}	2.30320×10^{-13}	2.30342×10^{-13}	1.42090×10^{-12}	1.42104×10^{-12}
4	1.15193×10^{-13}	1.15281×10^{-13}	4.60638×10^{-13}	4.60727×10^{-13}	2.84181×10^{-12}	2.84213×10^{-12}
6	1.72787×10^{-13}	1.72985×10^{-13}	6.90954×10^{-13}	6.91153×10^{-13}	4.26271×10^{-12}	4.26326×10^{-12}
8	2.30378×10^{-13}	2.30730×10^{-13}	9.21269×10^{-13}	9.21621×10^{-13}	5.68361×10^{-12}	5.68444×10^{-12}
10	2.87968×10^{-13}	2.88518×10^{-13}	1.15158×10^{-12}	1.15213×10^{-12}	7.10452×10^{-12}	7.10566×10^{-12}

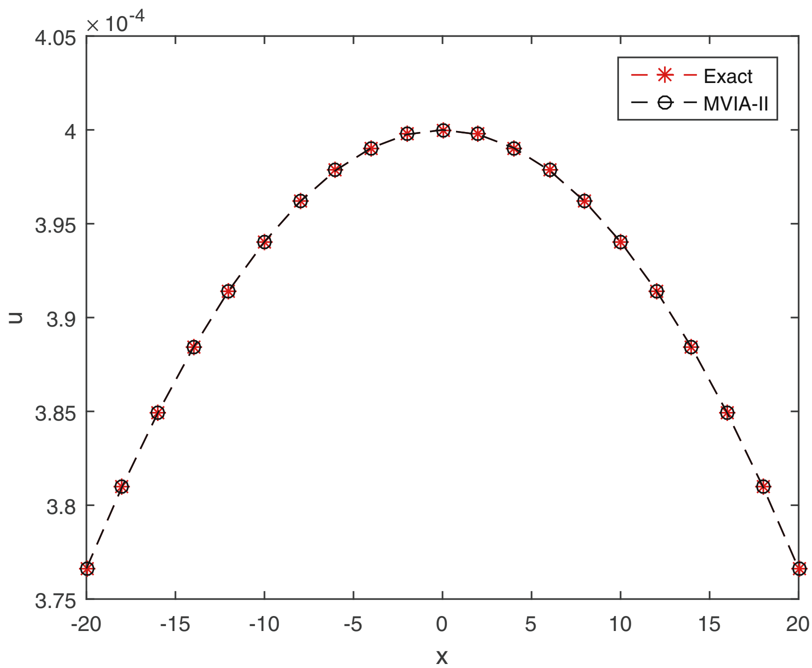

Figure 5: Exact and approximate solutions' comparison of Test Problem 3.3 for $t = 10$ and $k = 0.01$.

3.4 Test Problem 4

Consider the following KdV equation of fifth order (4)

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - u\frac{\partial^3 u}{\partial x^3} + \frac{\partial^5 u}{\partial x^5} = 0, \tag{19}$$

having the initial condition:

$$u(x, 0) = e^x, \tag{20}$$

and the exact solution of this fKdV equation (4) with condition (20) was given by [18]

$$u(x, t) = e^{x-t}. \tag{21}$$

The numerical simulation for different values of times t and x of Test Problem 3.4 using the MVIA-II is reported in Table 5. In the Tables 5 and 6, the comparison of numerical results of the MVIA-II, Laplace decomposition method [17] and homotopy perturbation transform method [18] for Test Problem 3.4 has been carried out. Comparison of numerical results is shown in Table 5 in terms of absolute errors for second-, fourth- and sixth-order approximations, as the order of approximation increases, the order of accuracy increases and results converge to exact solution and to show the efficiency and reliability of the

Table 5: Comparison of numerical solutions of fKdV equation (4) for Test Problem 3.4

x/t	E_2		E_4		E_6	
	MVIA-II	[**18**]	**MVIA-II**	[**18**]	**MVIA-II**	[**18**]
0.1	4.03000×10^{-7}	4.03001×10^{-7}	2.03168×10^{-10}	2.04000×10^{-10}	4.85726×10^{-14}	2.00000×10^{-12}
0.2	3.14615×10^{-6}	3.14615×10^{-6}	6.39581×10^{-9}	6.39800×10^{-9}	6.14129×10^{-12}	8.00000×10^{-12}
0.3	1.03656×10^{-5}	1.03656×10^{-5}	4.77886×10^{-8}	4.77890×10^{-8}	1.03657×10^{-10}	1.03000×10^{-10}
0.4	2.39942×10^{-5}	2.39942×10^{-5}	1.98186×10^{-7}	1.98186×10^{-7}	7.67223×10^{-10}	7.66000×10^{-10}
0.5	4.57809×10^{-5}	4.57809×10^{-5}	5.95330×10^{-7}	5.95333×10^{-7}	3.61487×10^{-9}	3.61500×10^{-9}

