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An Elastic-Viscous-Plastic Model for Sea Ice Dynamics

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Abstract. The standard model for sea ice dynamics treats the ice pack as a viscous-plastic material that flows plastically under typical stress conditions but behaves as a linear viscous fluid where strain rates are small and the ice becomes nearly rigid. Because of large viscosities in these regions, implicit numerical methods are necessary for timesteps larger than a few seconds. Current solution methods for these equations use iterative relaxation methods, which are time consuming, scale poorly with mesh resolution, and are not well adapted to parallel computation. To remedy this, we have developed and tested two separate methods. First, by demonstrating that the viscous-plastic rheology can be represented by a symmetric, negative definite matrix operator, we have implemented the much faster and better behaved preconditioned conjugate gradient method. Second, realizing that only the response of the ice on time scales associated with wind forcing need be accurately resolved, we have modified the model so that it reduces to the viscous-plastic model at these time scales, while at shorter time scales the adjustment process takes place by a numerically more efficient elastic wave mechanism. This modification leads to a fully explicit numerical scheme which further improves the model's computational efficiency and is a great advantage for implementations on parallel machines.
Furthermore, we observe that the standard viscous-plastic model has poor dynamic response to forcing on a daily time scale, given the standard time step (1 day) used by the ice modeling community. In contrast, the explicit discretization of the elastic wave mechanism allows the elastic-viscous-plastic model to capture the ice response to variations in the imposed stress more accurately. Thus, the elastic-viscous-plastic model provides more accurate results for shorter time scales associated with physical forcing, reproduces viscous-plastic model behavior on longer time scales, and is computationally more efficient overall.

1. Introduction

A model of sea ice dynamics predicts the movement of the ice pack based on winds, ocean currents, and a model of the material strength of the ice. Nonuniform motion of the ice is responsible for the thickness and extent of the ice pack, which in turn influences the exchange of energy between the atmosphere and polar oceans. The dynamic characteristics of sea ice thereby play an essential role in climate-related processes of the ocean and atmosphere.

Many models have been developed to describe the ice dynamics. Some early studies focused on free drift descriptions with no ice interaction (Felzenbaum, 1961; Bryan et al., 1975; Manabe et al., 1979; Parkinson and Washington, 1979); others included more complex sea ice rheologies, treating the ice as a Newtonian viscous fluid (Campbell, 1965), a linear viscous fluid (Hibler, 1974; Hibler and Tucker, 1979), or a plastic material. The Arctic Ice Dynamics Joint Experiment in the 1970's proposed an elastic-plastic rheology for the sea ice pack (Coon et al., 1974), and several other nonlinear plastic rheologies have been studied since then (e.g. Pritchard et al., 1977; Flato and Hibler, 1990; Ip et al., 1991). A nonlinear viscous-plastic (VP) rheology proposed by Hibler (1979) has become the standard sea ice dynamics model and the basis for many recent sea ice studies.

The VP model suffers from numerical difficulties related to the enormous range of effective viscosities present in the model, and requires large computational resources that become particularly cumbersome when the model is coupled to an ocean or atmophere model (Hibler and Bryan, 1987; Oberhuber, 1993a, b). To avoid the stringent time step restriction for stability of an explicit numerical scheme in regions where the ice is relatively rigid, the model equations are typically solved with implicit methods such as successive overrelaxation (Hibler, 1979) and line relaxation (Oberhuber, 1993a; Holland et al., 1993). However, these methods suffer from poor convergence characteristics as the mesh resolution is increased. Attempts to overcome the inherent problems of the model have included improved numerical methods as well as simplifications of the model itself. As part of this paper, we present a more efficient implicit numerical method for solving the VP model equations that uses preconditioned conjugate gradients.

Simpler versions of the VP model, such as free drift descriptions with no ice interaction and cavitating fluid models in which the ice has no resistance to shear forces (Nikiforov et al., 1967; Flato and Hibler, 1989, 1990, 1992), are more tractable numerically, but the model behavior is sensitive to these simplifications (Holland et al., 1993). Likewise, simulations with more complicated rheologies than the standard elliptical yield curve (Hibler, 1979), such as teardrop (Coon et al., 1974), sine wave lens (Bratchie, 1984), Mohr-Coulomb and square (Ip et al., 1991) shapes, show that the rheology can have a significant effect on long-term simulations of ice drift (Ip et al., 1991). Since an ice model need only simulate a viscous-plastic material at time scales on the order of those imposed by wind forcing (days), we also present a modification of the model, the addition of elastic behavior, that realizes significant gains in numerical efficiency, reduces to the original VP model behavior at long time scales, and is more accurate for transients. Our model avoids the difficulties of the early elastic-plastic models (Pritchard, 1975; Colony and Pritchard, 1975), because the elastic-like behavior is not intended to be physically realistic and is introduced for numerical expediency.

