Bulletin of the Seismological Society of America, Vol. 85, No. 4, pp. 1080–1093, August 1995

An Efficient Method for Computing Green's Functions for a Layered Half-Space with Sources and Receivers at Close Depths (Part 2)

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Abstract In this study, we improve Hisada's (1994) method to efficiently compute Green's functions for viscoelastic layered half-spaces with sources and receivers located at equal or nearly equal depths. Compared with Hisada (1994), we can significantly reduce the range of wavenumber integration especially for the case that sources and receivers are close to the free surface or to boundaries of the source layer. This can be done by deriving analytical asymptotic solutions for both the direct wave and the reflected/transmitted waves from the boundaries, which are neglected in Hisada (1994). We demonstrate the validity and efficiency of our new method for several cases. The FORTRAN codes for this method for both point and dipole sources are open to academic use through anonymous FTP.

Introduction

As described in Hisada (1994), it is difficult to compute Green's functions for layered half-spaces with sources and receivers at nearly equidistant depths, because their integrands oscillate with slowly decreasing and increasing amplitudes for displacements and stresses, respectively. To remedy this problem, Apsel and Luco (1983) proposed an asymptotic technique, in which we subtract asymptotic solutions from the integrands and integrate them analytically, and numerically integrate the remaining integrands. These procedures can be summarized in the following equation:

$$G = \int_0^\infty \{(V - \tilde{V})b_1 + (H - \tilde{H})b_2\}dkS + \Delta\tilde{V} + \Delta\tilde{H}, \tag{1}$$

where G is a displacement or stress Green's function, b_j is a Bessel function, S is a sinusoidal function, k is the horizontal wavenumber, and V and H are displacement-stress vectors for *P-SV* and *SH* waves, respectively. $\tilde{V}$ is the solution asymptotic to V as k goes to infinity, $\Delta\tilde{V}$ is the analytical integration corresponding to $\tilde{V}$. $\tilde{H}$ and $\Delta\tilde{H}$ are those for *SH* waves [see equations (20) and (21) in Hisada (1994) for details].

It is clear that the more accurate asymptotic solutions are, the shorter the range of integration is. However, more accurate solutions are also more complicated mathematically. Apsel and Luco (1983) used the static solutions of a homogeneous half-space as asymptotic solutions for layered half-spaces. Herrmann and Wang (1985) and Herrmann (1993) computed numerically asymptotic solutions using Haskell's propagator matrix. On the other hand, Zeng and Anderson (1995) recently used the analytic solutions of the direct waves from sources as asymptotic solutions. Hisada (1994) derived generalized forms for the direct waves and showed that the technique was very efficient even for the case that sources and/or receivers are located in layers different from the source layer. However, Hisada (1994) found that the convergences of the asymptotic solutions are rather slow, when sources and/or receivers are very close to layer boundaries, because the reflected waves from the boundaries are not included in the solutions [see case 3 in the Results section of Hisada (1994)]. Recently, Zeng (1995) independently improved Hisada's method by deriving a numerical generalized asymptotic solution including the reflected waves.

In this study, we derive the analytic asymptotic solutions of the direct, reflected, and transmitted waves from layer boundaries and show the procedure for computing Green's functions due to point and dipole sources. The FORTRAN codes of this method are open to the public. Finally, we demonstrate the validity and efficiency of our new method for a number of cases by comparing our results with Hisada (1994).

We will refer to Hisada (1994) as H94 hereafter, because this article uses many equations from Hisada (1994).

An Asymptotic Method to Compute Green's Function of Layered Half-Space

Green's Functions Due to Point Sources

As discussed in H94, we adopt the generalized R/T (reflection and transmission) coefficient method of Luco and Apsel (1983) rather than Kennett (1974) and Kennett and Kerry (1979). It should be noted that we improved Luco and

Apsel (1983) in H94 by completely eliminating the exponential terms that grow in amplitude with wavenumbers and/or frequencies and that hinder analytical evaluation of asymptotic solutions.

As shown in Figure 1, we employ the same layered half-space model and notations as those of H94. Point sources, with vector components $(Q_x,0,0)$, $(0,Q_y,0)$, and $(0,0,Q_z)$ in the Cartesian coordinate system, are located at $(0,0,h)$ in the **S**th layer. To express source conditions, we divide the **S**th layer into the upper ($\mathbf{S}^-$) and lower layer ($\mathbf{S}^+$) at the source depth h. A receiver is located at (r,θ,z) or (x,y,z) in the cylindrical or Cartesian coordinate system, respectively, in the **j**th layer. We assume that the receivers are located in the **S**th layer or its adjacent layers, $j = S-1, S^-, S^+$, or $S+1$, because our purpose is to derive the asymptotic solutions for the case in which the source depths are close to those of receivers.

Static and dynamic Green's functions due to point sources are summarized in equations (11) and (12) in H94, and their displacement-stress vectors are given in Appendix A of H94.

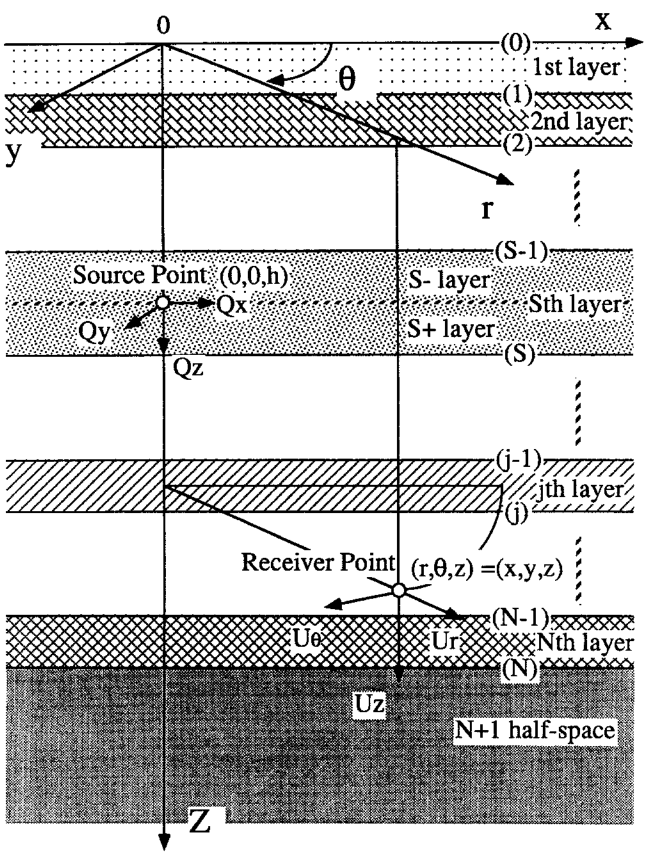

Figure 1. The multi-layered half-space model considered in this study. Point sources are located at $(0,0,h)$ in the **S**th layer, with the vector components $(Q_x,0,0)$, $(0,Q_y,0)$ and $(0,0,Q_z)$ in the Cartesian coordinate system. The receiver is located at (r,θ,z) in the **j**th layer, with the displacement components (U_r, U_θ, U_z) in the cylindrical coordinate system.

Asymptotic Solutions of the Displacement-Stress Vectors

SH Waves. We derive asymptotic solutions for the displacement-stress vectors including the reflected/transmitted waves from layer boundaries. We derive those of *SH* waves first in detail, because they have much simpler forms. From equations (A1), (A7), (A13), and (A14) in Appendix A of H94, the displacement-stress vectors of *SH* waves will be the following forms:

$$\begin{Bmatrix} H^j_{1q}(z;h) \\ H^j_{2q}(z;h) \end{Bmatrix} = \begin{bmatrix} E^j_{11} & E^j_{12} \\ E^j_{21} & E^j_{22} \end{bmatrix} \begin{bmatrix} \Lambda^j_d(z) & 0 \\ 0 & \Lambda^j_u(z) \end{bmatrix} \begin{Bmatrix} C^j_{dq}(h) \\ C^j_{uq}(h) \end{Bmatrix},$$

$$(q = x \text{ or } y), \tag{2}$$

where

$$\begin{bmatrix} E^j_{11} & E^j_{12} \\ E^j_{21} & E^j_{22} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -\mu^j k & \mu^j k \end{bmatrix},$$

$$\Lambda^j_d(z) = \exp\{-k(z - z^{(j-1)})\},$$

$$\text{and } \Lambda^j_u(z) = \exp\{-k(z^{(j)} - z)\}. \tag{3}$$

C^j_{dq} and C^j_{uq} are the down- and upgoing coefficients of the **j**th layer, the subscript q represents a direction of the point source, μ^j is the rigidity of the **j**th layer, and $z^{(j-1)}$ is the depth of the boundary between the $\mathbf{j}-\mathbf{1}$th and **j**th layers (see Fig. 1). We used the static solutions in the above equations, because dynamic solutions converge to static ones with increasing wavenumber (see, e.g., Luco and Apsel, 1983).

As shown in Appendix A of H94, the down/upgoing coefficients are determined from the boundary and source conditions using the generalized R/T coefficients. There are three cases for formulating asymptotic solutions.

Case 1 ($z^{(S-1)} \leqq h \leqq z^{(S)}$, and $S \neq 1$ nor $N+1$)

This case is same as case 1 of Appendix A in H94. Figure 2 shows all the down/upgoing coefficients existing in the $S-1$, S^-, S^+, and $S+1$ layers. We can assume incoming waves from the $S-1$ and $S+1$ layers quickly converge to zero with increasing wavenumber, because we do not have any growing exponential terms in our formulation:

$$C^{S-1}_{dq} \text{ and } C^{S+1}_{uq} \to 0. \tag{4}$$

Substituting equation (4) in equation (A16) of H94, we can derive the asymptotic solutions of the down/upgoing coefficients in the $S-1$, S^-, S^+, and $S+1$ layers:

$$C^{S-1}_{uq} = R^{(S-1)}_d C^{S-1}_{dq} + T^{(S-1)}_u C^{S^-}_{uq} \to T^{(S-1)}_u C^{S^-}_{uq},$$

$$C^{S^-}_{dq} = T^{(S-1)}_d C^{S-1}_{dq} + R^{(S-1)}_u C^{S^-}_{uq} \to R^{(S-1)}_u C^{S^-}_{uq},$$

$$C_{uq}^{S+} = R_d^{(S)} C_{dq}^{S+} + T_u^{(S)} C_{uq}^{S+1} \rightarrow R_d^{(S)} C_{dq}^{S+},$$

$$C_{dq}^{S+1} = T_d^{(S)} C_{dq}^{S+} + R_u^{(S)} C_{uq}^{S+1} \rightarrow T_d^{(S)} C_{dq}^{S+}, \tag{5}$$

where $T_u^{(S-1)}$ *etc.* are the modified R/T coefficients. Their asymptotic forms are analytically obtained by substituting equation (3) into equation (A18) of H94:

$$\begin{bmatrix} T_d^{(j)} & R_u^{(j)} \\ R_d^{(j)} & T_u^{(j)} \end{bmatrix} \rightarrow \frac{1}{\mu^{j+1} + \mu^j} \begin{bmatrix} 2\mu^j & \mu^{j+1} - \mu^j \\ -(\mu^{j+1} - \mu^j) & 2\mu^{j+1} \end{bmatrix} \times \begin{bmatrix} \exp\{-k(z^{(j)} - z^{(j-1)})\} & 0 \\ 0 & \exp\{-k(z^{(j+1)} - z^{(j)})\} \end{bmatrix}. \tag{6}$$

On the other hand, by comparing equation (5) with equations (A19) and (A20) of H94, we obtain

$$\tilde{T}_u^{(S-1)} \rightarrow T_u^{(S-1)}, \quad \tilde{R}_u^{(S-1)} \rightarrow R_u^{(S-1)},$$

$$\tilde{R}_d^{(S)} \rightarrow R_d^{(S)}, \quad \text{and } \tilde{T}_d^{(S)} \rightarrow T_d^{(S)}. \tag{7}$$

This shows that the generalized R/T coefficients ($\tilde{T}_u^{(S-1)}$ *etc.*) converge to the corresponding modified coefficients.