Table 6: Comparison of absolute errors of the MVIA-II and the Laplace decomposition method for sixth approximation

t/x	$x = 1.0$		$x = 1.5$		$x = 2.5$	
	MVIA-II	[17]	**MVIA-II**	[17]	**MVIA-II**	[17]
0.01	0	4.38198×10^{-11}	0	3.32204×10^{-10}	3.55271×10^{-15}	4.25802×10^{-9}
0.02	8.88178×10^{-16}	7.22467×10^{-11}	8.88178×10^{-16}	5.47711×10^{-10}	3.55271×10^{-15}	7.02029×10^{-9}
0.03	1.19904×10^{-14}	1.19114×10^{-10}	1.95399×10^{-14}	9.03023×10^{-10}	4.97379×10^{-14}	1.15745×10^{-8}
0.04	8.79296×10^{-14}	1.96387×10^{-10}	1.43884×10^{-13}	1.48883×10^{-9}	3.94351×10^{-13}	1.90831×10^{-8}
0.05	4.19220×10^{-13}	3.23787×10^{-10}	6.90114×10^{-13}	2.45467×10^{-9}	1.87405×10^{-12}	3.14627×10^{-8}

MVIA-II for different values of x and t, the comparison of errors with a well-known method for nonlinear PDEs, the Laplace decomposition method, is shown in Table 6.

3.5 Test Problem 5

Consider the Kawahara equation (5)

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} - \frac{\partial^5 u}{\partial x^5} = 0, \tag{22}$$

having the initial condition:

$$u(x, 0) = \frac{105}{169}\operatorname{sech}^4\left(\frac{x - x_0}{2\sqrt{13}}\right), \tag{23}$$

and the exact solution of equation (5) with condition (23) was given by [18]

$$u(x, t) = \frac{105}{169}\operatorname{sech}^4\left(\frac{1}{2\sqrt{13}}\left(x - x_0 + \frac{36t}{169}\right)\right). \tag{24}$$

To demonstrate the applicability and efficiency of our modified algorithm, we report the absolute errors for different values of x and t in Table 7. From the tabulated data and Figure 6, one can observe that the numerical and exact solutions by MVIA-II for the Kawahara equation (5) for different time levels with $x = 6.0$ are in good agreement for second-order approximation.

Table 7: Comparison of numerical solutions of Kawahara equation (5) for Test Problem 3.5

x/t	MVIA-II	[18]
0.1	2.96656×10^{-8}	1.64944×10^{-6}
0.2	2.37783×10^{-7}	6.67437×10^{-6}
0.3	8.04066×10^{-7}	1.51864×10^{-5}
0.4	1.90959×10^{-6}	2.72977×10^{-5}
0.5	3.73681×10^{-6}	4.31169×10^{-5}

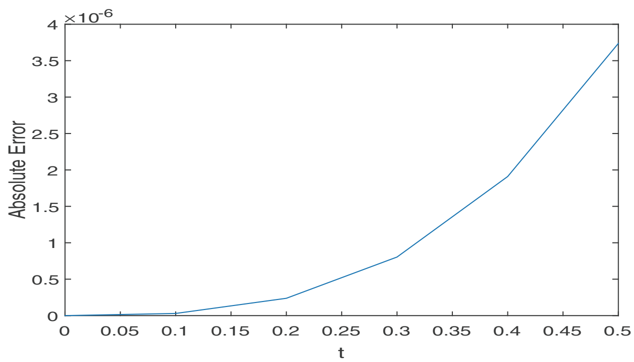

Figure 6: Absolute erorr graph for the Test Problem 3.5.

4 Conclusion

This article shows that the MVIA-II is very proficient, reliable and practically well suited for use in finding new traveling wave solutions for the higher order differential equations of KdV type. The reliability and accuracy of the method, as well as the decrease in the size of computational work, give this modified algorithm a more extensive pertinency. The results demonstrate that the algorithm is reliable and effective and gives more accurate solutions. This modified algorithm makes easy the computational work for solving nonlinear problems that emerge in science and engineering, and results of high degree accuracy can be obtained in few iterations as compared to earlier methods, and we hope that the obtained results will be useful for further studies in scientific materials science and engineering.