The VP model also suffers from inaccuracies in calculating transient behavior. For example, given daily time steps, the VP model behavior is acceptable only for surface stresses that vary on the order of a week or more. Hibler (1979) states that several time steps are needed between changes in the forcing (he uses 8-day averaged winds with a 1-day time step), and more recently, Stössel et al. (1994) have noted that the sea ice components of some ice-ocean coupled models are slow to converge, especially under daily forcing. The VP numerical model does produce correct transient behavior if the time step is taken sufficiently small, on the order of minutes for 1-day forcing time scales. Our implementation of the elastic-viscous-plastic (EVP) model is more accurate in resolving transients, even using relatively large time steps, and therefore will produce more accurate ice behavior.

The VP ice dynamics model is not well suited to parallel architectures. Implicit methods required for larger time steps typically entail a great deal of communication between processors, making parallel computation less attractive. Therefore, explicit models are generally preferable for parallel implementations. Ip et al. (1991) optimized the VP model for multiprocessor computers using an explicit, Euler time-stepping scheme, but stability requirements of the numerical method severely limited the time step. The new EVP model presented in this paper permits a fully explicit implementation with an acceptably long time step. Its efficiency is compared with three methods of solving the viscous-plastic equations: the preconditioned conjugate gradient method and two relaxation schemes (Hibler, 1979; Zhang and Hibler, 1994).

The present work is part of an effort to develop a computationally efficient sea ice component for a fully coupled atmosphere-ice-ocean global climate model. The sea ice model, which also includes thermodynamic and transport components, is designed to be compatible with the Parallel Ocean Program (POP), an ocean circulation model developed at Los Alamos National Laboratory for use on massively parallel computers (Smith et al., 1992; Dukowicz et al., 1993, 1994).

2. The Ice Dynamics Model

2.1. Viscous-plastic model equations

Pack ice typically consists of rigid plates which may drift freely in areas of relatively open water or be closely packed together in regions of high ice concentration. Although individual ice floes range from tens of meters to several kilometers across, the ice pack is considered to be a highly fractured two dimensional continuum, to make modeling it tractable (Pritchard, 1975; Rothrock, 1975b; Hibler, 1980; Gray and Morland, 1994). Roed and O'Brien (1983) assert that the continuum hypothesis is valid for grid lengths as short as 1 km.

The force balance per unit area in the ice pack is given by a two-dimensional momentum equation (Hibler, 1979), obtained by integrating the 3D equation through the thickness of the ice in the vertical direction and averaging in the horizontal directions:

$$m\frac{\partial u_i}{\partial t} = \frac{\partial \sigma_{ij}}{\partial x_j} + \tau_{ai} + \tau_{wi} + \epsilon_{ij3} m f u_j - mg\frac{\partial H_o}{\partial x_i}, \quad (1)$$

where $\vec{\tau}_a = (\tau_{ai}, \tau_{aj})$ and $\vec{\tau}_w = (\tau_{wi}, \tau_{wj})$ are wind and ocean stresses, respectively, assumed to be of the form

$$\begin{aligned} \vec{\tau}_a &= c_a \rho_a \left|\vec{U}_a\right| \left(\vec{U}_a \cos\phi + \hat{k} \times \vec{U}_a \sin\phi\right), \\ \vec{\tau}_w &= c_w \rho_w \left|\vec{U}_w - \vec{u}\right| \left[\left(\vec{U}_w - \vec{u}\right)\cos\theta \right. \\ &\quad \left. + \hat{k} \times \left(\vec{U}_w - \vec{u}\right)\sin\theta\right]. \end{aligned} \quad (2)$$

The strength of the ice is represented by the internal stress tensor σ_{ij}. Definitions of the other variables and constants are given in Tables 1 and 2.

