An Efficient Method to Compute Transfer Function of a Transformer From its Equivalent Circuit

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***Abstract*—The dynamics of an electrical network can completely be described from the knowledge of its poles and zeros. Computation of poles and zeros of the transfer function (TF) of a transformer winding, represented as a coupled ladder network, involves solution of a large-sized equivalent circuit. This paper presents a novel solution based on state space analysis approach. It is shown, how the linearly transformed state space formulation, together with algebraic manipulations, can become useful. In the proposed formulation, symbolic variables (i.e., Laplace variable, s) are suitably manipulated, so as to render computations purely numerical. With this feature, there is practically no limit on the size of networks and topologies (including resistances to model losses) that can be represented. So, virtually any number of windings of a transformer can be considered, permitting a comprehensive analysis of winding behavior and its interactions, that was until now severely limited, by the simplifying assumptions imposed by existing methods.**

***Index Terms*—Eigenvalues and diagonalization, poles and zeros, state space analysis, transfer function, transformer.**

I. Introduction

HV transformers are optimally designed to withstand a variety of overvoltages and short circuit forces. Occurrence of these events is natural, inevitable, and a main cause of transformer failures. Early detection and prevention of failures is paramount. To this end, diagnostic testing and condition monitoring is a means that enables utilities in early detection of incipient fault conditions. Among diagnostic methods that have evolved, transfer function (TF) has been found suitable to detect dielectric faults and mechanical deformations [1].

However, literature reveals that, improvement in interpretation of TF is essential, if a more meaningful assessment of the condition of transformer has to be made using TF tool. One possibility that appears promising, as shown in [2], is to analyze the transformer equivalent circuit i.e., poles and zeros of TF, by considering all windings and its interactions. Further, it is well known that, dynamics of any system or electrical network can completely be described, when all the poles and zeros are explicitly known. Somehow, the influence of zeros in shaping TF has so far not been fully unravelled. However, in [2], usefulness of zeros in interpreting TF was demonstrated. It was shown, how by knowing all zeros of TF (in addition to poles), occurrence of a near pole-zero cancellation could be observed. This in turn was shown responsible for imparting a unique shape to the TF of an interleaved winding (compared to disc winding). So, in the same way, knowledge of zeros is naturally expected to be useful, especially, when dealing with multiple windings and its interactions [3]. Such an analysis might yield newer insights, based on which, certain issues in TF and impulse behavior of windings could be better explained.

In this context, an equivalent circuit representation to accurately describe impulse phenomenon in transformers becomes a necessity. These matters have extensively been studied and many representations have been proposed. Amongst them, the equivalent circuit considering all relevant capacitive and inductive couplings is widely accepted and extensively used, as it is known to capture most of the subtle aspects of, the transient response due to full and chopped lightning impulses. Fig. 1 shows the equivalent circuit of a two winding transformer. Calculating its natural frequencies and voltage distribution are two major topics of interest. The following are instances during such investigations that necessitate modeling, representation and solution of large-sized equivalent circuits.

- Increased number of sections (corresponding to even an individual disk of the winding) might sometimes have to be considered, with the aim of achieving finer representation for resolving all the natural frequencies and also for higher accuracy.
- For an accurate understanding and interpretation of the measured transfer function of a transformer, it is essential that, all windings and its mutual interactions be considered [3]. Further, its TF can naturally be expected to possess a large number of natural frequencies and a fine structure as well. To adequately represent these finer aspects, as well as, to explain the genesis of TF of multi-winding transformer, all windings and their interactions need to be considered.
- Inclusion of losses (as resistances either in series or parallel or both) is required to make the terminal response more realistic. This inclusion only adds to complexity of solution, but not to network size.

As constants of the lumped parameter model are determined using quasi-static approximations, they are valid only up to about 2 MHz (an optimistic upper limit). So, this fundamental limitation will naturally restrict the number of sections that needs to be considered. Even if, one considers more number of sections, the natural frequencies estimated above 2 MHz are considered unreliable due to this limitation and hence have to be discarded.

Therefore, it emerges that, formulations that can give explicit information about all TF poles and zeros, and be suitable for large-sized networks, are desirable.

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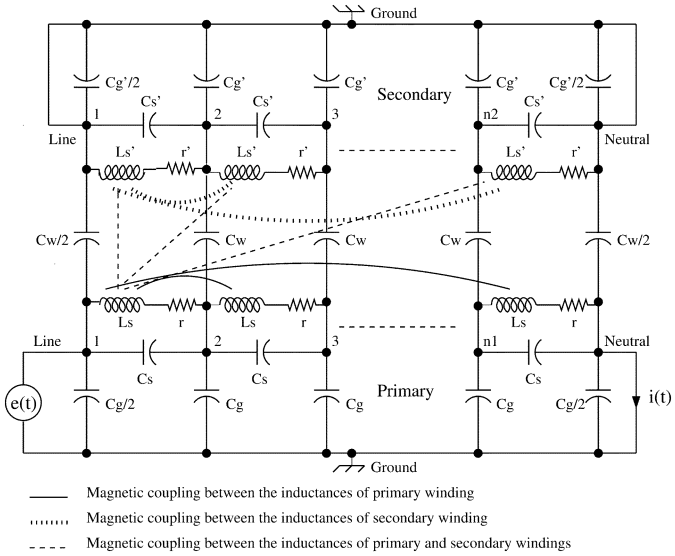

Fig. 1. Equivalent circuit of a two winding transformer with secondary short circuited (or opened as the case may be).

II. Existing Approaches and Motivation

The equivalent circuit considered is shown in Fig. 1. For sake of clarity, circuit elements of each section (in the two windings) are shown to be identical. However, a winding may be represented by sections, the elements of which have unequal values, to simulate conditions arising in partially/graded interleaved windings, discontinuity due to tap winding being different from main winding, etc. A brief mention of methods to solve circuit in Fig. 1, (for voltage distributions and natural frequencies) is made below.

