

An Efficient Wavelet Based Approximation Method to Time Fractional Black-Scholes European Option Pricing Problem Arising in Financial Market

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Abstract

In this paper, a wavelet based hybrid method is employed to provide the quick and accurate solutions of fractional Black-Scholes equation with boundary condition for a European option pricing (EOP) problem. The fractional Black-Scholes is used as a model for valuing European or American call and put options on a non-dividend paying stock. To the best of our knowledge, until now there is no rigorous Legendre wavelet solutions have been reported for the fractional Black-Scholes

equations (BSE). The fundamental idea of wavelet method is to convert the fractional Black-Scholes equations into a group of algebraic equations, which involves a finite number of variables. The illustrative examples are given to demonstrate the applicability and validity of the method. Moreover the use of wavelets is found to be simple, flexible, accurate and small computation costs.

Keywords: Black-Scholes equation, Legendre wavelets, Laplace transforms method, Operational matrices

1. Introduction

Wavelet Analysis, as a relatively new and emerging area in Applied Mathematical Research, has received considerable attention in dealing with PDEs [27-34]. In the last decades, fractional calculus found many applications in various fields of physical sciences such as viscoelasticity, diffusion, control, relaxation processes, signal processing, electromagnetism, biosciences, fluid mechanics, electrochemistry, fluid mechanics and so on. Fractional differential equations are extensively used in modeling phenomena in various fields of science and engineering [16,17,18-20,35].

In 1973, Fischer Black and Myron Scholes [8] established the famous theoretical scheme for options which earned them the 1997 Nobel Prize in Economics. The equilibrium condition between the expected return on the option, the expected return on the stock and the riskless interest rate based on the stochastic model is the Black-Scholes equation. An important key idea of Black and Scholes lie in the formation of a riskless portfolio taking positions in bonds (cash), option and the underlying stock. Such a model strengthens the use of the no-arbitrage principle as well. During the past few decades, many researchers studied the existence of solutions of the Black Scholes model using many methods [1-7,9-15,21-23-25,26,36]. Razzaghi and Yousefi [31] introduced the Legendre wavelet method for solving variational problems and constrained optimal control problems. Mohammadi and Hosseini [33] had showed a new Legendre wavelet operational matrix of derivative in solving singular ordinary differential equations. Yousefi [32] introduced the Legendre wavelets for solving Lane-Emden type differential equations. Recently, Fukang Yin et al. [34] introduced a coupled method of Laplace Transform and Legendre wavelets for Lane-Emden type differential Equations.

In this work, making use of nice properties of Haar wavelets and the operational matrix, we consider the following the Black-Scholes model (BSM) for the value of an option is described by the equation

$$\frac{\partial U}{\partial t} + \frac{\sigma^2 x^2}{2} \frac{\partial^2 U}{\partial x^2} + r(t)x \frac{\partial U}{\partial x} - r(t)U = 0, \quad (x,t) \in R^+ \times (0,T), \quad (1.1)$$

where $U(x,t)$ is the European call option price at asset price x and at time t , K

is the exercise price, T is the maturity, $r(t)$ is the risk free interest rate, and $\sigma(x,t)$ represents the volatility function of underlying asset. Let us denote $U_c(x,t)$ and $U_p(x,t)$ are the values of the European call and put options, respectively. Then, the payoff functions are

$$\left. \begin{aligned} U_c(x,t) &= \max(x - E, 0) \\ U_p(x,t) &= \max(E - x, 0) \end{aligned} \right\} \tag{1.2}$$

where E denotes the expiration price for the option and the function $\max(x,0)$ gives the larger value between x and 0. The main aim of this work is to apply the Laplace Legendre wavelet method (LLWM) to solve the fractional order Black-Scholes equation.

The paper is organized as follows: In Sec.2 the basic definitions and properties of the fractional Riemann-Liouville integral and Caputo fractional derivatives are briefly mentioned. For complete sake of Legendre wavelets methods and method of solution are presented in Sec.3. In Sec.4 Illustrative examples are presented for applicability and validity of the proposed wavelet methods. Finally, Sec.5 is dedicated to conclusion.

2. Definitions of fractional derivatives and integrals

The Caputo fractional derivative allows the utilization of initial and boundary conditions involving integer order derivatives, which have clear physically interpretations.