Substituting equations from (3) to (7) into equations (A26) and (A27) of H94, we obtain the asymptotic solutions of the down/upgoing coefficients of the $S-1$, S^-, S^+, and $S+1$ layers:

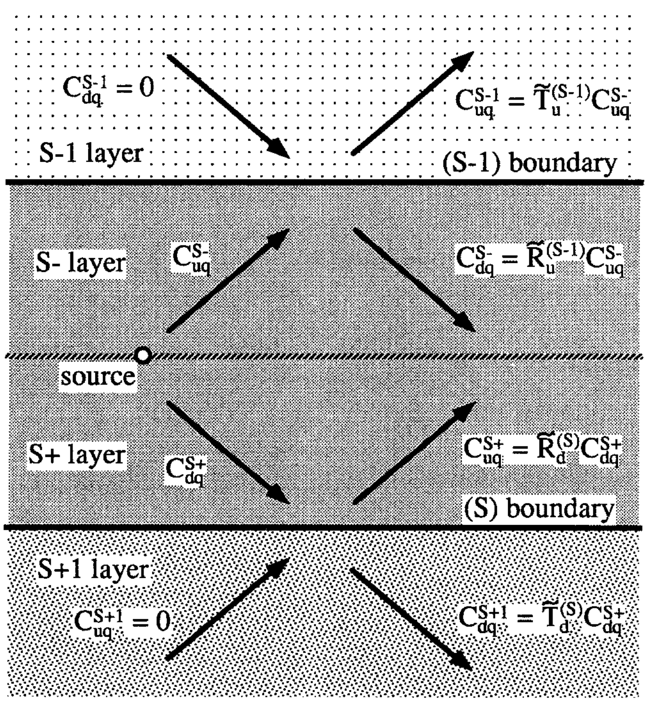

Figure 2. The asymptotic down/upgoing coefficients in the $S-1$, S^-, S^+, and $S+1$ layers for case 1: $Z^{(S-1)} \leqq h \leqq Z^{(S)}$, $S \neq 1$, and $S \neq N+1$.

$$C_{uq}^{S-} \rightarrow \frac{Q_q}{4\pi\mu^s}[1 - R^{(S)} \exp\{-2k(z^{(S)} - h)\}],$$

$$C_{dq}^{S+} \rightarrow \frac{Q_q}{4\pi\mu^s}[1 + R^{(S-1)} \exp\{-2k(h - z^{(S-1)})\}],$$

$$C_{dq}^{S-} \rightarrow \frac{Q_q}{4\pi\mu^s} R^{(S-1)} \exp\{-k(h - z^{(S-1)})\},$$

$$C_{uq}^{S+} \rightarrow -\frac{Q_q}{4\pi\mu^s} R^{(S)} \exp\{-k(z^{(S)} - h)\},$$

$$C_{uq}^{S-1} \rightarrow \frac{Q_q}{4\pi\mu^s} T^{(S-1)} \exp\{-k(h - z^{(S-1)})\},$$

$$C_{dq}^{S+1} \rightarrow \frac{Q_q}{4\pi\mu^s} T^{(S)} \exp\{-k(z^{(S)} - h)\}, \quad (q = x \text{ or } y), \tag{8}$$

where

$$R^{(S-1)} = \frac{\mu^S - \mu^{S-1}}{\mu^S + \mu^{S-1}}, \quad R^{(S)} = \frac{\mu^{S+1} - \mu^S}{\mu^{S+1} + \mu^S},$$

$$T^{(S-1)} = \frac{2\mu^S}{\mu^S + \mu^{S-1}}, \quad T^{(S)} = \frac{2\mu^S}{\mu^{S+1} + \mu^S}. \tag{9}$$

In the derivation of equation (8), we took only the first-order terms in k and neglected higher orders except $\exp\{-2k(z^{(S)} - h)\}$ in C_{uq}^{S-} and $\exp\{-2k(h - z^{(S-1)})\}$ in C_{dq}^{S+}, which represent the reflected waves from the (S) and ($S-1$) boundaries, respectively.

Finally, we obtain the asymptotic solutions of the displacement-stress vectors by substituting equations (3), (4), and (8) into equation (2):

$$\begin{Bmatrix} H_{1q}^{S-1}(z) \\ H_{2q}^{S-1}(z) \end{Bmatrix} \rightarrow \begin{Bmatrix} \tilde{H}_{1q}^{S-1}(z) \\ \tilde{H}_{2q}^{S-1}(z) \end{Bmatrix} \equiv \begin{Bmatrix} \bar{H}_{1q}^{S-1} \\ k\bar{H}_{2q}^{S-1} \end{Bmatrix} e^{-k(h-z)},$$

for the $S-1$ layer ($z^{(S-2)} \leqq z \leqq z^{(S-1)}$)

$$\begin{Bmatrix} H_{1q}^{S-}(z) \\ H_{2q}^{S-}(z) \end{Bmatrix} \rightarrow \begin{Bmatrix} \tilde{H}_{1q}^{S-}(z) \\ \tilde{H}_{2q}^{S-}(z) \end{Bmatrix} \equiv \begin{Bmatrix} \bar{H}_{1q}^{S} \\ k\bar{H}_{2q}^{S} \end{Bmatrix} e^{-k(h-z)} + \begin{Bmatrix} \bar{H}_{1q}^{(S-1)} \\ k\bar{H}_{2q}^{(S-1)} \end{Bmatrix} e^{-k(h+z-2z^{(S-1)})} + \begin{Bmatrix} \bar{H}_{1q}^{(S)} \\ k\bar{H}_{2q}^{(S)} \end{Bmatrix} e^{-k(2z^{(s)}-h-z)},$$

for the S^- layer ($z^{(S-1)} \leqq z \leqq h$).

$$\begin{Bmatrix} H_{1q}^{S+}(z) \\ H_{2q}^{S+}(z) \end{Bmatrix} \rightarrow \begin{Bmatrix} \tilde{H}_{1q}^{S+}(z) \\ \tilde{H}_{2q}^{S+}(z) \end{Bmatrix} \equiv \begin{Bmatrix} \bar{H}_{1q}^{S} \\ -k\bar{H}_{2q}^{S} \end{Bmatrix} e^{-k(z-h)} + \begin{Bmatrix} \bar{H}_{1q}^{(S-1)} \\ k\bar{H}_{2q}^{(S-1)} \end{Bmatrix} e^{-k(h+z-2z^{(S-1)})} + \begin{Bmatrix} \bar{H}_{1q}^{(S)} \\ k\bar{H}_{2q}^{(S)} \end{Bmatrix} e^{-k(2z^{(s)}-h-z)},$$

for the S^+ layer ($h \leqq z \leqq z^{(S)}$).

$$\begin{Bmatrix} H_{1q}^{S+1}(z) \\ H_{2q}^{S+1}(z) \end{Bmatrix} \to \begin{Bmatrix} \tilde{H}_{1q}^{S+1}(z) \\ \tilde{H}_{2q}^{S+1}(z) \end{Bmatrix} \equiv \begin{Bmatrix} \bar{H}_{1q}^{S+1} \\ k\bar{H}_{2q}^{S+1} \end{Bmatrix} e^{-k(z-h)}, \qquad (10)$$

for the $S + 1$ layer ($z^{(S)} \leqq z \leqq z^{(S+1)}$), where

$$\bar{H}_{1q}^{S} = \frac{Q_q}{4\pi\mu^S}, \quad \bar{H}_{2q}^{S} = \frac{Q_q}{4\pi},$$

$$\bar{H}_{1q}^{S-1} = T^{(S-1)}\bar{H}_{1q}^{S}, \quad \bar{H}_{2q}^{S-1} = \mu^{S-1}\bar{H}_{1q}^{S-1},$$

$$\bar{H}_{1q}^{(S-1)} = R^{(S-1)}\bar{H}_{1q}^{S}, \quad \bar{H}_{2q}^{(S-1)} = -R^{(S-1)}\bar{H}_{2q}^{S},$$

$$\bar{H}_{1q}^{(S)} = -R^{(S)}\bar{H}_{1q}^{S}, \quad \bar{H}_{2q}^{(S)} = -R^{(S)}\bar{H}_{2q}^{S},$$

$$\bar{H}_{1q}^{S+1} = T^{(S)}\bar{H}_{1q}^{S}, \quad \bar{H}_{2q}^{S+1} = -\mu^{S+1}\bar{H}_{1q}^{S+1},$$

$$(q = x \text{ or } y). \qquad (11)$$

In equation (10), the first terms for the S^- and S^+ layers represent the direct waves from the source, and the second and third terms are the reflected waves from the ($S - 1$) and (S) boundaries, respectively. On the other hand, the asymptotic solutions for the $S - 1$ and $S + 1$ layers consist of the transmitted waves through the ($S - 1$) and (S) boundaries, respectively. Comparing equation (10) with equations (13) and (14) in H94, we confirm that those in H94 have a generalized form of the direct and transmitted waves, but not the reflected waves. It is also clear from equation (10) that the asymptotic solutions of H94 are also valid for $h = z^{(S-1)}$ when we neglect the reflected wave from the $z^{(S)}$ boundary. The same is true for $h = z^{(S)}$.

Case 2 ($0 \leqq h \leqq z^{(1)}$; $S = 1$)

This case includes case 2 of Appendix A in H94. Figure 2 illustrates all the down/upgoing coefficients in the first and second layers. The difference between this case and case 1 is the reflected coefficient from the upper boundary: the free surface. The asymptotic solution for the reflected coefficient is given from equation (A18) of H94 using equation (3):

$$R_u^{(0)} = -(E_{21}^1)^{-1}E_{22}^1\Lambda_u^1(0) \to \exp(-kh). \qquad (12)$$

Using the same procedure as equations (4) to (11), we obtain asymptotic solutions similar to equation (10). The only difference is the coefficients corresponding to the reflected wave from the free surface in equation (11):

$$\bar{H}_{1q}^{(0)} = \frac{Q_q}{4\pi\mu^1}, \quad \bar{H}_{2q}^{(0)} = -\frac{Q_q}{4\pi}, \quad (q = x \text{ or } y). \qquad (13)$$

Case 3 ($z^{(N)} \leqq h$; $S = N + 1$)

This case is same as case 3 of Appendix A in H94. Figure 4 illustrates all the down/upgoing coefficients in the N and $N + 1$ layers. The only difference between this case and case 1 is that there are no reflected waves from the lower boundary. Therefore, we obtain the asymptotic solutions for the two layers by substituting

$$\bar{H}_{1q}^{(N+1)} = \bar{H}_{2q}^{(N+1)} = 0, \quad (q = x \text{ or } y) \qquad (14)$$

in equation (10).

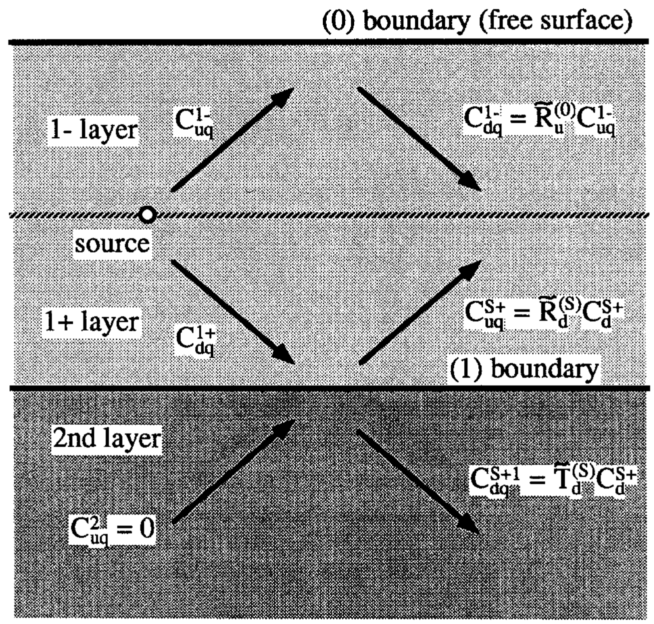

Figure 3. The asymptotic down/upgoing coefficients in the 1^-, 1^+, and 2nd layers for case 2: $Z^{(0)} \leqq h \leqq Z^{(1)}$ and $S = 1$.

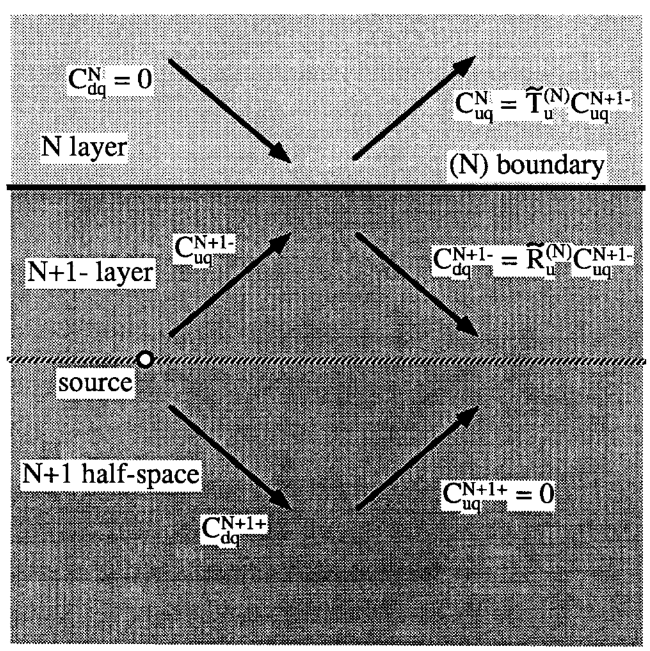

Figure 4. The asymptotic down/upgoing coefficients in the N, $N + 1^-$, and $N + 1^+$ layers for case 3: $Z^{(N)} \leqq h$ and $S = N + 1$.