1) Although circuit analysis software can be used, they yield only a plot of the resultant TF, and not the complete information about TF poles and zeros. This is because, from the TF plot, discrimination of two closely located poles, very low-height poles, and those pole-zero pairs located close to each other (resulting in near pole-zero cancellations), will be difficult.
2) In the mid 1950s, a method was presented by Abetti *et al.* [4] for the determination of the fundamental frequencies and wave shapes of the primary winding with short-circuited secondary (with and without iron core). Nodal analysis was used for analyzing the equivalent circuit model of the system with two coupled windings. The method described an iterative procedure and was shown to yield natural frequencies for only small-sized circuit models.
3) Solution methodology developed by Gururaj [5], in 1964, was based on eigenvalue approach. Essentially, the method initially starts with forming the nodal admittance matrix for the lossless case. The method described a procedure for determining natural frequencies and voltage distributions, wherein, computations are entirely numerical, i.e., without involving Laplace operator s, and hence applicable for large-sized networks. Recent discussions with him revealed that, the method can also be extended to cases where losses are included, as shown in Fig. 1. However, since for each change in topology of the circuit to be considered, a separate set of closed-form solutions have to be derived, this approach was not pursued further.
4) In 1977, Degeneff [6] reported a similar approach starting from the nodal admittance matrix expressed in the form-

$$Y = [G] + [C]s + [\Gamma]s^{-1} \tag{1}$$

where,
$[G]$ Nodal conductance matrix of dimension $n_1 \times n_1$
$[C]$ Nodal capacitance matrix of dimension $n_1 \times n_1$
$[\Gamma]$ Nodal reciprocal inductance matrix of dimension $n_1 \times n_1$
s Laplace operator
to obtain a general solution for determining voltage distribution in transformer windings. In this approach, the requirement was that, the nodal matrices $[G]$, $[C]$ and $[\Gamma]$ should be frequency independent (i.e., free from s), a condition that is naturally satisfied when losses are not considered. But, when losses are included, it is found that nodal reciprocal inductance matrix will consist of s in it. It was remarked that, computation is cumbersome and time-consuming when s is involved, and so diagonalization was suggested to reduce the burden.
5) In 1974, a solution approach based on state-space formulation for calculating voltage oscillations in transformer windings was presented by Fergestad *et al.* [7]. In this work, poles of TF are computed directly as the eigenvalues of the system matrix, while zeros of TF were computed by considering the inverse system, which, was numerically not a straightforward procedure. Further, it was stated that, the system becomes increasingly ill-conditioned whenever a finer representation of the windings (i.e., more number of sections) was used. This clearly sets an upper limit to the size of the circuit model that can be considered. Also, not all terminal conditions of the circuit can be easily accommodated, and hence, is another restriction.

Therefore, from the first two sections, it follows that it is desirable to develop a solution approach that has the following features.

- Must be entirely numerical in nature and yield values of TF poles and zeros.
- Impose no restriction on number of windings and sections, terminal condition of neighboring windings.
- Provide a single formulation suitable for any configuration of resistance (in series, parallel or both) for including loss.
- Be robust, free from numerical instabilities and ill-conditioned matrices, and unaffected by repeated or large spread in eigenvalues.

These also happen to be the objectives of this paper.

III. Computation of TF by State Space Analysis

A powerful method to describe the dynamic behavior of a system or network is the state-variable approach. Its universality has been demonstrated extensively for network analysis by Kuh and Rohrer [8]. Generality of the state space approach is due to the fact that, it can be extended to time-variable and nonlinear systems, where classic approaches would fail. Also, qualitative

measures of network behavior i.e., passivity, time-invariance, linearity, reciprocity and stability can be easily explained with state variable characterization. Another advantage is that, the system of first order differential equations obtained from this approach lends itself readily to programming for numerical solution on a digital computer.

A. Definitions

The state of a system may be considered as the least amount of information that must be known about the given system at a given time to determine its subsequent dynamics completely. A suitable selection of the independent variables result in a set of first order differential equations that are linearly independent [9]. These variables and equations are known as state variables and state equations respectively.

The most general form of the state equations of a linear, time-invariant network is as follows:

$$\dot{\mathbf{x}} = [A]\mathbf{x} + [B]\mathbf{u} \tag{2}$$

$$y = [C]\mathbf{x} + [D]\mathbf{u} \tag{3}$$

where,

$\mathbf{x}$	column vector of the state variables;
$\dot{\mathbf{x}}$	time derivatives of the state variables;
$\mathbf{u}$	excitation or input vector;
y	response or output vector;
(A, B, C, D)	matrices of constant coefficients.

B. Choice of State Variables

For a system to be analyzed, though the selection of state variables is not unique, they are to be chosen so as to enable formulation of state equations with least effort. A random selection of state variables may increase the complexity of the solution methodology and may also result in redundant equations. To avoid such situations, there exist some guidelines for selecting the state variables.

In physical terms, the state variables specify the energy stored in a set of independent energy storage elements. It is thus natural that, the number of state variables should be equal to the number of energy storing elements of the network, provided there are no tie sets of capacitors and cut sets of inductors. As the circuit model (Fig. 1) has tie sets of capacitors within the winding and between the windings, the count of state variables is less than the number of energy storing elements by the number of capacitance tie sets [10].

Based on this rationale, the currents through inductances and voltages across shunt capacitances (excluding the capacitance across the line terminal and datum of the primary winding) are chosen as the desired set of state variables. Since voltage across any series or interwinding capacitance can be expressed in terms of the nodal voltages, voltages across these capacitances are excluded from the list of state variables.