There has been a great deal of disagreement about the relative importance of the various terms in (1) (Parkinson and Washington, 1979). The primary components are the air and water stresses, Coriolis force, and ice interaction effects (Hibler, 1986); the most predominant of these is wind stress (Coon, 1980). Rothrock (1975a) demonstrated through scale analyses that the acceleration term is three orders of magnitude smaller than the stress terms. In contrast to Hibler (1979) and following Oberhuber (1993a), we neglect nonlinear advection, which is at least an order of magnitude smaller than the acceleration term. The ice interaction term is essential in balancing the stresses in much of the ice field (Hibler, 1979; Parkinson and Washington, 1979; Coon, 1980; Hibler, 1986), and although they are smaller in magnitude, current and tilt effects are significant over long periods of time (Hibler, 1986; Warn-Varnas et al., 1991).

The momentum equation must be consistent for any combination of ice and open water in a grid cell. Our particular model differentiates between thick and thin ice and tracks ice concentration with compactness, c, the fractional area of the cell covered with thick ice. When $c = 0$ there is no thick ice ($H = 0$), and there may be either thin ice ($h > 0$) or open water ($h = 0$). The mass m in (1) is the total mass of ice and snow per unit area, corresponding to $\int_0^H \rho\, dz$:

$$m = \rho_i \left[cH + (1-c)\,h\right] + \rho_s \left[cH_s + (1-c)\,h_s\right]. \quad (3)$$

Thin ice is assumed to have no strength. so that the internal stress tensor is nonzero only for thick ice. Since the surface stress terms τ_{ai} and τ_{wi} apply over the entire area, we see that thin ice in a cell that does not contain thick ice essentially exists in free drift, given by the momentum equation without ice interaction. In the special case when there is only open water, $m = 0$ and the "ice" velocity is that of the interface between atmosphere and ocean, calculated with $\vec{\tau}_a + \vec{\tau}_w = 0$.

The viscous-plastic rheology proposed by Hibler (1979) is given by a constitutive law that relates the internal ice stress σ_{ij} and the rates of strain $\dot{\epsilon}_{ij}$ through an internal ice pressure P and nonlinear bulk and shear viscosities, ζ and η, such that the principal components of stress lie on an elliptical yield curve with the ratio of major to minor axes e equal to 2. The constitutive law is given by

$$\sigma_{ij} = 2\eta\dot{\epsilon}_{ij} + (\zeta - \eta)\,\dot{\epsilon}_{kk}\delta_{ij} - P\delta_{ij}/2, \quad (4)$$

where

$$\dot{\epsilon}_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \quad (5)$$

Alternatively, this can be rewritten in the form

$$\frac{1}{2\eta}\sigma_{ij} + \frac{\eta - \zeta}{4\eta\zeta}\sigma_{kk}\delta_{ij} + \frac{P}{4\zeta}\delta_{ij} = \dot{\epsilon}_{ij}. \quad (6)$$

which will be useful to us later. This rheology allows the ice pack to diverge with little or no stress, yet

Table 1. Constants and parameters used in the dynamics equations.

c^*		20
c_a	air drag coefficient	0.0012
c_w	ocean drag coefficient	0.0055
δ_{ij}	Kronecker delta[a]	
ϵ_{ijk}	alternating tensor[b]	
e	yield curve axis ratio	2
g	gravitational acceleration	$980\,cm/s^2$
H_o	sea surface height	
$\hat{k}$	vertical unit vector	
P^*		$2.75 \times 10^5 dyne/cm^2$
ρ_a	air density	$0.0013\,g/cm^3$
ρ_i	sea ice density	$0.91\,g/cm^3$
ρ_s	snow density	$0.33\,g/cm^3$
ρ_w	seawater density	$1.03\,g/cm^3$
θ	water turning angle	25°
ϕ	air turning angle	25°

[a] $\delta_{ij} = 1$ if $i = j$ and 0 if $i \neq j$.
[b] $\epsilon_{ijk} = 0$ if any two indices are the same, 1 if the indices are in cyclical order, and -1 otherwise.

Table 2. Definitions of other symbols used in the dynamics equations, and their interdependencies.