It should be mentioned here that, in the case of the homogeneous half-space, we can easily derive the displacement-stress vectors by combining cases 2 and 3; we just substitute equations (13) and (14) in equation (10).

P-SV Waves. We can derive asymptotic solutions of the displacement-stress vectors for *P-SV* waves using the same procedures as those for *SH* waves, although they are much more complicated. From equations (A1), (A2), (A3), (A11), and (A12) in H94, the static displacement-stress vectors for *P-SV* waves are expressed as

$$\begin{Bmatrix}\mathbf{D}_q^j(z;h)\\ \mathbf{S}_q^j(z;h)\end{Bmatrix} = \begin{bmatrix}\mathbf{E}_{11}^j & \mathbf{E}_{12}^j\\ k\mu^j\bar{\mathbf{E}}_{21}^j & k\mu^j\bar{\mathbf{E}}_{22}^j\end{bmatrix}\begin{bmatrix}\Lambda_d^j(z) & 0\\ 0 & \Lambda_u^j(z)\end{bmatrix}\begin{Bmatrix}\mathbf{C}_{dq}^j(h)\\ \mathbf{C}_{uq}^j(h)\end{Bmatrix},$$

$$(q = x,\ y,\ \text{or } z), \tag{15}$$

where

$$\mathbf{D}_q^j(z;h) = \begin{Bmatrix}V_{1q}^j(z;h)\\ V_{2q}^j(z;h)\end{Bmatrix},\quad \mathbf{S}_q^j(z;h) = \begin{Bmatrix}V_{3q}^j(z;h)\\ V_{4q}^j(z;h)\end{Bmatrix}, \tag{16}$$

$$\mathbf{C}_{dq}^j(h) = \begin{Bmatrix}c_{d\alpha q}^j(h)\\ c_{d\beta q}^j(h)\end{Bmatrix},\quad \mathbf{C}_{uq}^j(h) = \begin{Bmatrix}c_{u\alpha q}^j(h)\\ c_{u\beta q}^j(h)\end{Bmatrix}, \tag{17}$$

$$\mathbf{E}_{11}^j = \begin{bmatrix}1 & 1\\ -(\kappa^j - 1) & 1\end{bmatrix},\quad \mathbf{E}_{12}^j = \begin{bmatrix}1 & 1\\ \kappa^j - 1 & -1\end{bmatrix},$$

$$\bar{\mathbf{E}}_{21}^j = \begin{bmatrix}\kappa^j - 3 & -2\\ \kappa^j - 1 & -2\end{bmatrix},\quad \bar{\mathbf{E}}_{22}^j = \begin{bmatrix}-(\kappa^j - 3) & 2\\ \kappa^j - 1 & -2\end{bmatrix},$$

$$\kappa^j = \frac{1 + (\bar{\beta}^j/\bar{\alpha}^j)^2}{1 - (\bar{\beta}^j/\bar{\alpha}^j)^2}, \tag{18}$$

$$\begin{bmatrix}\mathbf{T}_d^{(j)} & \mathbf{R}_u^{(j)}\\ \mathbf{R}_d^{(j)} & \mathbf{T}_u^{(j)}\end{bmatrix} \to \begin{bmatrix}\bar{\mathbf{T}}_d^{(j)} & \bar{\mathbf{R}}_u^{(j)}\\ \bar{\mathbf{R}}_d^{(j)} & \bar{\mathbf{T}}_u^{(j)}\end{bmatrix}\begin{bmatrix}\Lambda_d^j(z^{(j)}) & 0\\ 0 & \Lambda_u^{j+1}(z^{(j)})\end{bmatrix}, \tag{20}$$

where the components of $\bar{\mathbf{T}}_d^{(j)}$ *etc.* are given in Appendix B.

Using the same procedures as the *SH* case, we obtain the asymptotic solutions of the displacement-stress vectors in the $S - 1$, S^-, S^+, and $S + 1$ layers:

$$\begin{Bmatrix}\mathbf{D}_q^{S-1}(z;h)\\ \mathbf{S}_q^{S-1}(z;h)\end{Bmatrix} \to \begin{Bmatrix}\tilde{\mathbf{D}}_q^{S-1}(z;h)\\ \tilde{\mathbf{S}}_q^{S-1}(z;h)\end{Bmatrix} \equiv \begin{bmatrix}\mathbf{E}_{12}^{S-1}\\ k\mu^{S-1}\bar{\mathbf{E}}_{22}^{S-1}\end{bmatrix}\mathbf{C}_{uq}^{S-1},$$

for the $S - 1$ layer $(z^{(S-2)} \leqq z \leqq z^{(S-1)})$

$$\begin{Bmatrix}\mathbf{D}_q^{S-}(z;h)\\ \mathbf{S}_q^{S-}(z;h)\end{Bmatrix} \to \begin{Bmatrix}\tilde{\mathbf{D}}_q^{S-}(z;h)\\ \tilde{\mathbf{S}}_q^{S-}(z;h)\end{Bmatrix} \equiv \begin{bmatrix}\mathbf{E}_{11}^S & \mathbf{E}_{12}^S\\ k\mu^S\bar{\mathbf{E}}_{21}^S & k\mu^S\bar{\mathbf{E}}_{22}^S\end{bmatrix}\left(\begin{Bmatrix}0\\ \mathbf{C}_{uq}^{S-}\end{Bmatrix} + \begin{Bmatrix}\mathbf{C}_{dq}^{(S-1)}\\ 0\end{Bmatrix} + \begin{Bmatrix}0\\ \mathbf{C}_{uq}^{(S)}\end{Bmatrix}\right),$$

for the S^- layer $(z^{(S-1)} \leqq z \leqq h)$

$$\begin{Bmatrix}\mathbf{D}_q^{S+}(z;h)\\ \mathbf{S}_q^{S+}(z;h)\end{Bmatrix} \to \begin{Bmatrix}\tilde{\mathbf{D}}_q^{S+}(z;h)\\ \tilde{\mathbf{S}}_q^{S+}(z;h)\end{Bmatrix} \equiv \begin{bmatrix}\mathbf{E}_{11}^S & \mathbf{E}_{12}^S\\ k\mu^S\bar{\mathbf{E}}_{21}^S & k\mu^S\bar{\mathbf{E}}_{22}^S\end{bmatrix}\left(\begin{Bmatrix}\mathbf{C}_{dq}^{S+}\\ 0\end{Bmatrix} + \begin{Bmatrix}\mathbf{C}_{dq}^{(S-1)}\\ 0\end{Bmatrix} + \begin{Bmatrix}0\\ \mathbf{C}_{uq}^{(S)}\end{Bmatrix}\right),$$

for the S^+ layer $(h \leqq z \leqq z^{(S)})$

$$\begin{Bmatrix}\mathbf{D}_q^{S+1}(z;h)\\ \mathbf{S}_q^{S+1}(z;h)\end{Bmatrix} \to \begin{Bmatrix}\tilde{\mathbf{D}}_q^{S+1}(z;h)\\ \tilde{\mathbf{S}}_q^{S+1}(z;h)\end{Bmatrix} \equiv \begin{bmatrix}\mathbf{E}_{11}^{S+1}\\ k\mu^{S+1}\bar{\mathbf{E}}_{21}^{S+1}\end{bmatrix}\mathbf{C}_{dq}^{S+1}, \tag{21}$$

for the $S + 1$ layer $(z^{(S)} \leqq z \leqq z^{(S+1)})$, where

$$\mathbf{C}_{uq}^{S-1} = \begin{Bmatrix}\bar{C}_{u\alpha q1}^{S-1} - k(h - z^{(S-1)})\bar{C}_{u\alpha q2}^{S-1}\\ \bar{C}_{u\beta q1}^{S-1} - k(h - z^{(S-1)},\bar{C}_{u\beta q2}^{S-1} - k(z^{(S-1)} - z)\bar{C}_{u\alpha q1}^{S-1} + k^2(h - z^{(S-1)})(z^{(S-1)} - z)\bar{C}_{u\alpha q2}^{S-1}\end{Bmatrix}e^{-k(h-z)},$$

$$\Lambda_d^j(z) = \begin{bmatrix}1 & 0\\ -k(z - z^{(j-1)}) & 1\end{bmatrix}\exp\{-k(z - z^{(j-1)})\},$$

$$\Lambda_u^j(z) = \begin{bmatrix}1 & 0\\ -k(z^{(j)} - z) & 1\end{bmatrix}\exp\{-k(z^{(j)} - z)\}, \tag{19}$$

and the complex P and S velocities ($\bar{\alpha}^j$ and $\bar{\beta}^j$) are given in equation (A6) of H94.

Case 1 ($z^{(S-1)} \leqq h$) $z^{(S)}$, and $S \neq 1$ nor $N + 1$)

Equations (4) to (7) for *SH* waves are also true for this case. In order to obtain the modified R/T coefficients for *P-SV* waves, which are defined in equation (A18) of H94, we need to invert 4 by 4 matrices. Using the analytical inverse of a 4 by 4 matrix given in Appendix A, we obtain the asymptotic solutions of the modified R/T coefficients:

$$\mathbf{C}_{uq}^{S-} = \begin{Bmatrix}\bar{C}_{u\alpha q}^{S-}\\ \bar{C}_{u\beta q}^{S-} - k(h - z)\bar{C}_{u\alpha q}^{S-}\end{Bmatrix}e^{-k(h-z)},$$

$$\mathbf{C}_{dq}^{S+} = \begin{Bmatrix}\bar{C}_{d\alpha q}^{S+}\\ \bar{C}_{d\beta q}^{S+} - k(z - h)\bar{C}_{d\alpha q}^{S+}\end{Bmatrix}e^{-k(z-h)},$$

$$\mathbf{C}_{dq}^{(S-1)} = \begin{Bmatrix}\bar{C}_{d\alpha q1}^{S-} - k(h - z^{(S-1)})\bar{C}_{d\alpha q2}^{S-}\\ \bar{C}_{d\beta q1}^{S-} - k(h - z^{(S-1)})\bar{C}_{d\beta q2}^{S-} - k(z - z^{(S-1)})\bar{C}_{d\alpha q1}^{S-} + k^2(h - z^{(S-1)})(z - z^{(S-1)})\bar{C}_{d\alpha q2}^{S-}\end{Bmatrix}e^{-k(z+h-2z^{(S-1)})},$$

$$\mathbf{C}_{uq}^{(S)} = \begin{Bmatrix}\bar{C}_{u\alpha q1}^{S+} - k(z^{(S)} - h)\bar{C}_{u\alpha q2}^{S+}\\ \bar{C}_{u\beta q1}^{S+} - k(z^{(S)} - h)\bar{C}_{u\beta q2}^{S+} - k(z^{(S)} - z)\bar{C}_{u\alpha q1}^{S+} + k^2(z^{(S)} - h)(z^{(S)} - z)\bar{C}_{u\alpha q2}^{S+}\end{Bmatrix}e^{-k(2z^{(s)}-z-h)},$$

$$\mathbf{C}_{dq}^{S+1} = \begin{Bmatrix} \bar{C}_{daq1}^{S+1} - k(z^{(S)} - h)\bar{C}_{daq2}^{S+1} \\ \bar{C}_{d\beta q1}^{S+1} - k(z^{(S)} - h)\bar{C}_{d\beta q2}^{S+1} - k(z - z^{(S)})\bar{C}_{daq1}^{S+1} \\ + k^2(z^{(S)} - h)(z - z^{(S)})\bar{C}_{daq2}^{S+1} \end{Bmatrix} e^{-k(z-h)}, \tag{22}$$

and

$$\begin{Bmatrix} \bar{C}_{uaq1}^{S-1} \\ \bar{C}_{u\beta q1}^{S-1} \end{Bmatrix} = \bar{\mathbf{T}}_u^{(S-1)} \begin{Bmatrix} \bar{C}_{uaq}^{S-} \\ \bar{C}_{u\beta q}^{S-} \end{Bmatrix}, \quad \begin{Bmatrix} \bar{C}_{uaq2}^{S-1} \\ \bar{C}_{u\beta q2}^{S-1} \end{Bmatrix} = \bar{\mathbf{T}}_u^{(S-1)} \begin{Bmatrix} 0 \\ \bar{C}_{uaq}^{S-} \end{Bmatrix},$$

$$\begin{Bmatrix} \bar{C}_{daq1}^{S-} \\ \bar{C}_{d\beta q1}^{S-} \end{Bmatrix} = \bar{\mathbf{R}}_u^{(S-1)} \begin{Bmatrix} \bar{C}_{uaq}^{S-} \\ \bar{C}_{u\beta q}^{S-} \end{Bmatrix}, \quad \begin{Bmatrix} \bar{C}_{daq2}^{S-} \\ \bar{C}_{d\beta q2}^{S-} \end{Bmatrix} = \bar{\mathbf{R}}_u^{(S-1)} \begin{Bmatrix} 0 \\ \bar{C}_{uaq}^{S-} \end{Bmatrix},$$