(1) Riemann-Liouville definition:

$${}^R D_t^\alpha f(t) = \begin{cases} \frac{d^m f(t)}{dt^m}, & \alpha = m \in N; \\ \frac{d^m}{dt^m} \frac{1}{\Gamma(m-\alpha)} \int_a^t \frac{f(T)}{(t-T)^{\alpha-m+1}} dT, & 0 \leq m-1 < \alpha < m. \end{cases} \tag{2.1}$$

Fractional integral of order α is as follows:

$${}^R I_t^\alpha f(t) = \frac{1}{\Gamma(-\alpha)} \int_0^t (t-T)^{-\alpha-1} f(T) dT, \quad \alpha < 0. \tag{2.2}$$

(2) Caputo definition:

$${}^c D_t^\alpha f(t) = \begin{cases} \frac{d^m f(t)}{dt^m}, & \alpha = m \in N; \\ \frac{1}{\Gamma(m-\alpha)} \int_a^t \frac{f^{(m)}(T)}{(t-T)^{\alpha-m+1}} dT, & 0 \leq m-1 < \alpha < m. \end{cases} \tag{2.3}$$

3. Legendre wavelet method (LWM) preliminaries

The Legendre wavelets are defined by

$$\psi_{nm}(t) = \begin{cases} \sqrt{m + \frac{1}{2}} 2^{\frac{k}{2}} L_m(2^k t - \hat{n}), & \text{for } \frac{\hat{n}-1}{2^k} \leq t \leq \frac{\hat{n}+1}{2^k} \\ 0, & \text{otherwise} \end{cases}, \tag{3.1}$$

Where $m = 0, 1, 2, \dots, M-1$, and $n = 1, 2, \dots, 2^{k-1}$. The coefficient $\sqrt{m + \frac{1}{2}}$ is for orthonormality, then, the wavelets $\Psi_{k,m}(x)$ form an orthonormal basis for $L^2[0,1]$. In the above formulation of Legendre wavelets, the Legendre polynomials are in the following way:

$$\begin{aligned} p_0 &= 1, \\ p_1 &= x, \\ p_{m+1}(x) &= \frac{2m+1}{m+1} x p_m(x) - \frac{m}{m+1} p_{m-1}(x). \end{aligned}$$

(3.2)

and $\{p_{m+1}(x)\}$ are the orthogonal functions of order m , which is named the well-known shifted Legendre polynomials on the interval $[0,1]$. Note that, in the general form of Legendre wavelets, the dilation parameter is $a = 2^{-j}$ and the translation parameter is $b = n 2^j$.

Theorem 3.1:

Let $\Psi(x,y)$ be the two – dimensional Legendre wavelets vector, we have

$$\frac{\partial \Psi(x,y)}{\partial x} = D_x \Psi(x,y)$$

(3.3)

where D_x is $2^{k-1} \times 2^{k-1}$ matrix and has the form as follows:

$$D_x = \begin{bmatrix} D & 0' & \dots & 0' \\ 0' & D & \dots & 0' \\ \vdots & \vdots & \ddots & \vdots \\ 0' & 0' & \dots & D \end{bmatrix}$$

In which $0'$ and D is $2^{k-1} \times 2^{k-1}$ matrix and the element of D is defined as follows:

$$D_{r,s} = \begin{cases} 2^{k'}\sqrt{(2r-1)(2s-1)} & I, r = 2,3, \dots M; s = 1, \dots r-1; r+s \text{ is odd} \\ 0 & \text{otherwise} \end{cases} \tag{3.4}$$

and I, O are $2^{k'-1}M' \times 2^{k'-1}M'$ identity matrix.

Theorem 3.2: Let $\Psi(x,y)$ be the two-dimensional Legendre wavelets vector and we have $\frac{\partial \Psi(x,y)}{\partial x} = D_y \Psi(x,y)$, (3.5)

$$D_y = \begin{bmatrix} D & O' & \dots & O' \\ O' & D & \dots & O' \\ \vdots & \vdots & \ddots & \vdots \\ O' & O' & \dots & D \end{bmatrix},$$

where D_y is $2^{k'-1} \times 2^{k'-1}MM' \times 2^{k'-1} \times 2^{k'-1}MM'$ and O', D is $MM' \times MM'$ matrix is given as

$$D = \begin{bmatrix} F & O & \dots & O \\ O & F & \dots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \dots & F \end{bmatrix},$$

in which O and F is $M' \times M'$ matrix and F is defined as follows:

$$F_{r,s} = \begin{cases} 2^{k'}\sqrt{(2r-1)(2s-1)}, & r = 2, \dots, M'; S = 1, \dots, r-1; \text{and } r+s \text{ is odd} \\ 0, & \text{otherwise} \end{cases} \tag{3.6}$$

