

An Efficient Design of a Variable Fractional Delay Filter Using a First-Order Differentiator

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Abstract—In this letter, the Taylor series expansion is used to transform the design problem of a fractional delay filter into the one of a first-order differentiator such that the conventional finite-impulse response and infinite-impulse response differentiators can be applied to design a fractional delay filter directly. The proposed structure is more efficient than the well-known Farrow structure in terms of filter coefficient storage because only one first-order differentiator needs to be designed and implemented. Moreover, one design example is demonstrated to illustrate the effectiveness of this new design approach.

Index Terms—Differentiator, fractional delay filter.

I. INTRODUCTION

IN MANY applications of signal processing, there is a need for a delay that is a fraction of the sampling period. These applications include time adjustment in digital receivers, beam steering of antenna array, speech coding and synthesis, modeling of music instruments, sampling rate conversion, time delay estimation, comb filter design, analog–digital conversion, etc. [1]–[10]. An excellent survey of the fractional delay filter design is presented in tutorial papers [3], [4]. The desired frequency response of the variable fractional delay filter is given by

$$H_d(\omega, p) = e^{-j\omega(D+p)} \quad (1)$$

where delay D is an integer, and p is a variable or adjustable fractional number in the range $[-0.5, 0.5]$. So far, there have been several methods to design variable fractional delay finite-impulse response (FIR) filters. In [5], the transfer function of the FIR filter used to approximate this specification is chosen as follows:

$$H(z, p) = \sum_{n=0}^N a_n(p)z^{-n} \quad (2)$$

where $a_n(p)$ are the polynomial functions in p of degree M , i.e.,

$$a_n(p) = \sum_{k=0}^M a_{nk}p^k. \quad (3)$$

Manuscript received August 20, 2002; revised November 5, 2002. The associate editor coordinating the review of this manuscript and approving it for publication was Dr. Petr Tichavsky.

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Digital Object Identifier 10.1109/LSP.2003.815616

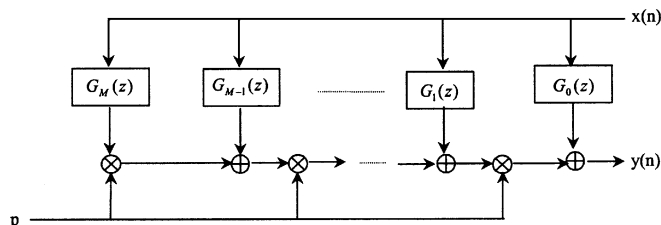


Fig. 1. Farrow structure for fractional delay filters with adjustable delay p .

Substituting (3) into (2), the transfer function can be rewritten as

$$H(z, p) = \sum_{k=0}^M \sum_{n=0}^N a_{nk}z^{-n}p^k = \sum_{k=0}^M G_k(z)p^k \quad (4)$$

where $G_k(z) = \sum_{n=0}^N a_{nk}z^{-n}$. In [5]–[10], several approaches have been proposed to design $M+1$ subfilters $G_k(z)$ for $k = 0, 1, \dots, M$ such that the filter $H(z, p)$ approximates the desired response $H_d(\omega, p)$ as well as possible. Once the $M+1$ subfilters $G_k(z)$ have been designed, the filter $H(z, p)$ can be implemented by the efficient Farrow structure shown in Fig. 1 [5].

On the other hand, the digital differentiator has been a very useful tool to determine and estimate the time derivatives of a given signal. For example, in radar and sonar applications, the velocity and acceleration are computed from position measurements using differentiators [11]. In biomedical engineering, it is often necessary to obtain the higher order derivatives of biomedical data, especially at low-frequency ranges [12]. Until now, several methods have been developed to design infinite-impulse response (IIR) and FIR digital differentiators such as the Remez exchange algorithm [13], eigenfilter method [14], least squares method [15], [16], quadratic programming [17], etc. In this letter, the Taylor series expansion will be used to transform the design problem of the fractional delay filter into that of a first-order differentiator such that conventional FIR and IIR differentiators can be applied to design the fractional delay filter directly. The proposed structure is more efficient than the Farrow structure in Fig. 1 in terms of filter coefficient storage because only one first-order differentiator needs to be designed and implemented instead of $M+1$ subfilters. Finally, it is worth mentioning that the idea of implementing a fractional delay filter or interpolation with various-order differentiators is not new. The related researches can be found in [18] and [19]. However, in this letter, the idea of only using the single first-order differentiator is novel.