For the circuit model shown in Fig. 1, the state variables are assigned as below:

1) Inductor currents of the primary winding $x_1 = i_1, x_2 = i_2, \ldots, x_{n_1} = i_{n_1}$
2) Inductor currents of the secondary winding $x_{n_1+1} = i'_1, x_{n_1+2} = i'_2, \ldots, x_{n_1+n_2} = i'_{n_2}$
3) Nodal voltages of the primary winding (excluding the line end which is energised by the source) $x_{n_1+n_2+1} = e_2, x_{n_1+n_2+2} = e_3, \ldots, x_{2n_1+n_2-1} = e_{n_1}$
4) Nodal voltages of the secondary winding (excluding the line end node which is grounded) $x_{2n_1+n_2} = e'_2, x_{2n_1+n_2+1} = e'_3, \ldots, x_{2n_1+2n_2-2} = e'_{n_2}$

Thus, the total number of state variables assigned to the circuit model is $\nu = 2n_1 + 2n_2 - 2$. Since, voltage across an inductor or current through a capacitor is governed by a first order differential equation, the formulation of the state equations becomes straightforward with the choice of these state variables.

C. State Model Formulation

State equations are to be formulated for a two winding transformer (Fig. 1) whose secondary is short circuited. Other secondary terminal conditions can similarly be considered. The objective here is to express the derivatives of the state variables in terms of the state variables and the excitation.

1) Time Derivatives of the Inductor Currents: Let $v_1, \ldots, v_{n1}$ and $v'_1, \ldots, v'_{n1}$ denote the voltages across the inductances of the primary and secondary circuits respectively. Let $[L]$ represent the self and mutual inductance matrix of the circuit. Since, time derivatives of the inductance currents are related to the voltages across the inductances as,

$$\begin{bmatrix} \frac{di_1}{dt} \\ \frac{di_2}{dt} \\ \vdots \\ \frac{di_{n_1-1}}{dt} \\ \frac{di_{n_1}}{dt} \\ \frac{di'_1}{dt} \\ \frac{di'_2}{dt} \\ \vdots \\ \frac{di'_{n_2-1}}{dt} \\ \frac{di'_{n_2}}{dt} \end{bmatrix} = [\Gamma_1] \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_{n_1-1} \\ v_{n_1} \\ v'_1 \\ v'_2 \\ \vdots \\ v'_{n_2-1} \\ v'_{n_2} \end{bmatrix} \tag{4}$$

where, $[\Gamma_1] = [L]^{-1}$. Since, the voltage across the inductances can be expressed in terms of the nodal voltages and with short circuited secondary $(e'_1 = 0)$, the above gets modified to,

$$\begin{bmatrix} \frac{di_1}{dt} \\ \frac{di_2}{dt} \\ \vdots \\ \frac{di_{n_1-1}}{dt} \\ \frac{di_{n_1}}{dt} \\ \frac{di'_1}{dt} \\ \frac{di'_2}{dt} \\ \vdots \\ \frac{di'_{n_2-1}}{dt} \\ \frac{di'_{n_2}}{dt} \end{bmatrix} = [\Gamma_1] \begin{bmatrix} e_1 - e_2 - i_1 r \\ e_2 - e_3 - i_2 r \\ \vdots \\ e_{n_1-1} - e_{n_1} - i_{n_1-1} r \\ e_{n_1} - i_{n_1} r \\ -e'_2 - i'_1 r' \\ e'_2 - e'_3 - i'_2 r' \\ \vdots \\ e'_{n_2-1} - e'_{n_2} - i'_{n_2-1} r' \\ e'_{n_2} - i'_{n_2} r' \end{bmatrix}. \tag{5}$$

Let $[R]$ be a diagonal matrix such that,

$$R(i,i) = \begin{cases} r, & 1 \le i \le n_1 \\ r', & n_1 < i \le n_1 + n_2 \end{cases} \tag{6}$$

and let Γ_2 be constructed from Γ_1 such that,

$$\Gamma_2(i,j) = \begin{cases} \Gamma_1(i,j), & \forall\, i, j = 1 \\ \Gamma_1(i,j) - \Gamma_1(i,j-1), & \forall\, i, 1 < j \le n_1 \\ \Gamma_1(i,j+1) - \Gamma_1(i,j), & \forall\, i, j > n_1. \end{cases} \quad (7)$$

Using (6) and (7) and separating the nodal voltages and the resistive drops, (5) simplifies to,

$$\begin{bmatrix} \frac{di_1}{dt} \\ \vdots \\ \frac{di_{n_1}}{dt} \\ \frac{di'_1}{dt} \\ \vdots \\ \frac{di'_{n_2}}{dt} \end{bmatrix} = -[\Gamma_1][R] \begin{bmatrix} i_1 \\ \vdots \\ i_{n_1} \\ i'_1 \\ \vdots \\ i'_{n_2} \end{bmatrix} + [\Gamma_2] \begin{bmatrix} e_1 \\ \vdots \\ e_{n_1} \\ e'_2 \\ \vdots \\ e'_{n_2} \end{bmatrix}. \quad (8)$$

Let $[I_i]$, $[I_e]$ and $[I_u]$ be the matrices relating the time derivatives of the currents to the inductor currents, the nodal voltages (state variables) and the input voltage respectively. They are expressed as,

$$[I_i] = -[\Gamma_1][R] \quad (9)$$

$$I_e(i,j) = \Gamma_2(i,j+1), \quad \forall\, i, \quad 1 \le j \le n_1 + n_2 - 2 \quad (10)$$

$$I_u(i,1) = \Gamma_2(i,1), \quad \forall\, i. \quad (11)$$

Hence (8) becomes,

$$\begin{bmatrix} \frac{di_1}{dt} \\ \vdots \\ \frac{di_{n1}}{dt} \\ \frac{di'_1}{dt} \\ \vdots \\ \frac{di'_{n2}}{dt} \end{bmatrix} = [I_i] \begin{bmatrix} i_1 \\ \vdots \\ i_{n1} \\ i'_1 \\ \vdots \\ i'_{n2} \end{bmatrix} + [I_e] \begin{bmatrix} e_2 \\ \vdots \\ e_{n1} \\ e'_2 \\ \vdots \\ e'_{n2} \end{bmatrix} + [I_u](e_1). \quad (12)$$

Thus, the time derivatives of the inductor currents are related to the inductor currents, the nodal voltages and the input voltage.