The operational matrices for nth derivative can be derived as

$$\frac{\partial^n \Psi(x,y)}{\partial x^n} = D_x^n \Psi(x,y), \frac{\partial^m \Psi(x,y)}{\partial y^m} = D_y^m \Psi(x,y)$$

$$\frac{\partial^{n+m} \Psi(x,y)}{\partial x^n \partial y^m} = D_x^n D_y^m \Psi(x,y),$$

Where D^n is the nth power of matrix D .

3.1 Block Pulse Functions (BPFs)

The block pulse functions form a complete set of orthogonal functions which defined on the interval $[0, b)$ by

$$b_i(t) = \begin{cases} 1, & \frac{i-1}{m} b \leq t < \frac{i}{m} b, \\ 0, & \text{elsewhere} \end{cases} \tag{3.7}$$

for $i = 1, 2, \dots, m$. It is also known that for any absolutely integrable function $f(t)$ on $[0, b)$ can be expanded in block pulse functions:

$$f(t) \cong \xi^T B_m(t) \tag{3.8}$$

$$\xi^T = [f_1, f_2, \dots, f_m], B_m(t) = [b_1(t), b_2(t), \dots, b_m(t)] \tag{3.9}$$

where f_i are the coefficients of the block-pulse function, given by

$$f_i = \frac{m}{b} \int_0^b f(t) b_i(t) dt \tag{3.10}$$

Remark 1: Let A and B are two matrices of m x m, then $A \otimes B = (a_{ij} \times b_{ij})_{mm}$.

Lemma 3.1: Assuming f(t) and g(t) are two absolutely integrable functions, which can be expanded in block pulse function as f(t) = FB(t) and g(t) = GB(t) respectively, then we have

$$f(t)g(t) = FB(t)B^T(t)G^T = HB(t) \tag{3.11}$$

where H = F ⊗ G.

3.2 Approximating the nonlinear term

The Legendre wavelets can be expanded into m- set of block-pulse Functions as

$$\Psi(t) = \Phi_{m \times m} B_m(t) \tag{3.12}$$

Taking the collocation points as following

$$t_i = \frac{i - 1/2}{2^{k-1}M}, i = 1, 2, \dots, 2^{k-1}M \tag{3.13}$$

The m-square Legendre matrix $\Phi_{m \times m}$ is defined as

$$\Phi_{m \times m} \cong [\Psi(t_1) \ \Psi(t_2) \ \dots \ \Psi(t_{2^{k-1}M})] \tag{3.14}$$

The operational matrix of product of Legendre wavelets can be obtained by using the properties of BPFs, let $f(x, t)$ and $g(x, t)$ are two absolutely integrable functions, which can be expanded by Legendre wavelets as $f(x, t) = \Psi^T(x)F\Psi(t)$ and $g(x, t) = \Psi^T(x)G\Psi(t)$ respectively. Then

$$f(x, t) = \Psi^T(x)F\Psi(t) = B^T(x)\Phi_{m \ m}^T F\Phi_{m \ m} B(t), \tag{3.15}$$

$$g(x, t) = \Psi^T(x)G\Psi(t) = B^T(x)\Phi_{m \ m}^T G\Phi_{m \ m} B(t), \tag{3.16}$$

and $F_b = \Phi_{m \ m}^T F\Phi_{m \ m}, G_b = \Phi_{m \ m}^T G\Phi_{m \ m}, H_b = F_b \otimes G_b$.