$$\begin{Bmatrix} \bar{C}_{uaq1}^{S+} \\ \bar{C}_{u\beta q1}^{S+} \end{Bmatrix} = \bar{\mathbf{R}}_d^{(S)} \begin{Bmatrix} \bar{C}_{daq}^{S+} \\ \bar{C}_{d\beta q}^{S+} \end{Bmatrix}, \quad \begin{Bmatrix} \bar{C}_{uaq2}^{S+} \\ \bar{C}_{u\beta q2}^{S+} \end{Bmatrix} = \bar{\mathbf{R}}_d^{(S)} \begin{Bmatrix} 0 \\ \bar{C}_{daq}^{S+} \end{Bmatrix},$$

$$\begin{Bmatrix} \bar{C}_{daq1}^{S+1} \\ \bar{C}_{d\beta q1}^{S+1} \end{Bmatrix} = \bar{\mathbf{T}}_d^{(S)} \begin{Bmatrix} \bar{C}_{daq}^{S+} \\ \bar{C}_{d\beta q}^{S+} \end{Bmatrix}, \quad \begin{Bmatrix} \bar{C}_{daq2}^{S+1} \\ \bar{C}_{d\beta q2}^{S+1} \end{Bmatrix} = \bar{\mathbf{T}}_d^{(S)} \begin{Bmatrix} 0 \\ \bar{C}_{daq}^{S+} \end{Bmatrix}, \tag{23}$$

$$\begin{Bmatrix} \bar{C}_{uaq}^{S-} \\ \bar{C}_{u\beta q}^{S-} \end{Bmatrix} = \begin{Bmatrix} \bar{C}_{daq}^{S+} \\ \bar{C}_{d\beta q}^{S+} \end{Bmatrix} = \frac{Q_q}{4\pi\mu^S} \frac{1}{\kappa^S + 1} \begin{Bmatrix} 1 \\ \kappa^S - 1 \end{Bmatrix},$$

for $q = x$ or y,

$$\begin{Bmatrix} \bar{C}_{uaz}^{S-} \\ \bar{C}_{u\beta z}^{S-} \end{Bmatrix} = -\begin{Bmatrix} \bar{C}_{daz}^{S+} \\ \bar{C}_{d\beta z}^{S+} \end{Bmatrix} = \frac{Q_z}{4\pi\mu^S} \frac{1}{\kappa^S + 1} \begin{Bmatrix} -1 \\ 1 \end{Bmatrix},$$

for $q = z$. (24)

In equation (21), the first terms in ( ) for the S^- and S^+ layers represent the direct waves from the source, and the second and third terms are the reflected waves from the $(S - 1)$ and (S) boundaries, respectively. On the other hand, the asymptotic solutions for the $S - 1$ and $S + 1$ layers consist of the transmitted waves through the $(S - 1)$ and (S) layers, respectively. As in the *SH* case, we find from equation (21) that equations (13) and (14) of H94 have generalized forms of the direct and transmitted waves, but not of the reflected waves.

Case 2 ($0 \leqq h \leqq z^{(1)}$; $S = 1$)

As in the *SH* case, the only difference between this case and case 1 is the reflected coefficients from the free surface. Their asymptotic solutions are given by substituting equations (16) to (19) into equation (A18) of H94:

$$\mathbf{R}_u^{(0)} = \bar{\mathbf{R}}_u^{(0)} \Lambda_u^1(0), \tag{25}$$

where

$$\bar{\mathbf{R}}_u^{(0)} = \begin{bmatrix} -(\kappa^1 - 2) & 2 \\ -0.5(\kappa^1 - 1)(\kappa^1 - 3) & \kappa^1 - 2 \end{bmatrix}. \tag{26}$$

Replacing $\bar{\mathbf{R}}_u^{(S-1)}$ in equation (23) by equation (26), we obtained an equation similar to equation (21) for this case.

Case 3 ($z^{(N)} \leqq h$: $S = N + 1$)

And again, as with the *SH* case, the difference between this case and case 1 is that there are no reflected waves from the lower boundary. Therefore, we can use equation (21) by substituting

$$\mathbf{C}_{uq}^{(S)} = \mathbf{C}_{uq}^{(N+1)} = \{0\}. \tag{27}$$

Analytical Integrations for Asymptotic Solutions

SH Waves. We derive the analytical wavenumber integrations of the asymptotic solutions obtained above, by substituting them into equations (11) and (12) in H94. We need the following integrations for *SH* waves:

$$\begin{Bmatrix} \Delta_1^{-1}\tilde{H}_{1q}^{S-1} \\ \Delta_1^{-1}\tilde{H}_{2q}^{S-1} \end{Bmatrix} \equiv \int_0^\infty \begin{Bmatrix} \tilde{H}_{1q}^{S-1}(z) \\ \tilde{H}_{2q}^{S-1}(z) \end{Bmatrix} \frac{J_1(kr)}{k} dk = \begin{Bmatrix} \bar{H}_{1q}^{S-1} I_1^{-1} \\ \bar{H}_{2q}^{S-1} I_1^0 \end{Bmatrix},$$

for the $S - 1$ layer ($z^{(S-2)} \leqq z \leqq z^{(S-1)}$)

$$\begin{Bmatrix} \Delta_1^{-1}\tilde{H}_{1q}^{S-} \\ \Delta_1^{-1}\tilde{H}_{2q}^{S-} \end{Bmatrix} \equiv \int_0^\infty \begin{Bmatrix} \tilde{H}_{1q}^{S-}(z) \\ \tilde{H}_{2q}^{S-}(z) \end{Bmatrix} \frac{J_1(kr)}{k} dk = \begin{Bmatrix} \bar{H}_{1q}^{S} I_1^{-1} \\ \bar{H}_{2q}^{S} I_1^0 \end{Bmatrix} + \begin{Bmatrix} \bar{H}_{1q}^{(S-1)} I_1^{-1(S-1)} \\ \bar{H}_{2q}^{(S-1)} I_1^{0(S-1)} \end{Bmatrix} + \begin{Bmatrix} \bar{H}_{1q}^{(S)} I_1^{-1(S)} \\ \bar{H}_{2q}^{(S)} I_1^{0(S)} \end{Bmatrix},$$

for the S^- layer ($z^{(S-1)} \leqq z \leqq h$)

$$\begin{Bmatrix} \Delta_1^{-1}\tilde{H}_{1q}^{S+} \\ \Delta_1^{-1}\tilde{H}_{2q}^{S+} \end{Bmatrix} \equiv \int_0^\infty \begin{Bmatrix} \tilde{H}_{1q}^{S+}(z) \\ \tilde{H}_{2q}^{S+}(z) \end{Bmatrix} \frac{J_1(kr)}{k} dk = \begin{Bmatrix} \bar{H}_{1q}^{S} I_1^{-1} \\ -\bar{H}_{2q}^{S} I_1^0 \end{Bmatrix} + \begin{Bmatrix} \bar{H}_{1q}^{(S-1)} I_1^{-1(S-1)} \\ \bar{H}_{2q}^{(S-1)} I_1^{0(S-1)} \end{Bmatrix} + \begin{Bmatrix} \bar{H}_{1q}^{(S)} I_1^{-1(S)} \\ \bar{H}_{2q}^{(S)} I_1^{0(S)} \end{Bmatrix},$$

for the S^+ layer ($h \leqq z \leqq z^{(S)}$)

$$\begin{Bmatrix} \Delta_1^{-1}\tilde{H}_{1q}^{S+1} \\ \Delta_1^{-1}\tilde{H}_{2q}^{S+1} \end{Bmatrix} \equiv \int_0^\infty \begin{Bmatrix} \tilde{H}_{1q}^{S+1}(z) \\ \tilde{H}_{2q}^{S+1}(z) \end{Bmatrix} \frac{J_1(kr)}{k} dk = \begin{Bmatrix} \bar{H}_{1q}^{S+1} I_1^{-1} \\ \bar{H}_{2q}^{S+1} I_1^0 \end{Bmatrix}, \tag{28}$$

for the $S + 1$ layer ($z^{(s)} \leqq z \leqq z^{(s+1)}$),

where I_1^{-1}, $I_1^{-1(S-1)}$ *etc.* are given in Appendix C. Coefficients $\bar{H}_{iq}^j$ are given in equation (11) for case 1, plus equations (13) and (14) for cases 2 and 3, respectively.

We also need

$$\begin{Bmatrix} \Delta_{d1}^0 \tilde{H}_{1q}^j \\ \Delta_{d1}^0 \tilde{H}_{2q}^j \end{Bmatrix} \equiv \int_0^\infty \begin{Bmatrix} \tilde{H}_{1q}^j(z) \\ \tilde{H}_{2q}^j(z) \end{Bmatrix} \frac{dJ_1(kr)}{dkr} dk,$$

$$(j = S - 1, S^-, S^+, S + 1). \tag{29}$$

The analytical solutions of the above integrations are obtained by replacing I_1^{-1}, $I_1^{-1(S-1)}$ *etc.* in equation (28) as follows:

$$I_1^{-1} \to I_0^0 - \frac{I_1^{-1}}{r}, \quad I_1^{-1(S-1)} \to I_0^{0(S-1)} - \frac{I_1^{-1(S-1)}}{r},$$

$$I_1^{-1(S)} \to I_0^{0(S)} - \frac{I_1^{-1(S)}}{r}, I_1^0 \to I_0^1 - \frac{I_1^0}{r},$$

$$I_1^{0(S-1)} \to I_0^{1(S-1)} - \frac{I_1^{0(S-1)}}{r}, \quad I_1^{0(S)} \to I_0^{1(S)} - \frac{I_1^{0(S)}}{r}, \qquad (30)$$

where I_0^0 *etc.* are also given in Appendix C.

P-SV Waves. From equations (11) and (12) of H94, we need the following integrations:

$$\begin{Bmatrix} \Delta_1^{-1}\tilde{\mathbf{D}}_q^{S-1} \\ \Delta_1^{-1}\tilde{\mathbf{S}}_q^{S-1} \end{Bmatrix} \equiv \int_0^\infty \begin{Bmatrix} \tilde{\mathbf{D}}_q^{S-1}(z;h) \\ \tilde{\mathbf{S}}_q^{S-1}(z;h) \end{Bmatrix} \frac{J_1(kr)}{k}\, dk$$

$$= \begin{Bmatrix} \mathbf{E}_{12}^{S-1}\,\Delta_1\,\mathbf{C}_{uq}^{S-1} \\ \mu^{S-1}\,\bar{\mathbf{E}}_{22}^{S-1}\,\Delta_2\mathbf{C}_{uq}^{S-1} \end{Bmatrix},$$

for the $S - 1$ layer ($z^{(S-2)} \leqq z \leqq z^{(S-1)}$)

$$\Delta_1^{-1}\,\tilde{\mathbf{D}}_q^{S-} \equiv \int_0^\infty \tilde{\mathbf{D}}_q^{S-}(z;h)\frac{J_1(kr)}{k}\, dk = [\mathbf{E}_{11}^S \quad \mathbf{E}_{12}^S]$$

$$\left(\begin{Bmatrix} 0 \\ \Delta_1\mathbf{C}_{uq}^{S-} \end{Bmatrix} + \begin{Bmatrix} \Delta_1\mathbf{C}_{dq}^{(S-1)} \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ \Delta_1\mathbf{C}_{uq}^{(S)} \end{Bmatrix}\right),$$

$$\Delta_1^{-1}\,\tilde{\mathbf{S}}_q^{S-} \equiv \int_0^\infty \tilde{\mathbf{S}}_q^{S-}(z;h)\frac{J_1(kr)}{k}\, dk$$

$$= \mu^s[\bar{\mathbf{E}}_{21}^S \quad \bar{\mathbf{E}}_{22}^S]\left(\begin{Bmatrix} 0 \\ \Delta_2\mathbf{C}_{uq}^{S-} \end{Bmatrix} + \begin{Bmatrix} \Delta_2\mathbf{C}_{dq}^{(S-1)} \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ \Delta_2\mathbf{C}_{uq}^{(S)} \end{Bmatrix}\right),$$

for the S^- layer ($z^{(S-1)} \leqq z \leqq h$)

$$\Delta_1^{-1}\,\tilde{\mathbf{D}}_q^{S+} \equiv \int_0^\infty \tilde{\mathbf{D}}_q^{S+}(z;h)\frac{J_1(kr)}{k}\, dk$$