II. DESIGN METHOD

In this section, we will use the Taylor series expansion to transform the design problem of the fractional delay filter into the one of a first-order differentiator. The main idea is based on the following fact.

Fact: If the frequency response of the first-order differentiator is denoted by $F(\omega) = (j\omega)e^{-j\omega n_0}$ and delay $D = Mn_0$, then it can be shown that the fractional delay filter $H_d(\omega, p)$ can be written as

$$H_d(\omega, p) = \sum_{k=0}^M \frac{(-1)^k}{k!} [F(\omega)]^k (e^{-j\omega n_0})^{M-k} p^k + O(p^{M+1}) \quad (5)$$

where M, n_0 are two prescribed integers, and $O(x)$ denotes a term which goes to zero at least as x when x approaches zero.

Proof: Using the Taylor series expansion, the term $e^{-j\omega p}$ can be expressed as a polynomial of p as follows:

$$\begin{aligned} e^{-j\omega p} &= \sum_{k=0}^{\infty} \frac{(-p)^k}{k!} (j\omega)^k \\ &= \sum_{k=0}^M \frac{(-1)^k}{k!} (j\omega)^k p^k \\ &\quad + p^{M+1} \left[\sum_{k=M+1}^{\infty} \frac{(-1)^k}{k!} (j\omega)^k p^{k-(M+1)} \right] \\ &= \sum_{k=0}^M \frac{(-1)^k}{k!} (j\omega)^k p^k + O(p^{M+1}). \end{aligned} \quad (6)$$

By multiplying both sides by the factor $e^{-j\omega D}$, we get the following equality:

$$e^{-j\omega(D+p)} = \sum_{k=0}^M \frac{(-1)^k}{k!} (j\omega)^k e^{-j\omega D} p^k + O(p^{M+1}). \quad (7)$$

Substituting $D = Mn_0$ into (7) and using equality $F(\omega) = (j\omega)e^{-j\omega n_0}$, we get

$$\begin{aligned} e^{-j\omega(D+p)} &= \sum_{k=0}^M \frac{(-1)^k}{k!} (j\omega)^k e^{-j\omega Mn_0} p^k + O(p^{M+1}) \\ &= \sum_{k=0}^M \frac{(-1)^k}{k!} (j\omega)^k e^{-j\omega kn_0} e^{-j\omega(M-k)n_0} p^k \\ &\quad + O(p^{M+1}) \\ &= \sum_{k=0}^M \frac{(-1)^k}{k!} [F(\omega)]^k (e^{-j\omega n_0})^{M-k} p^k \\ &\quad + O(p^{M+1}). \end{aligned} \quad (8)$$

Because the fractional number p is in the range $[-0.5, 0.5]$, the term $O(p^{M+1})$ approaches zero when M is very large. Thus,

TABLE I
THE NRMS ERROR FOR VARIOUS M

| M | NRMS (%) |
|-----|-----------|
| 1 | 20.526934 |
| 2 | 7.156246 |
| 3 | 2.034796 |
| 4 | 0.486079 |
| 5 | 0.099930 |
| 6 | 0.018016 |
| 7 | 0.002890 |
| 8 | 0.000417 |
| 9 | 0.000054 |
| 10 | 0.000006 |

the ideal response of the fractional delay filter can be approximated by the following form:

$$\hat{H}_d(\omega, p) = \sum_{k=0}^M \frac{(-1)^k}{k!} [F(\omega)]^k (e^{-j\omega n_0})^{M-k} p^k. \quad (9)$$