2) Time Derivatives of the Nodal Voltages: Applying KCL for the circuit of Fig. 1, that is, the sum of the currents diverging from the nodes through the inductances and the capacitances equals the currents fed to the nodes, results in,

$$\begin{bmatrix} i_1 \\ i_2 - i_1 \\ \vdots \\ i_{n1} - i_{n1-1} \\ i'_1 \\ i'_2 - i'_1 \\ \vdots \\ i'_{n2} - i'_{n2-1} \end{bmatrix} + [K] \begin{bmatrix} \frac{de_1}{dt} \\ \frac{de_2}{dt} \\ \vdots \\ \frac{de_{n1}}{dt} \\ \frac{de'_1}{dt} \\ \frac{de'_2}{dt} \\ \vdots \\ de'_{n2}dt \end{bmatrix} = \begin{bmatrix} i_s \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad (13)$$

where, $[K]$ represents nodal capacitance matrix of the circuit. The above equation can be written as,

$$[T] \begin{bmatrix} i_1 \\ \vdots \\ i_{n1} \\ i'_1 \\ \vdots \\ i'_{n2} \end{bmatrix} + [K] \begin{bmatrix} \frac{de_1}{dt} \\ \vdots \\ \frac{de_{n1}}{dt} \\ \frac{de'_1}{dt} \\ \vdots \\ \frac{de'_{n2}}{dt} \end{bmatrix} = \begin{bmatrix} i_s \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad (14)$$

where, $[T]$ is a matrix of dimension $(n_1 + n_2) \times (n_1 + n_2)$ and is expressed as,

$$[T] = \begin{bmatrix} [T_1] & 0 \\ 0 & [T_2] \end{bmatrix}. \quad (15)$$

Here, $[T_1]$ is a matrix of dimension $n_1 \times n_1$ and is expressed as,

$$[T_1] = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ -1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & 1 \end{bmatrix} \quad (16)$$

and $[T_2]$ will be of the same form as that of $[T_1]$ but with a dimension of $n_2 \times n_2$. As the secondary is short circuited, $e'_1 = 0$. Hence, (14) becomes,

$$[T] \begin{bmatrix} i_1 \\ \vdots \\ i_{n1} \\ i'_1 \\ \vdots \\ i'_{n2} \end{bmatrix} + [K_1] \begin{bmatrix} \frac{de_1}{dt} \\ \vdots \\ \frac{de_{n1}}{dt} \\ \frac{de'_2}{dt} \\ \vdots \\ \frac{de'_{n2}}{dt} \end{bmatrix} = \begin{bmatrix} i_s \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad (17)$$

where, $[K_1]$ is extracted from $[K]$ by eliminating its $(n_1 + 1)$th column.

As the line end nodes of both the primary and secondary are clamped to potentials of e_1 (input voltage) and 0 (datum potential) respectively, the application of KCL to these nodes results in redundant equations. In order to avoid redundancy, the corresponding rows namely 1st and $(n_1 + 1)$th are eliminated from the matrix equation formulated so far. Elimination of these two rows modifies (17) to,

$$[T_a] \begin{bmatrix} i_1 \\ \vdots \\ i_{n1} \\ i'_1 \\ \vdots \\ i'_{n2} \end{bmatrix} + [K_2] \begin{bmatrix} \frac{de_1}{dt} \\ \vdots \\ \frac{de_{n1}}{dt} \\ \frac{de'_2}{dt} \\ \vdots \\ \frac{de'_{n2}}{dt} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad (18)$$

where, $[T_a]$ and $[K_2]$ correspond to the matrices obtained by eliminating 1st and $(n_1 + 1)$th rows from $[T]$ and $[K_1]$ respec-

tively. Separating the derivatives of the state variables and the input voltage, the above equation can be modified as,

$$[T_a]\begin{bmatrix} i_1 \\ \vdots \\ i_{n1} \\ i'_1 \\ \vdots \\ i'_{n2} \end{bmatrix} + [K_3]\left(\frac{de_1}{dt}\right) + [K_4]\begin{bmatrix} \frac{de_2}{dt} \\ \vdots \\ \frac{de_{n1}}{dt} \\ \frac{de'_2}{dt} \\ \vdots \\ \frac{de'_{n2}}{dt} \end{bmatrix} = 0 \qquad (19)$$

where, $[K_3]$ and $[K_4]$ are derived from $[K_2]$ such that,

$$K_3(i,1) = K_2(i,1), \quad \forall\, i \qquad (20)$$

$$K_4(i,j) = K_2(i,j+1), \quad \forall\, i,\ 1 \le j \le n_1 + n_2 - 2. \qquad (21)$$

Let $[E_i]$ and $[E_u]$ be the matrices relating the time derivatives of the nodal voltages to the inductor currents and the time derivative of the input voltage respectively, that is,

$$[E_i] = -[K_4]^{-1}[T_a] \qquad (22)$$

$$[E_u] = -[K_4]^{-1}[K_3]. \qquad (23)$$

With these assignments (19) can be written as,

$$\begin{bmatrix} \frac{de_2}{dt} \\ \vdots \\ \frac{de_{n1}}{dt} \\ \frac{de'_2}{dt} \\ \vdots \\ \frac{de'_{n2}}{dt} \end{bmatrix} = [E_i]\begin{bmatrix} i_1 \\ \vdots \\ i_{n1} \\ i'_1 \\ \vdots \\ i'_{n2} \end{bmatrix} + [E_u]\left(\frac{de_1}{dt}\right). \qquad (24)$$

Thus, the time derivatives of the nodal voltages are expressed in terms of the inductor currents and the time derivative of the input voltage.

