AN EFFICIENT LINEAR METHOD OF ARMA SPECTRAL ESTIMATION

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ABSTRACT

A three step method for obtaining nearly maximum likelihood ARMA spectral estimates is presented. The computational complexity of the algorithm is comparable to Yule-Walker methods, but the method gives asymptotically efficient estimates. The implementation of the algorithm is discussed, and numerical examples are presented to illustrate its performance.

I. INTRODUCTION

Spectral estimation is a topic that continually receives a great deal of attention. Of the many techniques available, parametric techniques, and in particular the use of an autoregressive moving average (ARMA) model have become very popular. Two main types of ARMA spectral estimation methods have emerged. One type is optimization-based, and includes maximum likelihood (ML) methods, prediction error methods, and various nonlinear least square methods [1,2]. These procedures can sometimes be computationally intensive, and suffer from problems associated with convergence to false (local) minima. The other main type of estimator is the class of Yule-Walker based methods [2,3]. These techniques are generally much less computationally burdensome, but can produce estimates with poor accuracy unless special steps are taken.

This paper discusses an ARMA spectral estimation procedure that combines the simplicity of Yule-Walker based methods with the accuracy of ML methods. In this algorithm, initial covariance estimates are obtained, and an initial estimate of the ARMA spectral parameters are computed. These initial estimates are then used to correct the covariance estimates to improve their accuracy. This correction requires only linear operations, and the corrected estimates give asymptotically efficient estimates of their corresponding spectral parameters. This paper focuses on implementation aspects; we develop a numerically reliable version of the algorithm, and present simulation examples to illustrate its performance.

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II. AN ASYMPTOTICALLY EFFICIENT ALGORITHM

Consider the following ARMA process of order (na, nc):

$$A(q^{-1})y(t) = C(q^{-1})e(t) \tag{1}$$

where $e(t)$ is white noise with zero mean and variance λ^2 and

$$\begin{aligned} A(q^{-1}) &= 1 + a_1 q^{-1} + \cdots + a_{na} q^{-na} \\ C(q^{-1}) &= 1 + c_1 q^{-1} + \cdots + c_{nc} q^{-nc} \end{aligned} \tag{2}$$

We assume that the ARMA representation is minimal, stable, and invertible, and that the orders na and nc are given.

Next, we introduce the following notation:

$$r_k = E\{y(t)y(t-k)\} \tag{3a}$$

$$\phi(z) = \sum_{k=-\infty}^{\infty} r_k z^{-k} = \text{the spectral density of } y(t) \tag{3b}$$

Here, $E\{\cdot\}$ denotes expected value and z is a complex variable. It is well known that

$$\phi(z) = \lambda^2 \frac{C(z)C(z^{-1})}{A(z)A(z^{-1})} \tag{4}$$

The problem of parametric spectral estimation consists of first parameterizing the spectral density function, then estimating those parameters. One can parameterize $\phi(z)$ by $\{\lambda^2, a_1, \ldots, a_{na}, c_1, \ldots, c_{nc}\}$. However, statistically efficient estimation of these parameters is not an easy task, and nonlinear optimization routines are generally employed. In this paper the spectral density is instead parameterized by

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$$\theta = [r_0, \ldots, r_{n\theta}]^T \quad n\theta \triangleq na + nc \tag{5}$$

The a_i coefficients are uniquely determined from θ by solving

$$\begin{bmatrix} r_{nc} & \cdots & r_{nc-na+1} \\ \vdots & \ddots & \vdots \\ r_{nc+na-1} & \cdots & r_{nc} \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_{na} \end{bmatrix} = - \begin{bmatrix} r_{nc+1} \\ \vdots \\ r_{nc+na} \end{bmatrix} \tag{6}$$

or $Ra = -r$. Moreover, defining

$$b_k = \sum_{i=0}^{na} \sum_{j=0}^{na} a_i a_j r_{k+j-i} \tag{7}$$

where $a_0 = 1$, it readily follows from (1), (4), and (7) that

$$\phi(z) = \frac{\sum_{k=-nc}^{nc} b_k z^{-k}}{A(z)A(z^{-1})} \tag{8}$$

In the algorithm that follows, we will derive a statistically efficient estimate of θ. From θ we will obtain the AR coefficient estimate using (6), and the spectral estimate from (6)–(8). Thus, the problem of spectral estimation reduces to estimating the parameter vector θ from data measurements $\{y(t)\}_{t=1}^{N}$. For brevity, we present only the needed results, and refer the reader to [4] for details.