The larger M is, the better approximation that $\hat{H}_d(\omega, p)$ has. In order to evaluate the performance of this approximation, the normalized root mean square (NRMS) error is defined by

$$\text{NRMS} = \frac{\left[\int_0^{\alpha\pi} \int_{-0.5}^{0.5} |\hat{H}_d(\omega, p) - H_d(\omega, p)|^2 dp d\omega \right]^{1/2}}{\left[\int_0^{\alpha\pi} \int_{-0.5}^{0.5} |H_d(\omega, p)|^2 dp d\omega \right]^{1/2}} \times 100\%. \quad (10)$$

It is easy to show $|\hat{H}_d(\omega, p) - H_d(\omega, p)| = \left| \sum_{k=0}^M \frac{(-1)^k}{k!} (j\omega)^k p^k - e^{-j\omega p} \right|$ and $|H_d(\omega, p)| = 1$, so the NRMS only depends on the choice of M and α . Table I lists the NRMS for various M and $\alpha = 0.9$. From this result, it can be found that when $M \geq 5$, the NRMS is less than 0.1%. Thus, the $\hat{H}_d(\omega, p)$ approximates the ideal response $H_d(\omega, p)$ very well for $M \geq 5$.

Now, let us describe how to design a variable filter $H(z, p)$ to approximate $\hat{H}_d(\omega, p)$. From (9), we see that if a filter $G(z)$ is designed to approximate the first-order differentiator response $F(\omega) = (j\omega)e^{-j\omega n_0}$, then the following filter

$$H(z, p) = \sum_{k=0}^M \frac{(-1)^k}{k!} G(z)^k z^{-n_0(M-k)} p^k \quad (11)$$

approximates $\hat{H}_d(\omega, p)$ well. Based on (11), the fractional delay filter can be implemented by the M same first-order differentiator $G(z)$ and M integer delay $z^{-n_0(M-k)}$ shown in Fig. 2. Thus, the design problem reduces to the design of first-order differentiator $G(z)$. In the literature, several methods have been proposed to design FIR and IIR differentiator $G(z)$ [13]–[17]. Once $G(z)$ has been designed and inserted into the structure in Fig. 2, we can easily adjust the fractional number p to obtain the desired delay response. Now, three aspects of the efficiency are used to compare the Farrow structure in Fig. 1 with the proposed structure in Fig. 2.

1) *Computational complexity:* The Farrow structure has $M + 1$ subfilters $G_k(z)$, but our structure has M filters

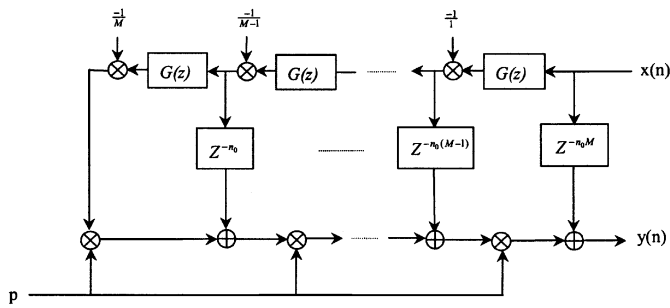


Fig. 2. Proposed structure for the fractional delay filter. The $G(z)$ is the first-order differentiator.

$G(z)$ and M scalar multiplications. Thus, both structure almost have the same arithmetic complexity.

- 2) *Delay of filter:* In Farrow structure, the integer delay D is fixed and specified in advance, but the delay D in our structure is equal to Mn_0 . Thus, when the number of subfilters M is large, the delay of the proposed structure is longer than the delay of Farrow structure.
- 3) *Storage requirement:* For the implementation of Farrow structure, there are the coefficients of $M + 1$ subfilters necessary to be stored in the memory. However, for the proposed structure, only the coefficients of a single first-order differentiator need to be stored in the memory. Thus, the proposed structure is more efficient than the Farrow structure in terms of the filter coefficient storage.