Then,

$$\begin{aligned} f(x, t)g(x, t) &= B^T H_b B(t), \\ &= B^T(x)\Phi_{m \ m}^T \text{inv}(\Phi_{m \ m}^T)H_b \text{inv}(\text{inv}(\Phi_{m \ m}^T)H_b \text{inv}(\Phi_{m \ m}))\Phi_{m \ m} B(t) \\ &= \Psi^T(x)H\Psi(t) \end{aligned} \tag{3.17}$$

where $H = \text{inv}(\Phi_{m \ m}^T)H_b \text{inv}(\Phi_{m \ m})$

3.3 Function Approximation.

A given function f(x) with the domain [0,1] can be approximated by:

$$f(x) = \sum_{k=1}^{\infty} \sum_{m=0}^{\infty} c_{k,m} \Psi_{k,m}(x) = C^T \cdot \Psi(x). \tag{3.18}$$

If the infinite series in Eq. (3.18) is truncated, then this equation can be written as:

$$f(x) \simeq \sum_{k=1}^{\infty} \sum_{m=0}^{\infty} c_{k,m} \Psi_{k,m}(x) = C^T \cdot \Psi(x). \tag{3.19}$$

where C and Ψ are the matrices of size $(2^{j-1}M \times 1)$.

$$C = [c_{1,0}, c_{1,1}, \dots, c_{1,M-1}, c_{2,0}, c_{2,1}, \dots, c_{2,M-1}, \dots, c_{2^{j-1},1}, \dots, c_{2^{j-1},M-1}]^T \tag{3.20}$$

$$\Psi(x) = [\Psi_{1,0}, \Psi_{1,1}, \Psi_{2,0}, \Psi_{2,1}, \dots, \Psi_{2^{j-1},M-1}]^T. \tag{3.21}$$

Consider the Black-Scholes option pricing equation as follows:

$$\frac{\partial U(x,t)}{\partial t} = \frac{\partial^2 U(x,t)}{\partial x^2} + (k-1) \frac{\partial U}{\partial x} - kU(x,t), \quad 0 < \alpha \leq 1, \tag{3.22}$$

Taking Laplace transform on both sides of Eq. (3.22), we get

$$sL(U) - U(x,0) = L[U_{xx} + (k-1)U_x - kU] \tag{3.23}$$

$$L(U) = \frac{U(x,0)}{s} + \frac{1}{s} L(U_{xx} + (k-1)U_x - kU) \tag{3.24}$$

Taking inverse Laplace transform to Eq. (3.24) we get

$$U(x,t) = U(x,0) + L^{-1} \left(\frac{1}{s} L(U_{xx} + (k-1)U_x - kU) \right) \tag{3.25}$$

Because

$$\begin{aligned} L^{-1} \left[\frac{1}{s} L(t^n) \right] &= L^{-1} \left(\frac{n!}{s^{n+2}} \right) \\ &= \frac{1}{n+1} t^{n+1}; (n = 0, 1, 2, \dots) \end{aligned} \tag{3.26}$$

We have

$$L^{-1}[s^{-1}L()] = \int_0^t (.) dt \tag{3.27}$$

From Eq.(3.25)

$$U(x,t) = U(x,0) + L^{-1} \left(\frac{1}{s} L(U_{xx} + f(x) + g(U)) \right) \tag{3.28}$$

Where

$$\left. \begin{aligned} g(U) &= -kU \\ f(x) &= (k-1)U_x \end{aligned} \right\} \tag{3.29}$$

$$U(x,t) = U(x,0) + L^{-1} \left(\frac{1}{s} L(U_{xx} + f(x) + g(U)) \right) \tag{3.30}$$

By using the Legendre wavelets method,

$$\left. \begin{aligned} U(x,t) &= C^T \psi(x,t) \\ U(x,0) &= S^T \psi(x,t) \\ g(U) &= G^T \psi(x,t) \\ f(x) &= F^T \psi(x,t) \end{aligned} \right\} \tag{3.31}$$

Substituting Eq.(3.31) in Eq.(3.30), we obtain

$$C^T = S^T + (C^T D \hat{x} + \hat{F} + \hat{G}) P_t^2 \tag{3.32}$$

Here G^T has a linear relation with C. When we solve a linear algebraic system, we get the solution is more complex and large computation time. In order to overcome

the above drawbacks, we introduce an approximation formula as follows:

$$U_{n+1} = U(x, 0) + \Pi \left[\frac{\partial^2 U_n}{\partial x^2} + k \frac{\partial U_n}{\partial x} - \frac{\partial U_n}{\partial x} + g(U_n) \right] \quad (3.33)$$

where $g(U) = -kU$

Expanding $u(x,t)$ by Legendre wavelets using the following relation

$$C_{n+1}^T = C_0^T + [C_n^T D_x^2 + F^T - G_n^T] P_t^2 \quad (3.34)$$

From the above formula, the wavelet coefficients can be calculated successively.