It is well-known that the standard unbiased sample covariances

$$\tilde{r}_k = \frac{1}{N-k} \sum_{t=1}^{N-k} y(t)y(t-k) \qquad k = 0, 1, \ldots \tag{9}$$

are consistent estimates, but are in general not efficient estimates, and can have very bad accuracy; see e.g. [5]. A more accurate estimator is given by the following three-step algorithm:

Step 1: Compute an initial $\tilde{\theta}$ vector using (9)

Step 2:

a) Find $\tilde{a}$ using $\tilde{r}$ in (6) or by solving an overdetermined set of Yule-Walker equations.

b) Find $\tilde{b}$ using $\tilde{a}$ and $\tilde{r}$ in (7).

c) Form z, W_{12}, W_{22} where:

$$\begin{aligned} z &= [z_1, z_2, \ldots, z_{nz}]^T \\ z_k &= \sum_{i=0}^{na} \sum_{j=0}^{na} \tilde{a}_i \, \tilde{a}_j \, \tilde{r}_{na+nc+k-i-j} \\ [W_{12}]_{jk} &= \alpha_{k+j} + \alpha_{k-j} \qquad (n\theta + 1 \times nz) \\ \alpha_s &= \text{the coeff. of } z^s \text{ in } z^{-(na+nc)} \frac{[\sum_{k=-nc}^{nc} \tilde{b}_k z^k]^2}{\tilde{A}^2(z^{-1})} \\ [W_{22}]_{ij} &= \beta_{i-j} \qquad (nz \times nz) \\ \beta_s &= \text{the coeff. of } z^s \text{ in } [\sum_{k=-nc}^{nc} \tilde{b}_k z^k]^2 \end{aligned}$$

d) Compute the updated estimate

$$\hat{\theta} = \tilde{\theta} - W_{12} W_{22}^{-1} z \tag{10}$$

Step3:

a) Compute $\hat{a}$ using $\hat{\theta}$ in (7).

b) Compute $\hat{b}$ using $\hat{a}$ and $\hat{\theta}$ in (7).

c) Compute $\hat{\phi}$ using $\hat{a}$ and $\hat{b}$ in (8).

Note that W_{22} is a banded, symmetric Toeplitz matrix with the bandwidth $2nc + 1$, and that $[W_{12}]_{ij} = 0$ for $j > nc - na + i$. Thus, computation of the updated estimate can be carried out in a computationally efficient manner. In fact, the computation of z, α, and β requires simple multiplication and addition steps. And since W_{22} is banded and Toeplitz, $W_{22}^{-1} z$ can be computed in an efficient manner using the method in [6] or [7].

The estimate $\hat{\theta}$ in (10) has the following properties [4]:

P1: $\hat{\theta}$ is a minimum variance estimate in the class of all estimators which are based on the sample covariances $\{\tilde{r}_0, \ldots, \tilde{r}_{n\theta+nz}\}$. Under the Gaussian hypothesis it is a large-sample approximation of the ML estimate of θ which uses the sample covariances as a data statistic.

P2: Let P_{nz}^{θ} denote the asymptotic (for N $\rightarrow \infty$) covariance matrix of $\sqrt{N}\hat{\theta}$. Then

$$P_{nz}^{\theta} \geq P_{nz+1}^{\theta} \quad \text{and} \quad \lim_{nz \to \infty} P_{nz}^{\theta} = P_{CR}^{\theta}$$

where P_{CR}^{θ} is the Cramèr-Rao lower bound (CRLB) for the covariance matrix of any consistent estimator of θ under the Gaussian hypothesis.

P3: Let P_{nz}^{a} denote the asymptotic covariance matrix of $\sqrt{N}\hat{a}$, and let P_{CR}^{a} denote its CRLB matrix under the Gaussian hypothesis. Then

$$P_{nz}^{a} \geq P_{nz+1}^{a} \quad \text{and} \quad \lim_{nz \to \infty} P_{nz}^{a} = P_{CR}^{a}$$

Explicit expressions for P_{nz}^{θ}, P_{CR}^{θ}, P_{nz}^{a}, and P_{CR}^{a} may be found in [4].

III. IMPLEMENTATION OF THE ALGORITHM

The properties obtained for the above algorithm are asymptotic (as $N \rightarrow \infty$) results. The derivations of those results rely on consistency properties to show that certain matrices are invertible, or that stable polynomial estimates result. For finite N, these propoerties may not hold, and the performance of the algorithm may not approximate its large sample properties. Specific problems which arise are:

- W_{22} imay be nearly singular.
- $\tilde{A}(z^{-1})$ may not stable, so the resulting α_k is sequence exponentially increasing.

The result of these effects is that the statistical properties of the estimates do not improve monotonically as nz is increased, unless N is very large. The "best" value of nz to use varies widely with data length and the ARMA process parameters.