$$= [\mathbf{E}_{11}^S \quad \mathbf{E}_{12}^S]\left(\begin{Bmatrix} \Delta_1\mathbf{C}_{dq}^{S+} \\ 0 \end{Bmatrix} + \begin{Bmatrix} \Delta_1\mathbf{C}_{dq}^{(S-1)} \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ \Delta_1\mathbf{C}_{uq}^{(S)} \end{Bmatrix}\right),$$

$$\Delta_1^{-1}\,\tilde{\mathbf{S}}_q^{S+} \equiv \int_0^\infty \tilde{\mathbf{S}}_q^{S+}(z;h)\frac{J_1(kr)}{k}\, dk$$

$$= \mu^S[\bar{\mathbf{E}}_{21}^S \quad \bar{\mathbf{E}}_{22}^S]\left(\begin{Bmatrix} \Delta_2\mathbf{C}_{dq}^{S+} \\ 0 \end{Bmatrix} + \begin{Bmatrix} \Delta_2\mathbf{C}_{dq}^{(S-1)} \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ \Delta_2\mathbf{C}_{uq}^{(S)} \end{Bmatrix}\right),$$

for the S^+ layer ($h \leqq z \leqq z^{(S)}$)

$$\begin{Bmatrix} \Delta_1^{-1}\tilde{\mathbf{D}}_q^{S+1} \\ \Delta_1^{-1}\tilde{\mathbf{S}}_q^{S+1} \end{Bmatrix} \equiv \int_0^\infty \begin{Bmatrix} \tilde{\mathbf{D}}_q^{S+1}(z;h) \\ \tilde{\mathbf{S}}_q^{S+1}(z;h) \end{Bmatrix} \frac{J_1(kr)}{k}\, dk$$

$$= \begin{Bmatrix} \mathbf{E}_{12}^{S+1}\Delta_1\mathbf{C}_{dq}^{S+1} \\ \mu^{S+1}\bar{\mathbf{E}}_{22}^{S+1}\Delta_2\mathbf{C}_{dq}^{S+1} \end{Bmatrix}, \qquad (31)$$

for the $S + 1$ layer ($z^{(s)} \leqq z \leqq z^{(s+1)}$) where

$$\Delta_1^{-1}\,\tilde{\mathbf{D}}_q^j(z;h) \equiv \begin{Bmatrix} \Delta_1^{-1}\tilde{V}_{1q}^j(z;h) \\ \Delta_1^{-1}\tilde{V}_{2q}^j(z;h) \end{Bmatrix},$$

$$\Delta_1^{-1}\,\tilde{\mathbf{S}}_q^j(z;h) \equiv \begin{Bmatrix} \Delta_1^{-1}\tilde{V}_{3q}^j(z;h) \\ \Delta_1^{-1}\tilde{V}_{4q}^j(z;h) \end{Bmatrix}, \qquad (32)$$

and

$$\Delta_1\mathbf{C}_{uq}^{S-1} = \begin{Bmatrix} \bar{C}_{u\alpha q1}^{S-1}\,I_1^{-1} - (h - z^{(S-1)})\,\bar{C}_{u\alpha q2}^{S-1}I_1^0 \\ \bar{C}_{u\beta q1}^{S-1}\,I_1^{-1} - \{(h - z^{(S-1)})\,\bar{C}_{u\beta q2}^{S-1} + (z^{(S-1)} - z)\,\bar{C}_{u\alpha q1}^{S-1}\}I_1^0 + (h - z^{(S-1)})(z^{(S-1)} - z)\,\bar{C}_{u\alpha q2}^{S-1}I_1^1 \end{Bmatrix},$$

$$\Delta_1\mathbf{C}_{uq}^{S-} \begin{Bmatrix} \bar{C}_{u\alpha q}^{S-}I_1^{-1} \\ \bar{C}_{u\beta q}^{S-}\,I_1^{-1} - (h - z)\,\bar{C}_{u\alpha q}^{S-}I_1^0 \end{Bmatrix},$$

$$\Delta_1\mathbf{C}_{dq}^{S+} = \begin{Bmatrix} \bar{C}_{d\alpha q}^{S+}I_1^{-1} \\ \bar{C}_{d\beta q}^{S+}I_1^{-1} - (z - h)\,\bar{C}_{d\alpha q}^{S+}I_1^0 \end{Bmatrix},$$

$$\Delta_1\mathbf{C}_{dq}^{(S-1)} = \begin{Bmatrix} \bar{C}_{d\alpha q1}^{S-}I_1^{-1(S-1)} - (h - z^{(S-1)})\,\bar{C}_{d\alpha q2}^{S-}I_1^{0(S-1)} \\ \bar{C}_{d\beta q1}^{S-}I_1^{-1(S-1)} - \{(h - z^{(S-1)})\,\bar{C}_{d\beta q2}^{S-} + (z - z^{(S-1)})\,\bar{C}_{d\alpha q1}^{S-}\}I_1^{0(S-1)} + (h - z^{(S-1)})(z - z^{(S-1)})\,\bar{C}_{d\alpha q2}^{S-}I_1^{1(S-1)} \end{Bmatrix},$$

$$\Delta_1\mathbf{C}_{uq}^{(S)} = \begin{Bmatrix} \bar{C}_{u\alpha q1}^{S+}I_1^{-1(S)} - (z^{(S)} - h)\,\bar{C}_{u\alpha q2}^{S+}I_1^{0(S)} \\ \bar{C}_{u\beta q1}^{S+}I_1^{-1(S)} - \{(z^{(S)} - h)\,\bar{C}_{u\beta q2}^{S+} + (z^{(S)} - z)\,\bar{C}_{u\alpha q1}^{S+}\}I_1^{0(S)} + (z^{(S)} - h)(z^{(S)} - z)\,\bar{C}_{u\alpha q2}^{S+}I_1^{1(S)} \end{Bmatrix},$$

$$\Delta_1\mathbf{C}_{dq}^{S+1} = \begin{Bmatrix} \bar{C}_{d\alpha q1}^{S+1}I_1^{-1} - (z^{(S)} - h)\,\bar{C}_{d\alpha q2}^{S+1}I_1^0 \\ \bar{C}_{d\beta q1}^{S+1}\,I_1^{-1} - \{(z^{(S)} - h)\,\bar{C}_{d\beta q2}^{S+1} + (z - z^{(S)})\,\bar{C}_{d\alpha q1}^{S+1}\}I_1^0 + (z^{(S)} - h)(z - z^{(S)})\,\bar{C}_{d\alpha q2}^{S+1}I_1^1 \end{Bmatrix}, \qquad (33)$$

where coefficients $\bar{C}^j_{uaq1}$ *etc.* are given in equations (23) to (26).

We obtain $\Delta_2 \mathbf{C}^j_{iq}$ for $\Delta_1^{-1}\tilde{\mathbf{S}}^j_q$ in equation (31) by performing the following replacements in equation (33):

$$I_1^{-1} \to I_1^0, \quad I_1^0 \to I_1^1, \quad I_1^1 \to I_1^2,$$

$$I_1^{-1(S-1)} \to I_1^{0(S-1)}, \quad I_1^{0(S-1)} \to I_1^{1(S-1)}, \quad I_1^{1(S-1)} \to I_1^{2(S-1)},$$

$$I_1^{-1(S)} \to I_1^{0(S)}, \quad I_1^{0(S)} \to I_1^{1(S)}, \quad I_1^{1(S)} \to I_1^{2(S)}, \qquad (34)$$

where I_1^0 etc. are given in Appendix C.

We also need to perform the following integrations:

$$\begin{Bmatrix} \Delta^0_{d1}\tilde{\mathbf{D}}^j_q \\ \Delta^0_{d1}\tilde{\mathbf{S}}^j_q \end{Bmatrix} \equiv \int_0^\infty \begin{Bmatrix} \tilde{\mathbf{D}}^j_q(z;h) \\ \tilde{\mathbf{S}}^j_q(z;h) \end{Bmatrix} \frac{dJ_1(kr)}{dkr}\, dk, \qquad (35)$$

$$\begin{Bmatrix} \Delta^0_{1}\tilde{\mathbf{D}}^j_q \\ \Delta^0_{1}\tilde{\mathbf{S}}^j_q \end{Bmatrix} \equiv \int_0^\infty \begin{Bmatrix} \tilde{\mathbf{D}}^j_q(z;h) \\ \tilde{\mathbf{S}}^j_q(z;h) \end{Bmatrix} J_1(kr)dk, \qquad (36)$$

$$\begin{Bmatrix} \Delta^0_{0}\tilde{\mathbf{D}}^j_q \\ \Delta^0_{0}\tilde{\mathbf{S}}^j_q \end{Bmatrix} \equiv \int_0^\infty \begin{Bmatrix} \tilde{\mathbf{D}}^j_q(z;h) \\ \tilde{\mathbf{S}}^j_q(z;h) \end{Bmatrix} J_0(kr)dk. \qquad (37)$$

We obtain the analytical integrations of $\Delta^0_{d1}\tilde{\mathbf{D}}^j_q$ in equation (35) using equation (30) plus the following replacements in equation (33):

$$I_1^1 \to I_0^2 - \frac{I_1^1}{r}, \quad I_1^{1(S-1)} \to I_0^{2(S-1)} - \frac{I_1^{1(S-1)}}{r},$$

$$I_1^{1(S)} \to I_0^{2(S)} - \frac{I_1^{1(S)}}{r}. \qquad (38)$$

Similarly, those for $\Delta^0_{d1}\tilde{\mathbf{S}}^j_q$ in equation (35) are obtained by the following replacements in equation (33):

$$I_1^{-1} \to I_0^1 - \frac{I_1^0}{r}, \quad I_1^0 \to I_0^2 - \frac{I_1^1}{r}, \quad I_1^1 \to I_0^3 - \frac{I_1^2}{r},$$

$$I_1^{-1(S-1)} \to I_0^{1(S-1)} - \frac{I_1^{0(S-1)}}{r},$$

$$I_1^{0(S-1)} \to I_0^{2(S-1)} - \frac{I_1^{1(S-1)}}{r}, \quad I_1^{1(S-1)} \to I_0^{3(S-1)} - \frac{I_1^{2(S-1)}}{r},$$

$$I_1^{-1(S)} \to I_0^{1(S)} - \frac{I_1^{0(S)}}{r},$$

$$I_1^{0(S)} \to I_0^{2(S)} - \frac{I_1^{1(S)}}{r}, \quad I_1^{1(S)} \to I_0^{3(S)} - \frac{I_1^{2(S)}}{r}. \qquad (39)$$

Also, we obtain the analytical integrations of $\Delta_1^0\tilde{\mathbf{D}}^j_q$ in equation (36) using equation (34). The rest of integrations in equations (36) and (37) are similarly obtained by the following replacements in equation (33):

$$I_1^{-1} \to I_1^1, \quad I_1^0 \to I_1^2, \quad I_1^1 \to I_1^3,$$

$$I_1^{-1(S-1)} \to I_1^{1(S-1)}, \quad I_1^{0(S-1)} \to I_1^{2(S-1)}, \quad I_1^{1(S-1)} \to I_1^{3(S-1)},$$

$$I_1^{-1(S)} \to I_1^{1(S)}, \quad I_1^{0(S)} \to I_1^{2(S)}, \quad I_1^{1(S)} \to I_1^{3(S)}, \qquad (40)$$

for $\Delta_1^0\tilde{\mathbf{S}}^j_q$ in equation (36),

$$I_1^{-1} \to I_0^0, \quad I_1^0 \to I_0^1, \quad I_1^1 \to I_0^2,$$

$$I_1^{-1(S-1)} \to I_0^{0(S-1)}, \quad I_1^{0(S-1)} \to I_0^{1(S-1)}, \quad I_1^{1(S-1)} \to I_0^{2(S-1)},$$

$$I_1^{-1(S)} \to I_0^{0(S)}, \quad I_1^{0(S)} \to I_0^{1(S)}, \quad I_1^{1(S)} \to I_0^{2(S)}, \qquad (41)$$

for $\Delta_0^0\tilde{\mathbf{D}}^j_q$ in equation (37), and

$$I_1^{-1} \to I_0^1, \quad I_1^0 \to I_0^2, \quad I_1^1 \to I_0^3,$$

$$I_1^{-1(S-1)} \to I_0^{1(S-1)}, \quad I_1^{0(S-1)} \to I_0^{2(S-1)}, \quad I_1^{1(S-1)} \to I_0^{3(S-1)},$$

$$I_1^{-1(S)} \to I_0^{1(S)}, \quad I_1^{0(S)} \to I_0^{2(S)}, \quad I_1^{1(S)} \to I_0^{3(S)}, \qquad (42)$$

for $\Delta_0^0\tilde{\mathbf{S}}^j_q$ in equation (37).