4. Numerical examples

In this section, two examples are given for demonstrating the validity and applicability of the proposed method.

Example 1. Consider the fractional Black-Scholes option pricing equation as follows [2,25,36]:

$$\frac{\partial^\alpha U(x,t)}{\partial t^\alpha} = \frac{\partial^2 U(x,t)}{\partial x^2} + (k-1) \frac{\partial U}{\partial x} - kU(x,t), \quad 0 < \alpha \leq 1, \quad (4.1)$$

with the initial condition $U(x, 0) = \max(e^x - 1, 0)$.
(4.2)

We notice that this system of equations contains just two dimensionless parameters $k = 2r / \sigma^2$, where k represents the balance between the rate of interests and the variability of stock returns and the dimensionless time to expiry $\frac{1}{2} \sigma^2 T$.

Using Homotopy perturbation method (HPM), the exact solution is given by

$$U(x,t) = \max(e^x - 1, 0) E_\alpha(-kt^\alpha) + \max(e^x, 0) (1 - E_\alpha(-kt^\alpha)) \quad (4.3)$$

Setting, $\alpha = 1$, the exact solution in a closed form is given by

$$U(x,t) = \max(e^x - 1, 0) e^{-kt} + \max(e^x, 0) (1 - e^{-kt}) \quad (4.4)$$

Here $E_\alpha(z)$ is Mittag-Leffler function in one parameter.

The Laplace Legendre wavelet (LLW) scheme is given by

$$C_{n+1}^T = C_0^T + [C_n^T D_x^2 + F^T - G_n^T] P_t^2 \quad (4.5)$$

The calculating results show that combining with wavelet matrix, the method in this paper can be effectively used in numerical calculus for constant coefficient fractional differential equations, and that the method is feasibility.

Example 2. Consider the Black-Scholes equation [11,14]

$$\frac{\partial U(x,t)}{\partial t} + x^2 \frac{\partial^2 U(x,t)}{\partial x^2} + 0.5x \frac{\partial U}{\partial x} - U(x,t) = 0, \quad 0 < \alpha \leq 1, \tag{4.6}$$

with initial condition $U(x,0) = x^3$ (4.7)

The exact solution in a closed form is given by

$$U(x,t) = x^3 e^{-6.5t} \tag{4.8}$$

Table.1 Numerical results for Example.2

(x,t)	U_{VIM} [36]	U_{ADM} [9]	U_{HPM} [15]	U_{LLWM}
(0.1,0.13)	0.0063281	0.0092646	0.0073609	0.0045554
(0.2,0.18)	0.0064473	0.0093652	0.0074447	0.0045367
(0.3,0.27)	0.0064603	0.0094713	0.0075226	0.0046228
(0.4,0.32)	0.0065227	0.0094747	0.0075656	0.0046818
(0.5,0.38)	0.0065863	0.0095805	0.0076267	0.0047882
(0.7,0.43)	0.0066178	0.0096129	0.0076926	0.0047823

Table.1 shows the comparison between the proposed wavelet methods and other methods. Fig.1 shows the accuracy of the methods for Example.1. It is worth mentioning that the LLWM provide excellent results when compared with exact solution.

All the numerical experiments presented in this section were computed in double precision with some MATLAB codes on a personal computer System Vostro 1400 Processor x86 Family 6 Model 15 Stepping 13 Genuine Intel ~1596 Mhz.

5. Conclusion

In this work, the Laplace Legendre wavelet method (LLWM) has been successfully employed to obtain the numerical solutions of the linear fractional Black-Scholes equation with boundary condition for a European option pricing (EOP) problem. The proposed schemes are the capability to overcome the difficulty arising in calculating the integral values while dealing with nonlinear fractional partial differential equations. This method shows higher efficiency than the traditional Legendre wavelet method for solving fractional PDEs. The execution time for LLWM is less than that variational iteration method (VIM), homotopy perturbation method (HPM) and also the homotopy analysis method (HAM). These two wavelet methods can be easily extended to find the solution of all other non-linear differential equations. The proposed wavelet results are in excellent agreement with the exact solution and those obtained by the Adomian decomposition method (ADM), Homotopy perturbation method (HPM), Homotopy analysis method (HAM) and the differential transform method (DTM). The numerical solutions obtained using the proposed method show that the solutions are in very good coincidence with the exact

solution.

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