Acknowledgments: Hijaz Ahmad thanks Prof. Dr. Ji-Huan He for appreciation and continuous helpful discussions and advice. All the authors thank the Editor-in-Chief and unknown referees for the fruitful comments and significant remarks that helped them in improving the quality and readability of the paper, which led to a significant improvement of the paper. This work was supported by the National Natural Science Foundation of China (No. 71601072) and the Key Scientific Research Project of Higher Education Institutions in Henan Province of China (No. 20B110006).

References

[1] R. Gharechahi, M. Arab Ameri, and M. Bisheh-Niasar, *High order compact finite difference schemes for solving Bratu-type equations*, J. Appl Comput. Mech. **5** (2019), no. 1, 91–102.

[2] D. Kumar, J. R. Sharma, and C. Cesarano, *An efficient class of Traub-Steffensen-type methods for computing multiple zeros*, Axioms **8** (2019), no. 2, 65, DOI: 10.3390/axioms8020065.

[3] M. Nadeem, F. Li, and H. Ahmad, *He's variational iteration method for solving non-homogeneous Cauchy Euler differential equations*, Nonlin. Sci. Lett. A Math. Phys. Mech. **9** (2018), no. 3, 231–237.

[4] J. R. Sharma, S. Kumar, and C. Cesarano, *An efficient derivative free one-point method with memory for solving nonlinear equations*, Mathematics **7** (2019), no. 7, 604, DOI: 10.3390/math7070604.
[5] H. Ahmad, *Auxiliary parameter in the variational iteration algorithm-ii and its optimal determination*, Nonlin. Sci. Lett. A **9** (2018), no. 1, 62–72.
[6] J.-H. He and X.-H. Wu, *Variational iteration method: new development and applications*, Comput. Math. Appl. **54** (2007), no. 7–8, 881–894.
[7] H. Ahmad and T. A. Khan, *Variational iteration algorithm i with an auxiliary parameter for the solution of differential equations of motion for simple and damped mass-spring systems*, NVWW **51** (2020), no. 1–2, 12–20.
[8] J. Umavathi, A. J. Chamkha, and M. Shekar, *Free convection flow of an electrically-conducting micropolar fluid between parallel porous vertical plates using differential transform*, J. Appl. Comput. Mech. **4** (2018), no. 4, 286–298.
[9] J.-H. He and C. Sun, *A variational principle for a thin film equation*, J. Math. Chem. **57** (2019), no. 9, 2075–2081.
[10] H. Ahmad, T. A. Khan, and C. Cesarano, *Numerical solutions of coupled Burgers' equations*, Axioms **8** (2019), no. 4, 119, DOI: 10.3390/axioms8040119.
[11] J. Zhang, Y. Shen, and J. He, *Some analytical methods for singular boundary value problem in a fractal space: a review*, Appl. Comput. Math. **18** (2019), no. 3, 225–235.
[12] F.-Y. Ji, C.-H. He, J.-J. Zhang, and J.-H. He, *A fractal boussinesq equation for nonlinear transverse vibration of a nanofiber-reinforced concrete pillar*, Appl. Math. Model. **82** (2020), 437–448.
[13] K. Sawada and T. Kotera, *A method for finding n-soliton solutions of the KdV equation and KdV-like equation*, Prog. Theor. Phys. **51** (1974), no. 5, 1355–1367.
[14] P. Caudrey, R. Dodd, and J. Gibbon, *A new hierarchy of Korteweg-de Vries equations*, Proc. R. Soc. Lond. A Math. Phys. Sci. **351** (1976), no. 1666, 407–422.
[15] P. D. Lax, *Integrals of nonlinear equations of evolution and solitary waves*, Commun. Pure Appl. Math. **21** (1968), no. 5, 467–490.
[16] A.-M. Wazwaz, *Solitons and periodic solutions for the fifth-order KdV equation*, Appl. Math. Lett. **19** (2006), no. 11, 1162–1167.
[17] S. Handibag and B. Karande, *Existence the solutions of some fifth-order KdV equation by Laplace decomposition method*, Am. J. Comput. Math. **3** (2013), no. 1, 80–85, DOI: 10.4236/ajcm.2013.31013.
[18] A. Goswami, J. Singh, and D. Kumar, *Numerical simulation of fifth order KdV equations occurring in magneto-acoustic waves*, Ain Shams Eng. J. **9** (2018), no. 4, 2265–2273.
[19] Y. Yamamoto and É. I. Takizawa, *On a solution on non-linear time-evolution equation of fifth order*, J. Phys. Soc. Jpn. **50** (1981), no. 5, 1421–1422.
[20] K. Djidjeli, W. G. Price, E. H. Twizell, and Y. Wang, *Numerical methods for the solution of the third-and fifth-order dispersive Korteweg-de Vries equations*, J. Comput. Appl. Math. **58** (1995), no. 3, 307–336.
[21] C. Chun, *Solitons and periodic solutions for the fifth-order KdV equation with the exp-function method*, Phys. Lett. A **372** (2008), no. 16, 2760–2766.
[22] S. Abbasbandy and F. S. Zakaria, *Soliton solutions for the fifth-order KdV equation with the homotopy analysis method*, Nonlin. Dyn. **51** (2008), no. 1–2, 83–87.
[23] D. Kaya, *An explicit and numerical solutions of some fifth-order KdV equation by decomposition method*, Appl. Math. Comput. **144** (2003), no. 2–3, 353–363.
[24] D. Kaya and S. M. El-Sayed, *On a generalized fifth order KdV equations*, Phys. Lett. A **310** (2003), no. 1, 44–51.
[25] M. Rafei and D. Ganji, *Explicit solutions of Helmholtz equation and fifth-order KdV equation using homotopy perturbation method*, Int. J. Nonlin. Sci. Num. Simul. **7** (2006), no. 3, 321–328.
[26] Y. Lei, Z. Fajiang, and W. Yinghai, *The homogeneous balance method, lax pair, Hirota transformation and a general fifth-order KdV equation*, Chaos Soliton Fract. **13** (2002), no. 2, 337–340.
[27] M. Helal and M. Mehanna, *A comparative study between two different methods for solving the general Korteweg-de Vries equation* (*gKdV*), Chaos Soliton Fract. **33** (2007), no. 3, 725–739.
[28] H. O. Bakodah, *Modified adomain decomposition method for the generalized fifth order KdV equations*, Am. J. Comput. Math. **3** (2013), 53–58, DOI: 10.4236/ajcm.2013.31008.
[29] M. Inokuti, H. Sekine, and T. Mura, *General use of the Lagrange multiplier in nonlinear mathematical physics*, in: S. Nemat Nasser, Ed., Variational Method in the Mechanics of Solid, vol. 33, 1978, Pergamom Press, New York, pp. 156–162.
[30] J.-H. He, *Notes on the optimal variational iteration method*, Appl. Math. Lett. **25** (2012), no. 10, 1579–1581.
[31] H. Ahmad, A. R. Seadawy, and T. A. Khan, *Numerical solution of Korteweg-de Vries-Burgers equation by the modified variational iteration algorithm-ii arising in shallow water waves*, Phys. Scr. **95** (2020), no. 4, 045210, DOI: 10.1088/1402-4896/ab6070.
[32] M. Hosseini, S. T. Mohyud-Din, H. Ghaneai, and M. Usman, *Auxiliary parameter in the variational iteration method and its optimal determination*, Int. J. Nonlin. Sci. Num. Simul. **11** (2010), no. 7, 495–502.
[33] H. Ahmad, A. R. Seadawy, T. A. Khan, and P. Thounthong, *Analytic approximate solutions for some nonlinear parabolic dynamical wave equations*, J. Taibah Univ. Sci. **14** (2020), no. 1, 346–358.
[34] H. Ghaneai and M. Hosseini, *Variational iteration method with an auxiliary parameter for solving wave-like and heat-like equations in large domains*, Comput. Math. Appl. **69** (2015), no. 5, 363–373.