III. DESIGN EXAMPLE

In this section, an example performed with MATLAB language in an IBM-compatible personal computer is presented to illustrate the effectiveness of the proposed design method. To evaluate the performance, the maximum absolute error ϵ_{\max} and rms error ϵ_{rms} are defined by

$$\epsilon_{\max} = \max\{|e(\omega, p)| | \omega \in [0, \alpha\pi], p \in [-0.5, 0.5]\}$$

$$\epsilon_{\text{rms}} = \left[\int_0^{\alpha\pi} \int_{-0.5}^{0.5} |e(\omega, p)|^2 d\omega dp \right]^{\frac{1}{2}} \quad (12)$$

where error

$$e(\omega, p) = H_d(\omega, p) - H(e^{j\omega}, p). \quad (13)$$

In this example, the parameters are chosen as $n_0 = 29$, $M = 7$, and $\alpha = 0.9$. Thus, the integer delay $D = Mn_0 = 203$. Now, the least squares method in [16] is used to design linear phase FIR differentiator $G(z)$ with length $2n_0 + 1$ and passband edge frequency $\omega_p = \alpha\pi$. Fig. 3 shows the magnitude response of the designed first-order differentiator $G(z)$. Clearly, $G(z)$ approximates the ideal response ω well in the range $[0, 0.9\pi]$. By inserting the designed differentiator $G(z)$ into the structure in Fig. 2, the variable fractional delay filter $H(z, p)$ can be obtained. Figs. 4 and 5 depict the magnitude response in decibel scale and group delay of the designed variable fractional delay filter $H(z, p)$ in the frequency range $[0, 0.9\pi]$ for different $p \in [-0.5, 0.5]$. The maximum absolute error ϵ_{\max} is

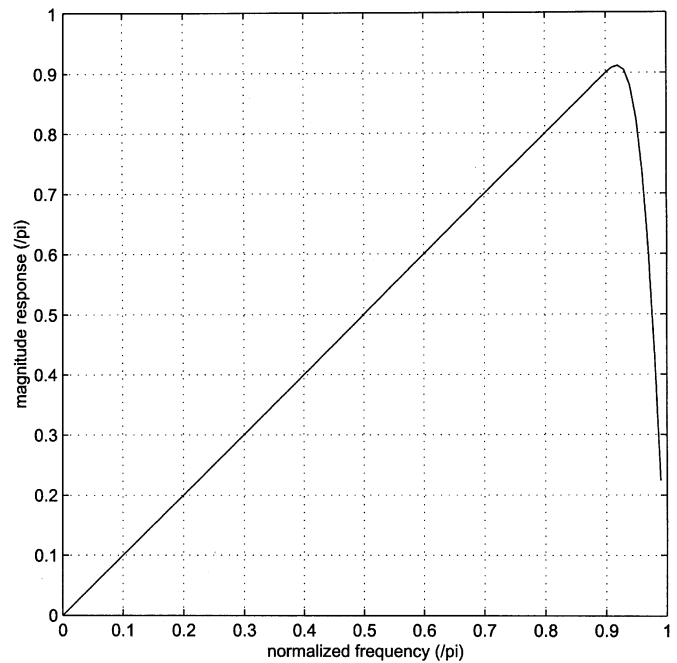


Fig. 3. Magnitude response of the designed first-order differentiator $G(z)$.