3) State Equation: For the circuit model, state equation is formulated by merging (12) and (24) as,

$$\dot{\mathbf{x}} = [A]\mathbf{x} + [B]\mathbf{u} \qquad (25)$$

where, $\mathbf{x}$ and $\dot{\mathbf{x}}$ represent the state variable vector and its first derivative respectively, $\mathbf{u}$ denotes the input vector and is expressed as,

$$\mathbf{u} = \begin{bmatrix} e_1 \\ \frac{de_1}{dt} \end{bmatrix} \qquad (26)$$

and the coefficient matrices $[A]$ and $[B]$ are expressed as,

$$[A] = \begin{bmatrix} [I_i] & [I_e] \\ [E_i] & 0 \end{bmatrix} \qquad (27)$$

$$[B] = \begin{bmatrix} [I_u] & 0 \\ 0 & [E_u] \end{bmatrix}. \qquad (28)$$

Thus, the time derivatives of the state variables are expressed in terms of the state variables and the input vector.

Generally, state equations relate the first derivative of the state variables to the state variables and the excitation, provided there are no cut sets of inductors and tie sets of capacitors. Since capacitance tie sets are present in the circuit model shown in Fig. 1, the state model developed has the input vector (26) consisting of the first derivative of the input voltage in addition to the input voltage [11].

4) Output Equation: Neutral current of the primary winding is chosen as the output variable 'y' of the state model, so it is to be expressed in terms of the state variables $(\mathbf{x})$ and the input $(\mathbf{u})$. From Fig. 1, it is obvious that, neutral current (i) is the sum of the currents through the inductance and the series capacitance of n_1th section of the primary winding, that is,

$$i = i_{n_1} + C_s\frac{de_{n_1}}{dt}. \qquad (29)$$

Using the definition of state variables and making use of (25), the above equation can be rewritten as,

$$y = x_{n_1} + C_s\left([A_y]\mathbf{x} + [B_y]\mathbf{u}\right) \qquad (30)$$

where, $[A_y]$ and $[B_y]$ represent $(2n_1 + n_2 - 1)$th row of $[A]$ and $[B]$ respectively. The first two terms of the above equation can be merged together by assigning $[C]\mathbf{x} = x_{n_1} + C_s[A_y]\mathbf{x}$ and let, $D = C_s[B_y]$. Now, (30) modifies to,

$$y = [C]\mathbf{x} + [D]\mathbf{u}. \qquad (31)$$

Thus, the output variable of the state model is expressed in terms of the state variables and the input vector. Equations (25) and (31) together constitute the state model of the network.

IV. Determination of TF Poles and Zeros

To find an analytical expression for transfer function, the time domain state equations are to be transformed to s-domain. Applying Láplace transform to (25) and (31), and simplifying them, leads to,

$$Y(s) = \left([C][sI - A]^{-1}[B] + [D]\right)U(s) \qquad (32)$$

where, $[I]$ is an identity matrix of dimension same as that of $[A]$. Since,

$$\mathcal{L}(\mathbf{u}) = U(s) = \begin{bmatrix} 1 \\ s \end{bmatrix} E_1(s). \qquad (33)$$

Transfer function, which is defined as a ratio of the output $Y(s)$ to the input voltage $E_1(s)$, is obtained as,

$$TF = \frac{Y(s)}{E_1(s)} = \left([C][sI - A]^{-1}[B] + [D]\right)\begin{bmatrix} 1 \\ s \end{bmatrix}. \qquad (34)$$

Using the above equation, it is possible to obtain a plot of TF by assigning numerical values to s ($s = j\omega$, complex frequency) over a wide range of frequencies. As already mentioned, such a plot will not directly yield the complete information of all the poles and zeros of TF.

A. Mathematical Intricacies in Computing TF

The evaluation of TF as per (34) requires the computation of the inverse of a symbolic matrix $[sI - A]$ which has symbolic variable s. Finding $[sI - A]^{-1}$ needs large time, even when $[A]$

is small. Hence, a method is explored for circumventing this bottleneck.

Finding the inverse of a matrix becomes a much simpler operation, if and only if, the matrix is diagonal. It is because the inverse of such matrix is a diagonal matrix, whose diagonal elements are just the reciprocal of the diagonal elements of the matrix. With this in mind, the existing state model needs to be transformed into another state model, where the system matrix will be diagonal in nature.

B. Diagonalization of System Matrix

Diagonalising the system matrix can be achieved through linear transformation, a well known technique [12]. It is to be mentioned that, the transfer function is invariant through linear transformation. If the basis of the vector space is changed through a transformation matrix constructed from the eigenvectors of the system matrix $[A]$, then it will result in a state model, whose system matrix will be diagonal.

Eigenvalues $\lambda_1, \lambda_2 \ldots, \lambda_\nu$ of the system matrix are determined, from which the transformation matrix ($[M]$, called modal matrix) is constructed such that, its columns are the eigenvectors of $[A]$. By performing linear transformation, $x = [M]z$, (25) and (31) get transformed to,

$$\dot{\mathbf{z}} = [\Lambda]\mathbf{z} + [\tilde{B}]\mathbf{u} \tag{35}$$

$$y = [\tilde{C}]\mathbf{z} + [D]\mathbf{u} \tag{36}$$

where, $[\Lambda] = [M]^{-1}[A][M]$, $[\tilde{B}] = [M]^{-1}[B]$ and $[\tilde{C}] = [C][M]$.