To improve the robustness of the algorithm for small N, the algorithm was augmented in two places. First, the computation of $W_{22}^{-1}z$ includes a singularity test and correction. Specifically, if a reflection coefficient in the inverse computation (see [6]) is too close to one, it is truncated to $1-\epsilon$. This step requires at most nz extra multiplies is the algorithm in [7] is used to compute $W_{22}^{-1}z$.

Second, the computation of α_s in Step 2 is replaced by:

- If $\tilde{A}(z^{-1})$ has unstable roots, replace it by $\bar{A}(z^{-1})$ where $\bar{a}_i = \tilde{a}_i(1-\delta)^i$ for some $\delta > 0$
- Compute $\bar{b}_k$ by $\bar{b}_k = \tilde{b}_k(1-\delta)^{2nc-k}$.
- Compute W_{12} using $\bar{b}_k$ and $\bar{a}_k$.

In our simulations $\delta = 10/N$, was used, but it was found that an increase or decrease of δ by an order of magnitude did not significantly affect the results. The change in the $\bar{b}$ sequence is a first order approximation which attempts to keep the residues of the poles unchanged. This step is also computationally efficient; the Levinson algorithm is used to check for stability of $\tilde{A}(z^{-1})$ (requiring $O(na^2)$ computations), and $\bar{a}$ and $\bar{b}$ can be computed using $2nc + na + 1$ multiplies.

It was experimentally found that the revised α_s sequence significantly improved the robustness of the algorithm mainly when N was small and nz was large; otherwise the poles $\tilde{A}(z^{-1})$ were not large enough in magnitued to produce an α_s sequence that grew too fast. The check in the $W_{22}^{-1}z$ computation had a much smaller effect on the statistics of the parameter estimates in relation to the α_s check.

IV. SIMULATIONS

We have tested this algorithm on a variety of numerical examples using several ARMA models; we show results of a typical example here. We have generated data from the ARMA(4,4) model with coefficients:

$$a_1 = 0.1 \quad a_2 = 1.66 \quad a_3 = 0.093 \quad a_4 = .8649$$
$$c_1 = 0.0226 \quad c_2 = 0.8175 \quad c_3 = 0.0595 \quad c_4 = 0.0764$$

This ARMA process is the same as in [8] and references therein. Experiments were conducted by forming 200 Monte-Carlo estimates from simulated data, and computing the average value of N times the sums-squared error (SSE) of various parameter estimates. Bias and variance were also computed, and in all cases the bias term was negligible compared to the variance; thus the SSE plots here are essentially normalized variance plots.

Figure 1 shows $N\cdot$SSE of the AR coefficient $\hat{a}_1$ as a function of nz and for varying data lengths (the $nz = 0$ point is the Yule-Walker estimate estimate $\tilde{a}_1$). As expected, the experimental SSE's more closely approximate the theoretical (asymptotic) result as N increases. Also, it can be seen that the SSE's stay fairly constant up to a 'threshold' value of nz, after which the SSE's become very large. This threshold value increases with increasing N. Thus, if good estimation of AR coefficients is needed, any value of nz in the range 3–9 provides similar (and near minimum) estimation variances.

For large values of N, the spectral estimates also exhibit behavior that closely matches theory. Of interest is the characteristics of the algorithm when N is relatively small. To this end, Figure 2 shows ten candidate spectral estimates obtained from $N = 200$, but using different values of nz. Figure 3 shows N times the standard deviation of 200 estimates for differing nz. It can be seen that the standard deviation of the estimator decreases substantially when the three step algorithm is used.

Other experiments included using an overdetermined Yule-Walker equation set instead of the minimal one in (6). The results were improved for short data lengths, but the improvement diminishes for larger N.

V. CONCLUSIONS

We have presented an asymptotically statistically efficient linear estimator for ARMA processes. The estimator is based on a three step algorithm which obtaines statistically efficient estimates of the first $na + nc$ autocovariances of the process; these coefficients can then be transformed to ARMA coefficients, or into a spectral estimate. The algorithm is computationally very efficient, requiring two more Toeplitz equation solutions and a small number of additional multiplies over the standard Yule-Walker estimator.

Simulations demonstrate that as N increases, the experimental parameter variances more closely match theoretical variances. When N is large, very good agreement with theory is noted. For smaller data lengths ($N = 200$ in the example shown), special numerical conditioning steps are added to the original algorithm; even in this case significant improvement in estimator variance can be obtained.