Procedure for Green's Function Due to Point Sources Using the Asymptotic Technique

The procedure for computing Green's functions due to point sources can be summarized in equations (20) and (21) of H94. In this study, the asymptotic solutions of the displacement-stress vectors are given in equations (10) to (14) for $\tilde{H}^j_{iq}$ and from equations (21) to (27) for $\tilde{V}^j_{iq}$. The analytical integrations of those asymptotic solutions, which correspond to equations (15) and (16) in H94, are expressed as follows:

$$\tilde{U}^j_{r\binom{x}{y}} = \left\{\Delta^0_{d1}\,\tilde{V}^j_{1\binom{x}{y}} + \frac{1}{r}\,\Delta_1^{-1}\,\tilde{H}^j_{1\binom{x}{y}}\right\}\begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix}$$

$$\equiv \tilde{U}^j_{r\binom{x}{y}}\begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix},$$

$$\tilde{U}^j_{rz} = -\Delta_1^0\tilde{V}^j_{1z},$$

$$\tilde{U}^j_{\theta\binom{x}{y}} = \begin{pmatrix} - \\ + \end{pmatrix}\left\{\frac{1}{r}\,\Delta_1^{-1}\,\tilde{V}^j_{1\binom{x}{y}} + \Delta^0_{d1}\,\tilde{H}^j_{1\binom{x}{y}}\right\}\begin{pmatrix}\sin\theta \\ \cos\theta\end{pmatrix}$$

$$\equiv \begin{pmatrix} - \\ + \end{pmatrix}\tilde{u}^j_{\theta\binom{x}{y}}\begin{pmatrix}\sin\theta \\ \cos\theta\end{pmatrix},$$

$$\tilde{U}^j_{z\binom{x}{y}} = -\Delta_1^0\tilde{V}^j_{2\binom{x}{y}}\begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix},$$

$$\tilde{U}^j_{zz} = -\Delta_0^0\tilde{V}^j_{2z}, \qquad (43)$$

for displacement, and

$$\tilde{\sigma}^j_{rz\binom{x}{y}} = \left\{\Delta^0_{d1}\,\tilde{V}^j_{3\binom{x}{y}} + \frac{1}{r}\,\Delta^{-1}_1\,\tilde{H}^j_{2\binom{x}{y}}\right\}\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix},$$

$$\tilde{\sigma}^j_{rzz} = -\Delta^0_1\tilde{V}^j_{3z},$$

$$\tilde{\sigma}^j_{\theta z\binom{x}{y}} = \begin{pmatrix}-\\+\end{pmatrix}\left\{\frac{1}{r}\,\Delta^{-1}_1\,\tilde{V}^j_{3\binom{x}{y}} + \Delta^0_{d1}\,\tilde{H}^j_{2\binom{x}{y}}\right\}\begin{pmatrix}\sin\theta\\ \cos\theta\end{pmatrix},$$

$$\tilde{\sigma}^j_{zz\binom{x}{y}} = -\Delta^0_1\,\tilde{V}^j_{4\binom{x}{y}}\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix},$$

$$\tilde{\sigma}^j_{zzz} = -\Delta^0_0\,\tilde{V}^j_{4z} \tag{44}$$

for stress, where the upper and lower values within the parentheses are allotted to the solutions due to Q_x and Q_y, respectively. $\Delta^0_{d1}\tilde{H}^j_{iq}$ and $\Delta^{-1}_1\tilde{H}^j_{iq}$ are given in equations (28), (29), and (30), respectively, and $\Delta^{-1}_1\tilde{V}^j_{iq}$ etc. are obtained from equations (31) to (42).

It is easy to confirm that the static Green's function for the homogeneous full-space is equal to the direct wave part in equations (43) and (44).

Green's Function Due to Dipole Sources Using the Asymptotic Technique

We obtain Green's functions due to dipole sources using the same procedure as equations (22) to (34) in H94. The only difference is the use of our new asymptotic solutions. We can derive the equations corresponding to (32) in H94 by differentiating equation (43) with respect to j $(=x$ or $y)$ and using equation (28) of H94, as follows:

$$\tilde{U}^j_{r\binom{x}{y},j} = r_{,j}\left\{\Delta^0_{d1}\,\tilde{V}^j_{1\binom{x}{y},r} - \frac{1}{r^2}\,\Delta^{-1}_1\,\tilde{H}^j_{1\binom{x}{y}} + \frac{1}{r}\,\Delta^{-1}_1\,\tilde{H}^j_{1\binom{x}{y},r}\right\}\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} + \tilde{u}^j_{r\binom{x}{y}}\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}_{,j},$$

$$\tilde{U}^j_{rz,j} = -r_{,j}\,\Delta^0_1\,\tilde{V}^j_{1z,r},$$

$$\tilde{U}^j_{\theta\binom{x}{y},j} = \begin{pmatrix}-\\+\end{pmatrix} r_{,j}\left\{-\frac{1}{r^2}\,\Delta^{-1}_1\,\tilde{V}^j_{1\binom{x}{y}} + \frac{1}{r}\,\Delta^{-1}_1\,\tilde{V}^j_{1\binom{x}{y},r} + \Delta^0_{d1}\,\tilde{H}^j_{1\binom{x}{y},r}\right\}\begin{pmatrix}\sin\theta\\ \cos\theta\end{pmatrix}\begin{pmatrix}-\\+\end{pmatrix}\tilde{u}^j_{\theta\binom{x}{y}}\begin{pmatrix}\sin\theta\\ \cos\theta\end{pmatrix}_{,j},$$

$$\tilde{U}^j_{z\binom{x}{y},j} = -r_{,j}\,\Delta^0_1\,\tilde{V}^j_{2\binom{x}{y},r}\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} - \Delta^0_1\,\tilde{V}^j_{2\binom{x}{y}}\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}_{,j},$$

$$\tilde{U}^j_{zz,j} = -r_{,j}\,\Delta^0_0\,\tilde{V}^j_{2z,r}, \tag{45}$$

where $\tilde{u}^j_{rq}$ and $\tilde{u}^j_{\theta q}$ are defined in equation (43), and

$$\Delta^{-1}_1\,\tilde{H}^j_{1q,r} = \Delta^0_{d1}\,\tilde{H}^j_{1q},$$

$$\Delta^0_{d1}\,\tilde{H}^j_{1q,r} = -\frac{1}{r}\,\Delta^0_{d1}\,\tilde{H}^j_{1q} + \frac{1}{r^2}\,\Delta^{-1}_1\,\tilde{H}^j_{1q} - \Delta^1_1\,\tilde{H}^j_{1q}, \tag{46}$$

for *SH* waves, and

$$\Delta^{-1}_1\,\tilde{V}^j_{1q,r} = \Delta^0_{d1}\tilde{V}^j_{1q},$$

$$\Delta^0_{d1}\tilde{V}^j_{1q,r} = -\frac{1}{r}\,\Delta^0_{d1}\tilde{V}^j_{1q} + \frac{1}{r^2}\,\Delta^{-1}_1\tilde{V}^j_{1q} - \Delta^1_1\tilde{V}^j_{1q},$$

$$\Delta^0_1\tilde{V}^j_{1z,r} = \Delta^1_0\tilde{V}^j_{1z} - \frac{1}{r}\,\Delta^0_1\tilde{V}^j_{1z},$$

$$\Delta^0_1\tilde{V}^j_{2q,r} = \Delta^1_0\tilde{V}^j_{2q} - \frac{1}{r}\,\Delta^0_1\tilde{V}^j_{2q},$$

$$\Delta^0_0\tilde{V}^j_{2z,r} = -\Delta^1_1\tilde{V}^j_{2z} \tag{47}$$

for *P-SV* waves. In equation (46), a new analytical integration $\Delta^1_1\tilde{H}^j_{1q}$ is obtained by replacing I^{-1}_1 etc. in equation (28) as follows:

$$I^{-1}_1 \to I^1_1,\quad I^{-1(S-1)}_1 \to I^{1(S-1)}_1,\quad I^{-1(S)}_1 \to I^{1(S)}_1. \tag{48}$$

Similarly, new analytical integrations in equation (47) are obtained from equations (31) and (32) by performing the following replacements in equation (33):

$$I^{-1}_1 \to I^1_0,\quad I^0_1 \to I^2_0,\quad I^1_1 \to I^3_0,$$

$$I^{-1(S-1)}_1 \to I^{1(S-1)}_0,\quad I^{0(S-1)}_1 \to I^{2(S-1)}_0,\quad I^{1(S-1)}_1 \to I^{3(S-1)}_0,$$

$$I^{-1(S)}_1 \to I^{1(S)}_0,\quad I^{0(S)}_1 \to I^{2(S)}_0,\ I^{1(S)}_1 \to I^{3(S)}_0, \tag{49}$$

for $\Delta^1_0\tilde{V}^j_{1z}$ and $\Delta^1_0\tilde{V}^j_{2q}$ and

$$I^{-1}_1 \to I^1_1,\quad I^0_1 \to I^2_1,\quad I^1_1 \to I^3_1,$$

$$I^{-1(S-1)}_1 \to I^{1(S-1)}_1,\quad I^{0(S-1)}_1 \to I^{2(S-1)}_1,\quad I^{1(S-1)}_1 \to I^{3(S-1)}_1,$$

$$I^{-1(S)}_1 \to I^{1(S)}_1,\quad I^{0(S)}_1 \to I^{2(S)}_1,\ I^{1(S)}_1 \to I^{3(S)}_1 \tag{50}$$

for $\Delta^1_1\tilde{V}^j_{1q}$ and $\Delta^1_1\tilde{V}^j_{2z}$

On the other hand, we can derive the equations corresponding to equation (34) in H94 by differentiating equation (43) with respect to z and using equations (30) and (31) of H94, as follows:

$$\tilde{U}^j_{r\binom{x}{y},z} = \left\{\Delta^0_{d1}\,\tilde{V}^j_{1\binom{x}{y},z} + \frac{1}{r}\,\Delta_1^{-1}\,\tilde{H}^j_{1\binom{x}{y},z}\right\}\binom{\cos\theta}{\sin\theta},$$

$$\tilde{U}^j_{rz,z} = -\Delta^0_1\,\tilde{V}^j_{1z,z},$$

$$\tilde{U}^j_{\theta\binom{x}{y},z} = \binom{-}{+}\left\{\frac{1}{r}\,\Delta_1^{-1}\,\tilde{V}^j_{1\binom{x}{y},z} + \Delta^0_{d1}\,\tilde{H}^j_{1\binom{x}{y},z}\right\}\binom{\sin\theta}{\cos\theta}$$

$$\tilde{U}^j_{z\binom{x}{y},z} = -\Delta^0_{1}\,\tilde{V}^j_{2\binom{x}{y},z}\binom{\cos\theta}{\sin\theta},$$

$$\tilde{U}^j_{zz,z} = -\Delta^0_0\,\tilde{V}^j_{2z,z}, \tag{51}$$

where

$$\Delta_1^{-1}\,\tilde{H}^j_{1q,z} = \frac{1}{\mu^j}\,\Delta_1^{-1}\,H^j_{2q},$$

$$\Delta^0_{d1}\,\tilde{H}^j_{1q,z} = \frac{1}{\mu^j}\,\Delta^0_{d1}\,\tilde{H}^j_{2q} \tag{52}$$

for *SH* waves, and

$$\Delta_1^{-1}\,\tilde{V}^j_{1q,z} = \frac{1}{\mu^j}\,\Delta_1^{-1}\,\tilde{V}^j_{3q} + \Delta^0_1\,\tilde{V}^j_{2q},$$

$$\Delta^0_{d1}\,\tilde{V}^j_{1q,z} = \frac{1}{\mu^j}\,\Delta^0_{d1}\,\tilde{V}^j_{3q} + \Delta^1_0\,\tilde{V}^j_{2q} - \frac{1}{r}\,\Delta^0_1\,\tilde{V}^j_{2q},$$

$$\Delta^0_1\,\tilde{V}^j_{1z,z} = \frac{1}{\mu^j}\,\Delta^0_1\,\tilde{V}^j_{3z} + \Delta^1_1\,\tilde{V}^j_{2z},$$

$$\Delta^0_1\,\tilde{V}^j_{2q,z} = \frac{1}{\lambda^j + 2\mu^j}\,\{\Delta^0_1\tilde{V}^j_{4q} - \lambda^j\,\Delta^1_1\,\tilde{V}^j_{1q}\},$$

$$\Delta^0_0\,\tilde{V}^j_{2z,z} = \frac{1}{\lambda^j + 2\mu^j}\,\{\Delta^0_0\,\tilde{V}^j_{4z} - \lambda^j\,\Delta^1_0\,\tilde{V}^j_{1z}\} \tag{53}$$

for *P-SV* waves.

Dissemination of the FORTRAN Codes

The FORTRAN codes described above are available to academic users using the anonymous FTP. The address is "coda.usc.edu" or "128.125.23.15," and the user name is "anonymous." The source codes for point and dislocation sources with examples of data are located in the directory "pub/hisada/green." Users can see manuals in the directory for the details.