It is to be mentioned here that, $[\Lambda]$ is a diagonal matrix whose elements are same as the eigenvalues $\lambda_1, \lambda_2 \ldots, \lambda_\nu$ of $[A]$. Equations (35) and (36) are now referred as the transformed state model of the network considered. TF of the system is obtained by using (34) with matrices $([A], [B], [C], [D])$ replaced by $([\Lambda], [\tilde{B}], [\tilde{C}], [D])$ as below,

$$TF = \left([\tilde{C}][sI - \Lambda]^{-1}[\tilde{B}] + [D]\right)\begin{bmatrix}1\\ s\end{bmatrix}. \tag{37}$$

$[sI - \Lambda]$ is a diagonal matrix. Although $[sI - \Lambda]^{-1}$ is computed efficiently, the symbolic variable s still continues to remain in the TF expression and poses problems, especially when larger networks are considered. A means of elimination of s from the actual computation process will be examined next.

C. Algebraic Method of Constructing TF

The procedure of constructing TF by means of extracting the coefficients of the numerator and denominator polynomials is explained here. Of the matrices involved in the TF expression, the only symbolic matrix is $[sI - \Lambda]^{-1}$, which is diagonal and the rest of them are numeric in nature. This diagonal form easily lends itself to algebraic manipulation, yielding the coefficients of numerator and denominator polynomials of TF. Once the coefficients of these polynomials are obtained, then it becomes straightforward to find their roots (poles and zeros). This manipulation is a crucial step and is explained below.

The transformed state model is characterized by the matrices $([\Lambda], [\tilde{B}], [\tilde{C}], [D])$ and they are of the form,

$$[\Lambda] = \begin{bmatrix}\lambda_1 & 0 & \cdots & 0\\ 0 & \lambda_2 & \cdots & 0\\ \vdots & \vdots & & \vdots\\ 0 & 0 & \cdots & \lambda_\nu\end{bmatrix}, \quad [\tilde{B}] = \begin{bmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\\ \vdots & \vdots\\ b_{\nu 1} & b_{\nu 2}\end{bmatrix}$$

$$[\tilde{C}] = [\,c_1 \quad c_2 \quad \cdots \quad c_\nu\,], \quad [D] = [\,d_1 \quad d_2\,]. \tag{38}$$

The inverse of the characteristic matrix becomes,

$$[sI - \Lambda]^{-1} = \begin{bmatrix}\frac{1}{s-\lambda_1} & 0 & \cdots & 0\\ 0 & \frac{1}{s-\lambda_2} & \cdots & 0\\ \vdots & \vdots & & 0\\ 0 & 0 & \cdots & \frac{1}{s-\lambda_\nu}\end{bmatrix}. \tag{39}$$

Let $[\Delta(s)] = [\tilde{C}][sI - \Lambda]^{-1}[\tilde{B}]$ and is evaluated as,

$$[\Delta(s)] = [\,\tfrac{c_1}{s-\lambda_1} \quad \tfrac{c_2}{s-\lambda_2} \quad \cdots \quad \tfrac{c_\nu}{s-\lambda_\nu}\,]\begin{bmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\\ \vdots & \vdots\\ b_{\nu 1} & b_{\nu 2}\end{bmatrix}. \tag{40}$$

The above can be simplified to,

$$[\Delta(s)] = [\,\delta_1(s) \quad \delta_2(s)\,] \tag{41}$$

where,

$$\delta_1(s) = \frac{b_{11}c_1}{s-\lambda_1} + \frac{b_{21}c_2}{s-\lambda_2} + \cdots + \frac{b_{\nu 1}c_\nu}{s-\lambda_\nu} = \frac{\alpha_1(s)}{\beta(s)} \tag{42}$$

$$\delta_2(s) = \frac{b_{12}c_1}{s-\lambda_1} + \frac{b_{22}c_2}{s-\lambda_2} + \cdots + \frac{b_{\nu 2}c_\nu}{s-\lambda_\nu} = \frac{\alpha_2(s)}{\beta(s)} \tag{43}$$

with $\alpha_1(s)$, $\alpha_2(s)$ and $\beta(s)$ expressed as,

$$\alpha_1(s) = \sum_{i=1}^{\nu} b_{i1}c_i \prod_{j=1, j\neq i}^{\nu}(s - \lambda_j) \tag{44}$$

$$\alpha_2(s) = \sum_{i=1}^{\nu} b_{i2}c_i \prod_{j=1, j\neq i}^{\nu}(s - \lambda_j) \tag{45}$$

$$\beta(s) = \prod_{i=1}^{\nu}(s - \lambda_i). \tag{46}$$

Equation (37) can be rewritten as,

$$TF = ([\Delta(s)] + [D])\begin{bmatrix}1\\ s\end{bmatrix}. \tag{47}$$

Substituting for $[\Delta(s)]$ and $[D]$ and simplifying the above equation results in,

$$TF = \frac{\alpha_1(s) + s\alpha_2(s) + d_1\beta(s) + d_2 s\beta(s)}{\beta(s)} = \frac{P(s)}{Q(s)} \tag{48}$$

where $P(s)$ and $Q(s)$ represent the numerator and denominator polynomials of TF respectively.

Determination of Poles

Poles of TF are the roots of the polynomial $Q(s)$. Referring to (48) and (46) yields,

$$Q(s) = \prod_{i=1}^{\nu}(s - \lambda_i). \tag{49}$$

Therefore, poles of TF are $\lambda_1, \lambda_2, \ldots, \lambda_\nu$ which are same as eigenvalues of the system matrix $[A]$ and have already been determined.