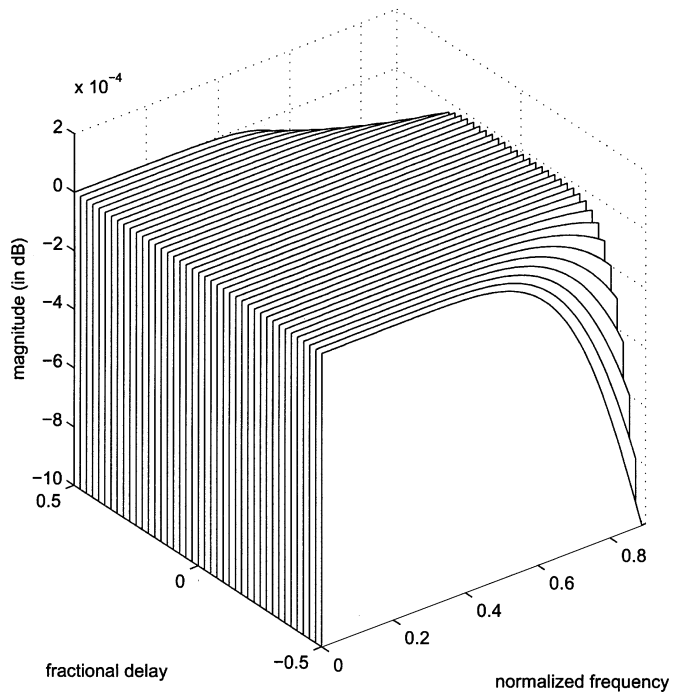


Fig. 4. Magnitude response of the designed variable fractional delay filter $H(z, p)$.

4.306×10^{-4} and rms error ϵ_{rms} is 4.272×10^{-5} . Because the errors are very small, the specification is well fitted.

Finally, it is interesting to compare the performance of the proposed structure with conventional Farrow structure under the same arithmetic complexity. Because filter $G_k(z)$ in Farrow structure is nonlinear phase, there are $N + 1$ multiplications needed to implement $G_k(z)$. In the above example, the first-order differentiator $G(z)$ is linear phase with length $2n_0 + 1$, so there are n_0 multiplications needed to implement differentiator

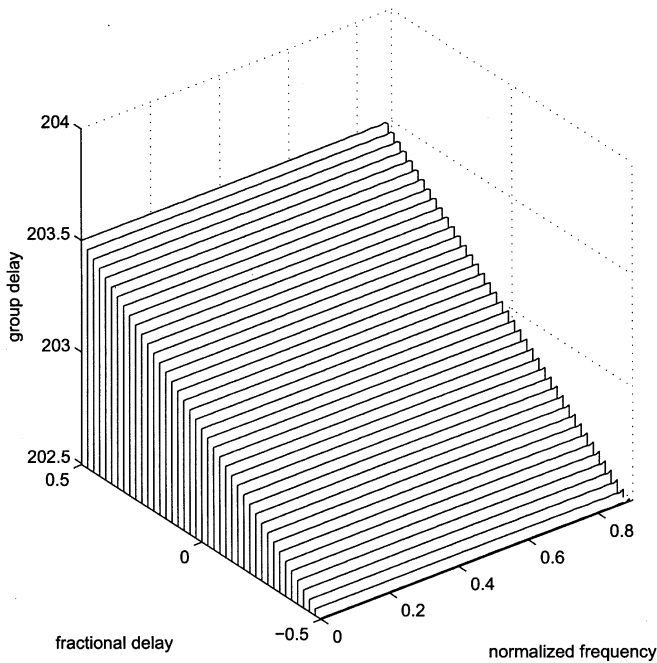


Fig. 5. Group delay response of the designed variable fractional delay filter $H(z, p)$.

$G(z)$. Thus, when we choose $N + 1 = n_0$, the filters in both structures have the same arithmetic complexity. Now, the conventional weighted least squares method in [7] is used to design filters $G_k(z)$ in Farrow structure with specification $N = 30$, $D = N/2 = 15$ and uniform weighting. As a result, the maximum absolute error ϵ_{\max} is 4.778×10^{-3} and rms error ϵ_{rms} is 7.845×10^{-4} . Thus, the proposed structure has smaller design errors than the Farrow structure under the same arithmetic complexity. However, the delay D of Farrow structure is 15, but the delay D of our structure is $Mn_0 = 203$. Thus, the proposed structure has longer delay than the Farrow structure.

IV. CONCLUSION

In this letter, the Taylor series expansion has been used to transform the design problem of the fractional delay filter into the one of a first-order differentiator such that the conventional digital differentiators can be directly applied to design a fractional delay filter. The proposed structure is more efficient than the well-known Farrow structure in terms of filter coefficient

storage because only a single first-order differentiator needs to be designed and implemented.