[35] H. Ahmad, *Variational iteration method with an auxiliary parameter for solving differential equations of the fifth order*, Nonlin. Sci. Lett. A **9** (2018), no. 1, 27–35.

[36] D.-N. Yu, J.-H. He, and A. G. Garca, *Homotopy perturbation method with an auxiliary parameter for nonlinear oscillators*, J. Low Freq. Noise V. A. **38** (2019), no. 3–4, 1540–1554, DOI: 10.1177/1461348418811028.

[37] H. Ahmad and T. A. Khan, *Variational iteration algorithm-i with an auxiliary parameter for wave-like vibration equations*, J. Low Freq. Noise V. A. **38** (2019), no. 3–4, 1113–1124, DOI: 10.1177/1461348418823126.

[38] J.-H. He, *Generalized variational principles for buckling analysis of circular cylinders*, Acta Mech. **231** (2020), 899–906.

[39] J.-H. He, *A fractal variational theory for one-dimensional compressible flow in a microgravity space*, Fractals **28** (2020), no. 2, 2050024, DOI: 10.1142/S0218348X20500243.

[40] C.-H. He, Y. Shen, F.-Y. Ji, and J.-H. He, *Taylor series solution for fractal Bratu-type equation arising in electrospinning process*, Fractals **28** (2020), no. 1, 2050011, DOI: 10.1142/S0218348X20500115.