Determination of Zeros

Zeros of TF are the roots of the polynomial $P(s)$. From (48),

$$P(s) = \alpha_1(s) + s\alpha_2(s) + d_1\beta(s) + d_2 s\beta(s). \tag{50}$$

Coefficients of the polynomial $P(s)$ can be obtained through the algebraic manipulation of the coefficients of polynomials $\alpha_1(s)$, $\alpha_2(s)$ and $\beta(s)$, which can be extracted, as illustrated below.

Referring to (46), since $\beta(s)$ is expressed as the product of ν factors (each of the form $(s - \lambda_i)$), the extraction of coefficients from the polynomial $\beta(s)$ having $\lambda_1, \lambda_2, \ldots, \lambda_\nu$ as its roots becomes a simple task (a built-in function for extracting coefficients of a polynomial with known roots, 'poly' exists in Matlab). The order of s in the polynomial $\beta(s)$ is ν and hence the extracted values will correspond to the coefficients of $s^\nu, s^{\nu-1}, \ldots, s^1, s^0$ in the polynomial $\beta(s)$.

Similarly, coefficients of the polynomials $\alpha_1(s)$ and $\alpha_2(s)$ can also be extracted from (44) and (45). The order of s in these polynomials is $\nu - 1$ and hence the extracted values will correspond to the coefficients of $s^{\nu-1}, s^{\nu-2}, \ldots, s^1, s^0$ in both the polynomials. Therefore, the extracted coefficients of the polynomials can be expressed in a form shown below,

$$\begin{matrix} Polynomial \\ \begin{bmatrix} \beta(s) \\ \alpha_1(s) \\ \alpha_2(s) \end{bmatrix} \end{matrix} \quad \mapsto \quad \begin{matrix} coefficients \\ \begin{bmatrix} \beta^\nu & \beta^{\nu-1} & \ldots & \beta^1 & \beta^0 \\ & \alpha_1^{\nu-1} & \ldots & \alpha_1^1 & \alpha_1^0 \\ & \alpha_2^{\nu-1} & \ldots & \alpha_2^1 & \alpha_2^0 \end{bmatrix} \end{matrix}. \tag{51}$$

Coefficients of $P(s)$ can be constructed easily from (51). Summation of the coefficients of like powers of s of the polynomials involved in $P(s)$ yields the coefficients of $P(s)$. For doing so, coefficients of the polynomials $\alpha_2(s)$, $\beta(s)$ are to be left-shifted by one position so as to have one-to-one correspondence between the polynomials $s\alpha_2(s)$, $s\beta(s)$ and the coefficients representing them. Finally, zeros of TF are the roots of the coefficients of $P(s)$ thus obtained.

Since zeros and poles of TF are computed through manipulation of the matrices which are numeric in nature, without involving any symbol, the process will require lesser computing time and memory. Thus, an entirely symbol-free computation of poles and zeros of TF has been achieved.

At this juncture it must be pointed out that, the above described method of manipulating matrices of TF equation to arrive at a symbol-free computation, appears to be also applicable (though not explicitly shown) for the approach described in [5] and [6].

TABLE I
CPU TIME FOR PROPOSED METHOD

Sections per winding ($n_1 = n_2$)	2	10	50	100	150	200	250
Time (s)	0.1	0.2	7	44	145	354	709

V. RESULTS AND DISCUSSION

The proposed method was implemented in MATLAB (version 6). The CPU time (system used was P4, 1.8 GHz, 256 MB RAM) needed to determine the poles and zeros of TF for the circuit in Fig. 1, with different number of sections per winding, are presented in Table I (typical values of circuit parameters were chosen).

1) It can be seen from Table I that, the state space analysis method is computationally very efficient and thus enables even large-sized networks to be solved within a reasonable time frame. The reason for considering 250 sections/winding is purely academic, and to show that, the proposed method can handle networks of even such huge dimensions without any difficulty. Thus, it implies that, detailed modeling of transformer windings becomes practical.
2) Sample results (TF plot, and its pole-zero plot) are shown in Figs. 2 and 3 corresponding to a circuit (as in Fig. 1) wherein the primary is interleaved and the secondary is a disc winding (short circuited). Both neutrals are grounded. With each winding represented by 10 sections, the highest natural frequency is about 1.2 MHz. Fig. 2 is the TF plot obtained by sweeping s. Fig. 3 shows the TF pole-zero plot where occurrence of near pole-zero cancellations can easily be detected (by inspection of numerical values), whereas the same is not so easy to comprehend from the TF plot, especially when the incremental frequency used to compute TF is not infinitesimally small. Those of the zeros that lie closer to the $j\omega$-axis and in close proximity to a pole will have a greater influence on the resultant TF shape. Further, the minima of the TF plot need not necessarily always correspond to the TF zero, and this fact is evident from the pole-zero plot (zeros have positive real parts). Thus, benefit of knowing both poles and zeros is briefly mentioned.
3) Case studies were also carried out by varying the series capacitance (C_s *or* C_s') so as to simulate different types of windings (disc, interleaved). No problems were faced in any of the cases (in spite of the existence of near pole-zero cancellations), illustrating the robustness of the proposed method.
4) It is also possible to determine the voltage distributions at various nodes (since node voltages are the state variables and known) without much effort. The voltage distribution at node 2 of the primary (for the system in Fig. 3) is shown in Fig. 4(b) for a 1 p.u standard lightning impulse (Fig. 4(a)) excitation.
5) The neutral current response to any full or chopped lightning impulse can also be easily determined. As an example, Fig. 4(c) shows the neutral current response.