REFERENCES

- [1] S. C. Pei and C. C. Tseng, "A comb filter design using fractional sample delay," *IEEE Trans. Circuits Syst. II*, vol. 45, pp. 649–653, June 1998.
- [2] K. Rajamani, Y. S. Lai, and C. W. Farrow, "An efficient algorithm for sample rate conversion from CD to DAT," *IEEE Signal Processing Lett.*, vol. 7, pp. 288–290, Oct. 2000.
- [3] T. I. Laakso, V. Valimaki, M. Karjalainen, and U. K. Laine, "Splitting the unit delay: Tools for fractional delay filter design," *IEEE Signal Processing Mag.*, vol. 13, pp. 30–60, Jan. 1996.
- [4] V. Valimaki and T. I. Laakso, "Principle of fractional delay filters," in *Proc. ICASSP*, May 2000, pp. 3870–3873.
- [5] C. W. Farrow, "A continuously variable digital delay element," in *Proc. Int. Symp. Circuits and Systems*, 1988, pp. 2641–2645.
- [6] A. Tarczyski, G. D. Cain, E. Hermanowicz, and M. Rojewski, "WLS design of variable frequency response FIR filters," in *Proc. Int. Symp. Circuits and Systems*, 1997, pp. 2244–2247.
- [7] W. S. Lu and T. B. Deng, "An improved weighted least-squares design for variable fractional delay FIR filters," *IEEE Trans. Circuits Syst. II*, vol. 46, pp. 1035–1040, Aug. 1999.
- [8] T. B. Deng, "Discretization-free design of variable fractional-delay FIR filters," *IEEE Trans. Circuits Syst. II*, vol. 48, pp. 637–644, June 2001.
- [9] J. Vesma and T. Saramaki, "Design and properties of polynomial-based fractional delay filters," in *Proc. Int. Symp. Circuits and Systems*, 2000, pp. I-104–I-107.
- [10] C. C. Tseng, "Design of variable fractional delay FIR filter using differentiator bank," in *Proc. Int. Symp. Circuits and Systems*, vol. 4, 2002, pp. 421–424.
- [11] M. I. Skolnik, *Introduction to Radar Systems*. New York: McGraw-Hill, 1980.
- [12] S. Usui and I. Amidror, "Digital lowpass differentiation for biological signal processing," *IEEE Trans. Biomed. Eng.*, vol. BME-29, pp. 686–693, 1982.
- [13] T. W. Parks and J. H. McClellan, "Chebyshev approximation for non recursive digital filters with linear phase," *IEEE Trans. Circuit Theory*, vol. CT-19, pp. 189–194, Mar. 1972.
- [14] S. C. Pei and J. J. Shyu, "Eigenfilter design of higher-order digital differentiators," *IEEE Trans. Acoust. Speech Signal Processing*, vol. 37, pp. 505–511, 1989.
- [15] S. Sunder and V. Ramachandra, "Design of equiripple nonrecursive digital differentiators and Hilbert transformers using a weighted least-squares technique," *IEEE Trans. Signal Processing*, vol. 42, pp. 2504–2509, Sept. 1994.
- [16] G. Mollova, "Compact formulas for least-squares design of digital differentiators," *Electron. Lett.*, vol. 35, no. 20, pp. 1695–1697, Sept. 1999.
- [17] C. C. Tseng, "Stable IIR digital differentiator design using iterative quadratic programming approach," *Signal Process.*, vol. 80, pp. 857–866, 2000.
- [18] F. Harris, "Performance and design considerations of the Farrow filter when used for arbitrary resampling of sampled time series," in *Proc. 31st Asilomar Conf. Signals, Systems, and Computers*, vol. 2, Nov. 1997, pp. 1745–1749.
- [19] R. Sudhakar, R. C. Agarwal, and S. C. Dutta Roy, "Time domain interpolation using differentiators," *IEEE Trans. Acoust. Speech Signal Process.*, vol. ASSP-30, pp. 992–997, Dec. 1982.