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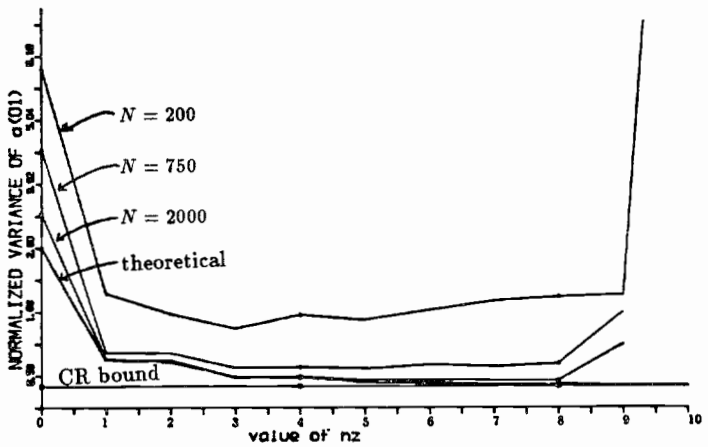

Figure 1: A comparison of N times the sum-squared error of a_1 for various data lengths.

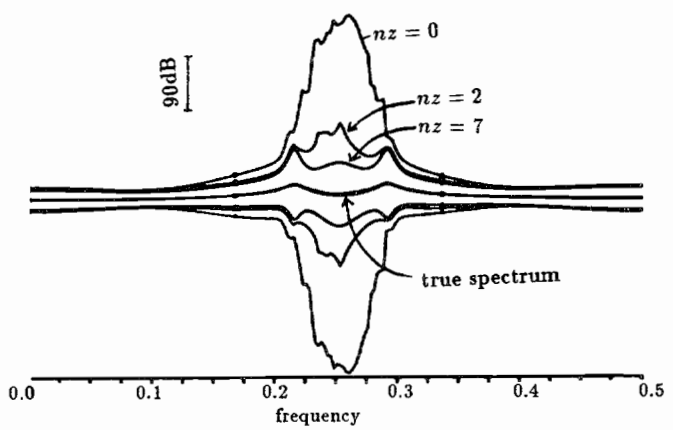

Figure 3: Sample estimate (from 200 realizations) of $\sqrt{N}$ times the standard deviation of the spectral estimates for $N = 200$ and various nz.

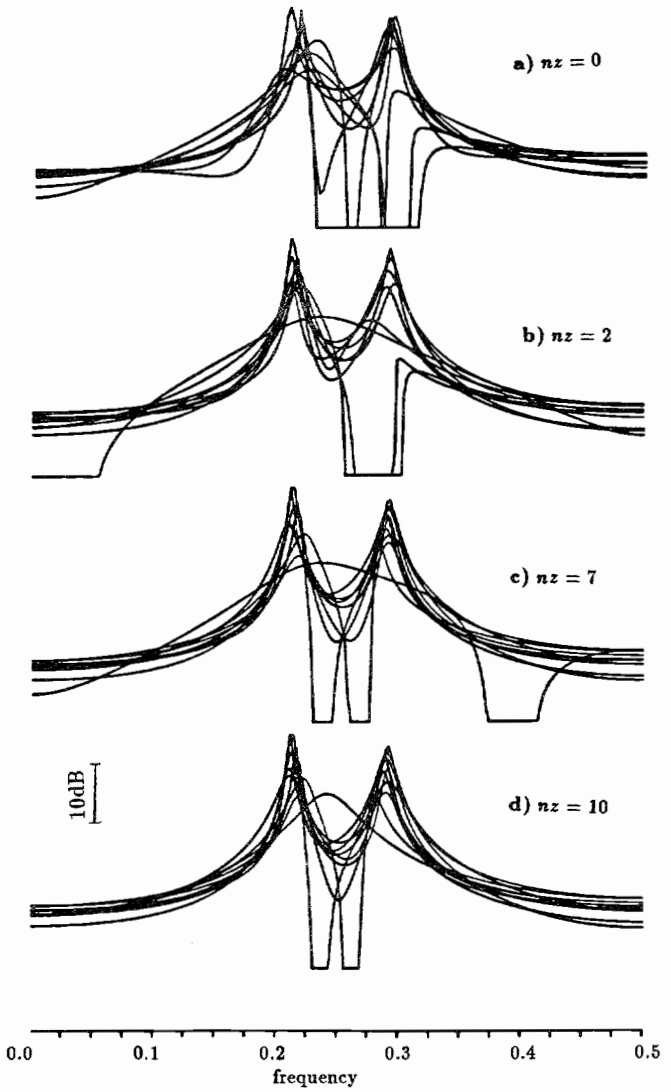

Figure 2: Ten overlapped spectral estimates using varying nz, each from 200 data points.