Results

We test our new method using the three-layer model shown in Figure 5, which is the same as the test model used in H94. We compare results obtained by our new method with those of H94. We only present a couple of results for the following limited cases here, which are for point sources and circular frequency $\omega = 1$, because we reached the same conclusions for all the other cases we tried.

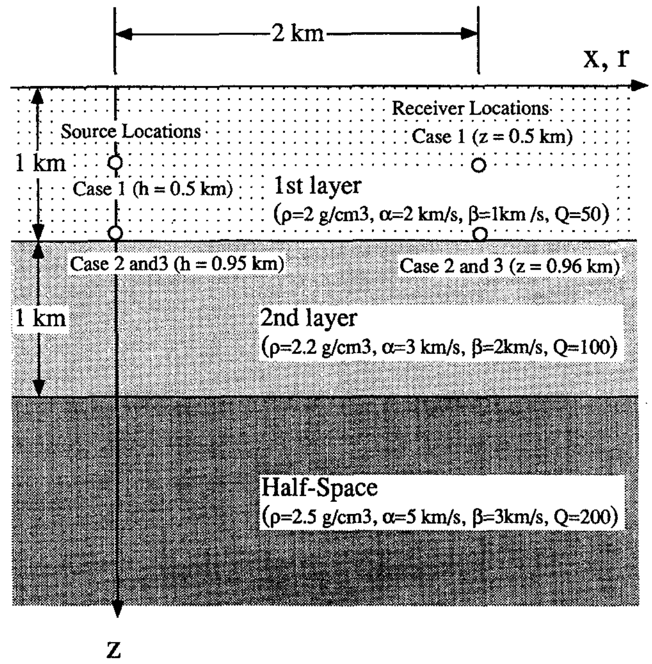

Figure 5. The layered half-space model to check our new method. In the following computations, we use a point source with $Q_x = Q_y = Q_z = 1$ for the case of $\omega = 1$.

Case 1 (a Source at the Middle of the First Layer)

In the first case, we fix a source and a receiver at the middle of the first layer ($h = z = 500$ m), and take $r = 2000$ m, $\theta = 0$, and $Q_x = Q_y = Q_z = 1$. In order to construct asymptotic solutions of H94, we take $\bar{k}_1 = 0.05$ and $\bar{k} = \bar{k}_2 = 0.06$.

Figure 6a shows the wavenumber versus the integrand of the original $U^j_{\theta x}$, the integrands applied by the asymptotic techniques of H94 and this study. Similarly, Figure 6b shows those for σ^j_{zzz}. As discussed in case 5 of H94, the original integrands oscillate with very slowly decreasing and increasing amplitude for displacement and stress, respectively. In contrast, the integrands by H94 and our new method quickly converge to zero. Note that the integrands by the new method converge more quickly than those of H94, which is clear for the stress shown in Figure 6b. This shows that the reflected waves from the free surface and the lower boundary still affect the asymptotic solutions even for the case in which a source and a receiver lie at the middle of a layer.

Case 2 (a Source Close to a Boundary)

In this case, we put the source 50 m above the first boundary ($h = 950$ m) and locate the receivers at two different depths ($z = 960$ and 1050 m). This case is the same as case 3 in H94. Note that the receiver at $z = 1050$ m is located in a different layer from the source layer. For the

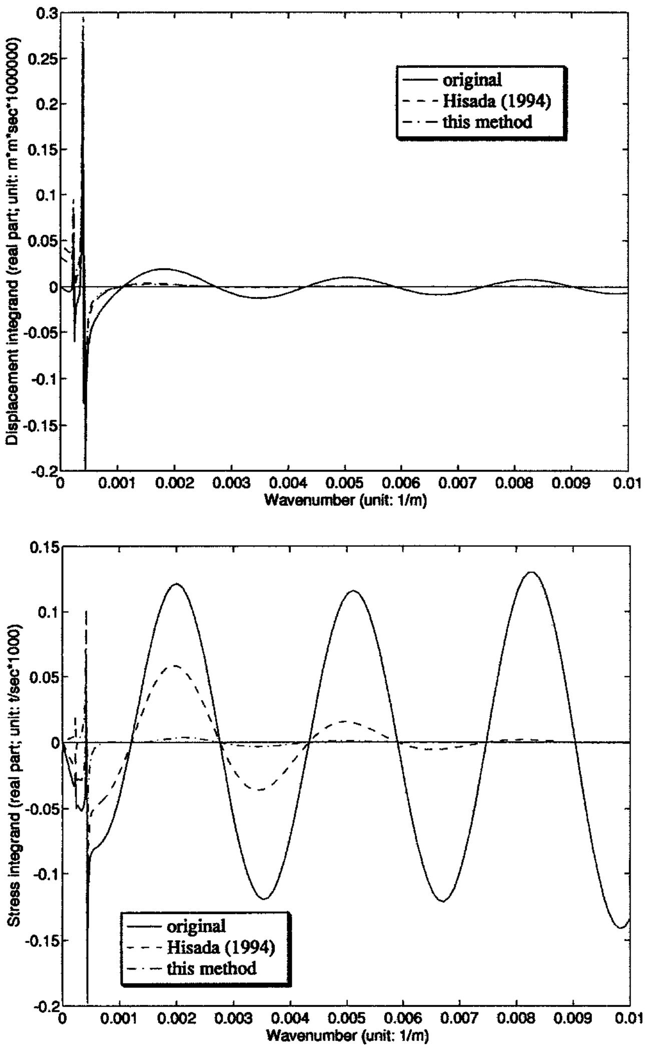

Figure 6. Wavenumber versus the real parts of the integrand (a: upper figure) $U^j_{\theta x}$ and (b: lower figure) σ^j_{zzz} for case 1 ($h = z = 500$ m). The solid lines represent the original integrands without asymptotic techniques, and the other kinds of lines represent the integrands with the methods in Hisada (1994) and this study.

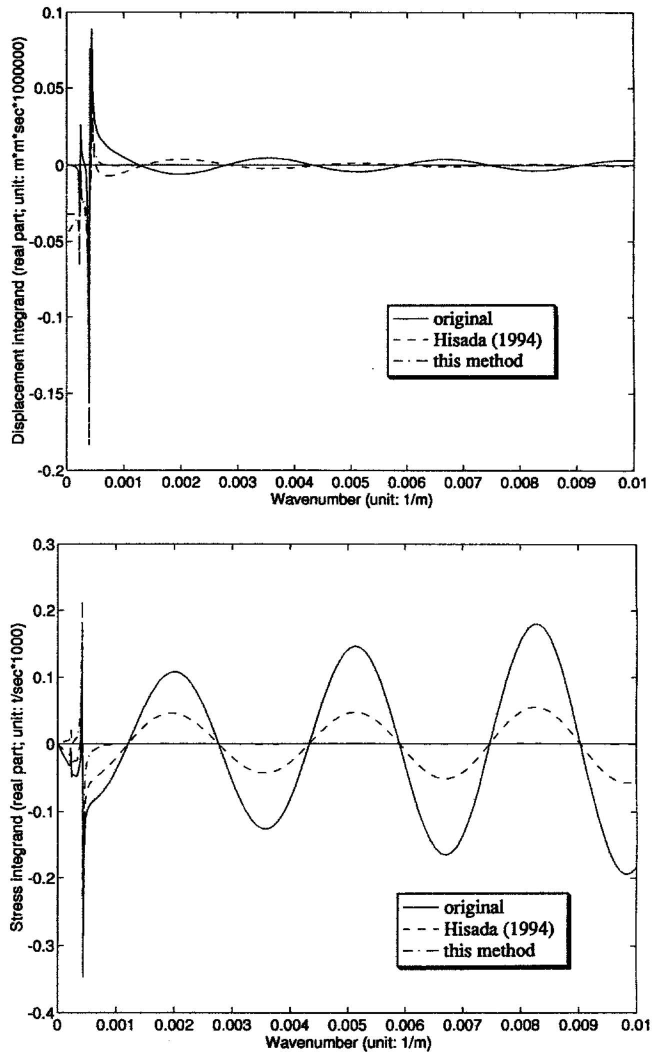

Figure 7. Same as Figure 6 but for (a) U^j_{rx} and (b) σ^j_{zzz} for case 2 ($h = 950$ m and $z = 960$ m). Note that the integrands by this method immediately converge to zero after passing the poles.

same reason mentioned for case 3 in H94, we use the larger values for the asymptotic solutions of H94: $\bar{k}_1 = 0.2$ and $\bar{k} = \bar{k}_2 = 0.21$.

Figure 7a shows the wavenumber versus the integrand of the original U^j_{rx}, the integrands applied by H94 and the new method for $z = 960$ m. Figure 7b shows those for σ^j_{zzz}. Similar to case 1, the original integrands oscillate with very slowly decreasing amplitudes. However, contrary to case 1, the integrands by H94 do not show quick convergence; they do not converge until about $k = 0.03$ for U^j_{rx} and 0.12 for σ^j_{rzx}, as shown in Figures 6 and 7 of H94 and also case 3 shown below. This is because H94 neglects the reflected waves from the lower boundary. On the contrary, the integrands by our new method immediately converge to zero after passing the Love and Rayleigh poles.

Similarly, Figure 8 shows the integrands σ^j_{rzz} for $z = 1050$ m. In this case, both integrands by H94 and the new method quickly converge to zero. This is because the asymptotic solution of H94 also correctly expresses the transmitted waves as explained in the paragraphs below equations (11) and (21).

Case 3 (Final Green's Functions)

Finally, we compare the final values of the Green's functions obtained by the original integrations, those by H94 and by the new method. We adopt the same parameters used in case 2 and carry out the numerical integrations using Simpson's rule with increments of 0.000001 for wavenumbers between 0.000001 and 0.0006, and with increments of 0.0001 for wavenumbers greater than 0.0006. These are same as those of case 6 of H94.

Figures 9a and 9b show the upper limit of the integration

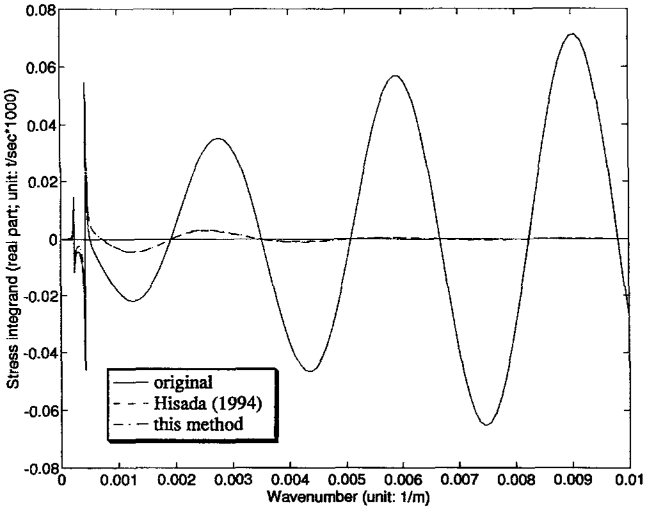

Figure 8. Same as Figure 7 but for σ^1_{zzz} for case 2 (h = 950 m and z = 1050 m). The integrands by H94 and this method share almost the same trajectories in this case.