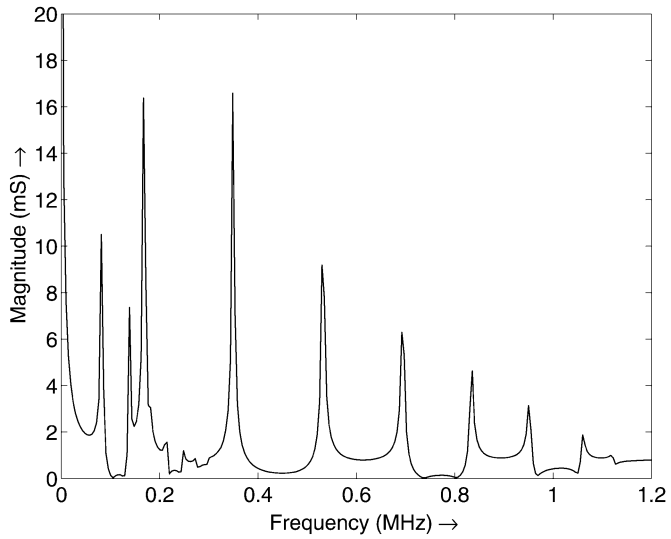

Fig. 2. TF plot.

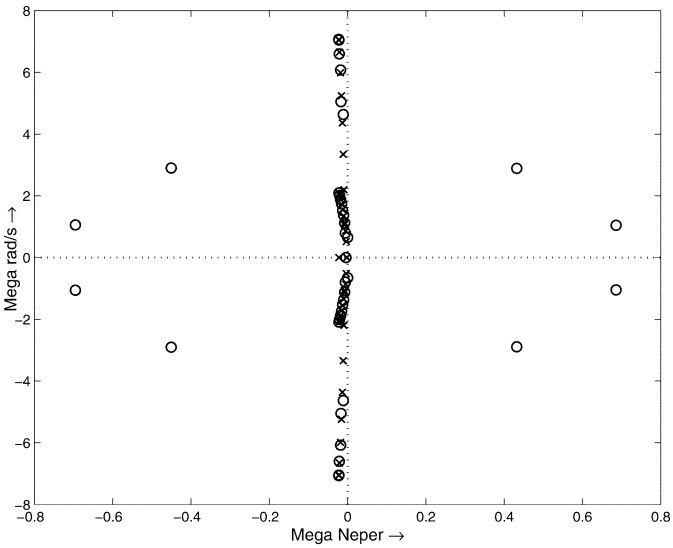

Fig. 3. TF pole-zero plot.

6) Diagonalization of the system matrix $[A]$ is a crucial step, and hence it has to be ascertained, whether or not, it is achievable under all conditions. Diagonalization is possible, if and only if, there exist ν independent eigenvectors corresponding to ν eigenvalues. With ν distinct eigenvalues, it is possible to find ν linearly independent eigenvectors, posing no problem for diagonalization. But, if an eigenvalue λ_m gets repeated 'm' times, it need not always be possible to find 'm' independent eigenvectors corresponding to λ_m. In such cases, suppose if, there is a shortage of eigenvectors, then diagonalization will fail. To find 'm' independent eigenvectors corresponding to the repeated eigenvalue λ_m, the necessary condition to be satisfied is that, the rank of $[A - \lambda_m I]$ should be equal to $(\nu - m)$ [12]. In order to verify these possibilities, simulations were carried out for different terminal conditions, from which the following points emerge.
   - With secondary open circuited (with or without loss) or secondary short circuited (with loss), there always existed ν distinct eigenvalues and hence ν independent eigenvectors.
   - With secondary short circuited (lossless case), the solution always results in an eigenvalue of $\lambda_2 = 0$ with a multiplicity of $m = 2$. In such lossless cases, the rank of $[A - \lambda_2 I]$ always equals $\nu - 2$, which satisfies the above requirement.

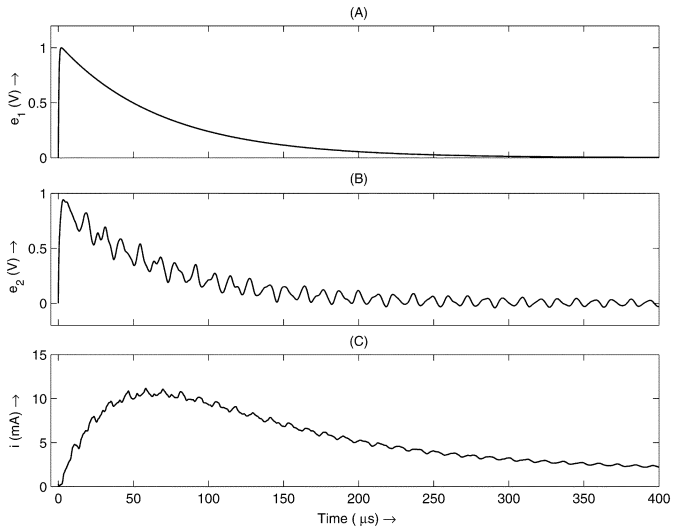

Fig. 4. (A) Std. lightning impulse, (B) voltage at node 2, (C) neutral current.

(Note: An eigenvalue, at the origin, of multiplicity 2 should not be interpreted as a double pole at the origin. The solution also includes a zero at the origin cancelling one of the poles, thus effectively leaving behind only one pole at the origin.)

The above findings demonstrate that, the system matrix is always diagonalisable, irrespective of the terminal conditions, thus, validating the solution method to be suitable for all terminal conditions.

VI. Conclusions

In this paper, a novel method based on state space approach to compute the poles and zeros of transfer function for an equivalent circuit model of a transformer has been presented. The versatility of the proposed method with reference to the number of windings, ability to represent very large number of sections and terminal conditions, require less computational time, and being free from computational problems reported in earlier methods, was clearly illustrated.

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