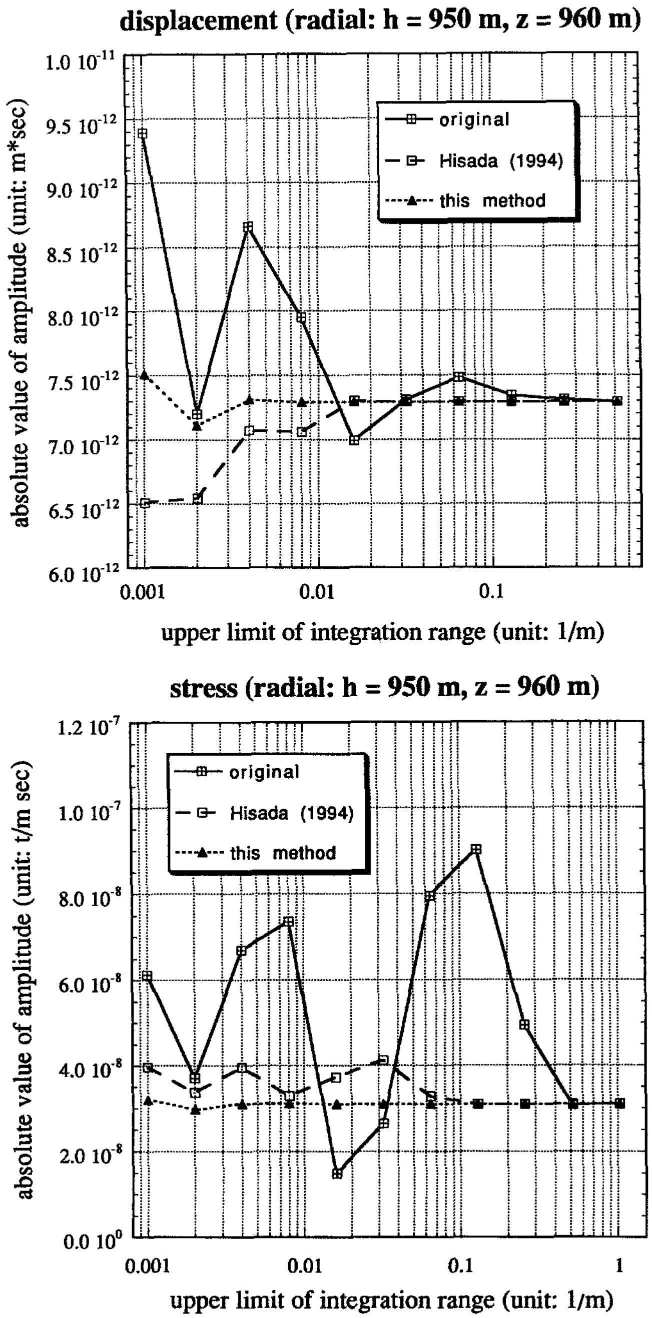

Figure 9. The relations between the upper limit of the integration range and the absolute values of (a) displacement $|U^1_{rx} + U^1_{ry} + U^1_{rz}|$ and (b) stress $|U^1_{rzx} + \sigma^1_{rzy} + \sigma^1_{rzz}|$ of Green's functions, for case 3 (h = 950 m and z = 960 m).

range (maximum wavenumber) versus the absolute values of $|U^1_{rx} + U^1_{ry} + U^1_{rz}|$, and $|\sigma^1_{rzx} + \sigma^1_{rzy} + \sigma^1_{rzz}|$, for z = 960 m. Both displacement and stress by our new method show much faster and more stable convergence with the increasing upper limit than those of H94. In this case, the new method can reduce the integration ranges down to about one-hundredth of those of the original integrations, and about one-tenth of H94.

Conclusions

We derived the analytical asymptotic solutions of the direct waves from a source and the reflected/transmitted waves from the layers adjacent to the source layer. This improves Hisada (1994) and more efficiently computes Green's functions due to point and dipole sources for viscoelastic layered half-spaces with near equal source and receiver depths. We confirmed that we can significantly reduce the range of wavenumber integration especially when a source and a receiver are close to the free surface or to the boundaries adjacent to the source layer. In one case (case 3), we could reduce the integration range down to about one-hundredth of the original and to one-tenth of those Hisada (1994). We made the FORTRAN codes of this method available to the public through the anonymous FTP.

Acknowledgments

The author is grateful to D. Adams, M. Forrest, and an anonymous referee for improving the English in the manuscript. This work was in part supported by NSF under Contract Number ACS-9318163 and Southern California Earthquake Center through NSF cooperative agreement EAR-8920136 and USGS cooperative agreement through 14-08-0001-A0899, and the Kajima Foundation's Research Grant. This study was done while the author was a research associate at the Earth Sciences University of Southern California.

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Appendix A

Analytical Solution of a 4 × 4 Inverse Matrix

One of the most CPU time consuming parts in the R/T coefficient method is to compute the inverse matrix of a 4 by 4 matrix, which is given in equation (A18) of H94 to obtain the modified R/T coefficients for *P-SV* waves. Therefore, it is convenient to give the analytical solution of the 4 by 4 inverse matrix.

We define a 4 by 4 inverse matrix of [**E**] as [**A**]:

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix} \equiv \begin{bmatrix} E_{11} & E_{12} & E_{13} & E_{14} \\ E_{21} & E_{22} & E_{23} & E_{24} \\ E_{31} & E_{32} & E_{33} & E_{34} \\ E_{41} & E_{42} & E_{43} & E_{44} \end{bmatrix}^{-1}. \tag{A1}$$

Using the elimination method, we obtain the elements in the first column of [**A**] as follows:

$$A_{11} = (-E_{22}d_3 + E_{23}d_2 - E_{24}c_1)/A_1,$$

$$A_{21} = (E_{21}d_3 - E_{23}d_1 - E_{24}c_2)/A_1,$$

$$A_{31} = (-E_{21}d_2 + E_{22}d_1 - E_{24}c_3)/A_1,$$

$$A_{41} = (E_{21}c_1 + E_{22}c_2 + E_{23}c_3)/A_1, \tag{A2}$$

where

$$A_1 = a_1c_1 + a_2c_2 + a_3c_3 + b_1d_1 + b_2d_2 + b_3d_3, \tag{A3}$$

$$a_1 = E_{14}E_{21} - E_{11}E_{24}, \quad a_2 = E_{14}E_{22} - E_{12}E_{24},$$

$$a_3 = E_{14}E_{23} - E_{13}E_{24},$$

$$b_1 = E_{13}E_{22} - E_{12}E_{23}, \quad b_2 = E_{11}E_{23} - E_{13}E_{21},$$

$$b_3 = E_{12}E_{21} - E_{11}E_{22},$$

$$c_1 = E_{33}E_{42} - E_{32}E_{43}, \quad c_2 = E_{31}E_{43} - E_{33}E_{41},$$

$$c_3 = E_{32}E_{41} - E_{31}E_{42},$$

$$d_1 = E_{34}E_{41} - E_{31}E_{44}, \quad d_2 = E_{34}E_{42} - E_{32}E_{44},$$

$$d_3 = E_{34}E_{43} - E_{33}E_{44}. \tag{A4}$$

The second column of [**A**] is obtained using the following replacements in equations (A2) to (A4):

$$A_{i1} \to A_{i2} \quad \text{in equation (A2)} \quad (i = 1, 2, 3, \text{ and } 4),$$

$$A_1 \to A_2 \quad \text{in equations (A2) and (A3)},$$

$$E_{1i} \to E_{2i},\ E_{2i} \to E_{3i},\ E_{3i} \to E_{4i},\ E_{4i} \to E_{1i}$$

$$\text{in equation (A4)} \quad (i = 1, 2, 3, \text{ and } 4). \tag{A5}$$

Similarly, we obtain the third and fourth columns of [**A**] using the following replacements:

$$A_{i1} \to A_{i3} \quad \text{in equation (A2)} \quad (i = 1, 2, 3, \text{ and } 4),$$

$$A_1 \to A_3 \quad \text{in equations (A2) and (A3)},$$

$$E_{1i} \to E_{3i},\ E_{2i}, \to E_{4i},\ E_{3i} \to E_{1i},\ E_{4i} \to E_{2i}$$

$$\text{in equation (A4)} \quad (i = 1, 2, 3, \text{ and } 4). \tag{A6}$$

for the third column of [**A**], and

$$A_{i1} \to A_{i4} \quad \text{in equation (A2)} \quad (i = 1, 2, 3, \text{ and } 4),$$

$$A_1 \to A_4 \quad \text{in equations (A2) and (A3)},$$

$$E_{1i} \to E_{4i},\ E_{2i}, \to E_{1i},\ E_{3i} \to E_{2i},\ E_{4i} \to E_{3i}$$

$$\text{in equation (A4)} \quad (i = 1, 2, 3, \text{ and } 4) \tag{A7}$$

for the fourth column of [**A**].

Appendix B

Analytical Asymptotic Solution of Equation (20)

We can analytically derive the asymptotic solutions of the 4 by 4 R/T coefficients given in equation (20). We express those elements as follows:

$$\begin{bmatrix} \bar{\mathbf{T}}_d^{(j)} & \bar{\mathbf{R}}_u^{(j)} \\ \bar{\mathbf{R}}_d^{(j)} & \bar{\mathbf{T}}_u^{(j)} \end{bmatrix} \equiv \begin{bmatrix} B_{11} & B_{12} & B_{13} & B_{14} \\ B_{21} & B_{22} & B_{23} & B_{24} \\ B_{31} & B_{32} & B_{33} & B_{34} \\ B_{41} & B_{42} & B_{43} & B_{44} \end{bmatrix}. \tag{B1}$$

Elements B_{ij} in the above equation are given using Appendix A in this study and equation (A18) in H94 as follows:

$$B_{11} = 2(\kappa^j + 1)a^j(a^j + \kappa^j)/B^j$$

$$B_{21} = (\kappa^j + 1)a^j\{(\kappa^j\kappa^{j+1} - 3\kappa^j + 2) - a^j(\kappa^j\kappa^{j+1} - 3\kappa^{j+1} + 2)\}/B^j,$$

$$B_{31} = 2(\kappa^j - 2)(1 - a^j)(1 + a^j\kappa^{j+1})/B^j,$$

$$B_{41} = [a^j(\kappa^{j+1} - 1)(\kappa^j\kappa^j - 3\kappa^j + 4) - (\kappa^j - 1)\{4 + a^ja^j(\kappa^j - 3)\,\kappa^{j+1}\}]/B^j,$$

$$B_{12} = 0,$$

$$B_{22} = 2(\kappa^j + 1)\,a^j(1 + a^j\kappa^{j+1})/B^j,$$

$$B_{32} = 4(a^j - 1)(1 + a^j\kappa^{j+1})/B^j,$$

$$B_{42} = -B_{31},$$

$$B_{13} = 2(\kappa^{j+1} - 2)(a^j - 1)(a^j + \kappa^j)/B^j,$$

$$B_{23} = [a^j(\kappa^j - 1)(\kappa^{j+1}\kappa^{j+1} - 3\kappa^{j+1} + 4) - (\kappa^{j+1} - 1)\{4a^ja^j + (\kappa^{j+1} - 3)\kappa^j\}]/B^j,$$

$$B_{33} = 2(\kappa^{j+1} + 1)(1 + a^j\kappa^{j+1})/B^j,$$

$$B_{43} = (\kappa^j + 1)\{a^j(\kappa^j\kappa^{j+1} - 3\kappa^{j+1} + 2) - (\kappa^j\kappa^{j+1} - 3\kappa^j + 2)\}/B^j,$$

$$B_{14} = -4(a^j - 1)(a^j + \kappa^j)/B^j,$$

$$B_{24} = -B_{13},$$

$$B_{34} = 0,$$

$$B_{44} = 2(\kappa^{j+1} + 1)(a^j + \kappa^j)/B^j, \tag{B2}$$

where

$$a^j = \mu^j/\mu^{j+1},$$

$$B^j = 2\{\kappa^j + a^ja^j\kappa^{j+1} + a^j(\kappa^j\kappa^{j+1} + 1)\}, \tag{B3}$$

and κ^j is given in equation (A11) in H94.

Appendix C

The analytical integrations, which are used in the formulation in this study, are given as follows:

$$I_0^0 \equiv \frac{1}{R} = \int_0^\infty [\exp\{-k|z - h|\}J_0(kr)]dk,$$

$$I_0^1 \equiv \frac{|z - h|}{R^3} = \int_0^\infty [\exp\{-k|z - h|\}kJ_0(kr)]dk,$$

$$I_0^2 \equiv \frac{1}{R^3}\left\{3\frac{(z - h)^2}{R^2} - 1\right\} = \int_0^\infty [\exp\{-k|z - h|\}k^2J_0(kr)]dk,$$

$$I_0^3 \equiv 3\frac{|z - h|}{R^5}\left\{5\frac{(z - h)^2}{R^2} - 3\right\} = \int_0^\infty [\exp\{-k|z - h|\}k^3J_0(kr)]dk, \tag{C1}$$

and

$$I_1^{-1} \equiv \frac{r}{R + |z - h|} = \int_0^\infty \left[\exp\{-k|z - h|\}\frac{J_1(kr)}{k}\right]dk,$$

$$I_1^0 \equiv \frac{r}{R(R + |z - h|)} = \int_0^\infty [\exp\{-k|z - h|\}J_1(kr)]dk,$$

$$I_1^1 \equiv \frac{r}{R^3} = \int_0^\infty [\exp\{-k|z - h|\}kJ_1(kr)]dk,$$

$$I_1^2 \equiv 3\frac{r|z - h|}{R^5} = \int_0^\infty [\exp\{-k|z - h|\}k^2J_1(kr)]dk,$$

$$I_1^3 \equiv 3\frac{r}{R^5}\left\{5\frac{(z - h)^2}{R^2} - 1\right\} = \int_0^\infty [\exp\{-k|z - h|\}k^3J_1(kr)]dk, \tag{C2}$$

where

$$R = \sqrt{r^2 + (z - h)^2}. \tag{C3}$$

Similarly, $I_0^{0(S-1)}$, $I_0^{0(S)}$ *etc.* corresponding to the reflected waves are obtained using the following replacements in equations (C1), (C2), and (C3):

$$z - h \rightarrow z + h - 2z^{(S-1)} \tag{C4}$$

for $I_0^{0(S-1)}$ *etc.* and

$$z - h \rightarrow 2z^{(S)} - z - h \tag{C5}$$

for $I_0^{0(S)}$ *etc.*

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Manuscript received 21 June 1994.