Variable Quantities		Interdependence
c	compactness	$\vec{u}$
E	Young's modulus	c, H
$\dot{\epsilon}_{ij}$	strain rate tensor	$\vec{u}$
f	Coriolis parameter	
h	thickness of thin ice	$\vec{u}$
h_s	snow depth on thin ice	$\vec{u}$
H	thickness of thick ice	$\vec{u}$
H_s	snow depth on thick ice	$\vec{u}$
m	mass per unit area	c, H, H_s, h, h_s
P	pressure	c, H
ν	damping coefficient	ζ, m
η	shear viscosity	P, $\dot{\epsilon}_{ij}$
ζ	bulk viscosity	P, $\dot{\epsilon}_{ij}$
σ_{ij}	stress tensor	η, ζ, P, $\dot{\epsilon}_{ij}$
$\vec{\tau}_a$	wind stress	
$\vec{\tau}_w$	ocean stress	$\vec{u}$
$\vec{u}$	ice velocity	σ_{ij}, $\vec{\tau}_w$, m
$\vec{U}_a$	geostrophic wind	
$\vec{U}_w$	geostrophic ocean current	

resist compression and shearing motion under convergent conditions. The pressure P, a measure of ice strength, depends on both thickness and compactness: $P = P^* c H e^{-c^*(1-c)}$, where P^* and c^* are constants given in Table 1. The viscosities increase with pressure and with decreasing strain rates:

$$\zeta = \frac{P}{2\Delta}, \tag{7}$$

$$\eta = \frac{P}{2\Delta e^2}, \tag{8}$$

$$\Delta = \left[\left(\dot\epsilon_{11}^2 + \dot\epsilon_{22}^2\right)\left(1 + e^{-2}\right) + 4e^{-2}\dot\epsilon_{12}^2 + 2\dot\epsilon_{11}\dot\epsilon_{22}\left(1 - e^{-2}\right)\right]^{1/2}. \tag{9}$$

These parameters represent an idealized visco-plastic material whose effective viscosities become infinite in the limit of zero strain rate. Hibler (1979) chose to regularize this behavior by bounding the viscosities when the rates of strain are small and the ice pack moves as an essentially rigid solid; the viscosities are set to large, constant values so that the ice pack is treated as a linear viscous fluid undergoing very slow creep. The maximum value for ζ is $2.5 \times 10^8 P\, g/s$; η is similarly bounded through equations (7) and (8). He also set minimum values to provide against nonlinear instabilities, with $\zeta_{\min} = 4 \times 10^{11}\, g/s$. For a sufficiently small value of c, $\zeta_{\max} < \zeta_{\min}$, in which case $\zeta = \zeta_{\min}$. For a general account of constitutive laws for sea ice, see Hibler (1986).

Equations (1), (2), (5) and (6) may be combined as follows.

$$\begin{aligned} m\partial_t u = {} & \partial_x\left[(\eta+\zeta)\,\partial_x u\right] + \partial_y\left(\eta\partial_y u\right) \\ & + \partial_x\left[(\zeta-\eta)\,\partial_y v\right] + \partial_y\left(\eta\partial_x v\right) - \partial_x P/2 \\ & + c'\left[(U_w - u)\cos\theta - (V_w - v)\sin\theta\right] \\ & + \tau_{ai} + mfv - mg\partial_x H_o, \end{aligned} \tag{10}$$

$$\begin{aligned} m\partial_t v = {} & \partial_y\left[(\eta+\zeta)\,\partial_y v\right] + \partial_x\left(\eta\partial_x v\right) \\ & + \partial_y\left[(\zeta-\eta)\,\partial_x u\right] + \partial_x\left(\eta\partial_y u\right) - \partial_y P/2 \\ & + c'\left[(V_w - v)\cos\theta + (U_w - u)\sin\theta\right] \\ & + \tau_{aj} - mfu - mg\partial_y H_o, \end{aligned} \tag{11}$$

where $c' = \rho_w C_w \left|\vec{U}_w - \vec{u}\right|$.

2.2. Motivation for alternative methods and an elastic formulation

The difficulty in solving (10) and (11) is associated with the presence of shear strength ($\eta \neq 0$). The case $\eta = 0$ corresponds to the much simpler and easier to solve cavitating fluid model (e.g. Flato and Hibler, 1990). This difficulty may be illustrated for the case of divergence-free velocity ($\nabla\cdot\vec{u} = 0$) and constant η and m. Setting the pressure, surface stresses, Coriolis and tilt terms equal to $\vec{R}$, assumed known, the equations decouple to give

$$m\frac{\partial\vec{u}}{\partial t} = \eta\nabla^2\vec{u} + \vec{R}, \tag{12}$$

a simple parabolic equation. The one-dimensional stability condition for an explicit discretization of (12) is

$$\Delta t \le \frac{m}{2\eta}\Delta x^2. \tag{13}$$

Given the maximum value of viscosity allowed in the VP model, the time step is on the order of a second for a mesh spacing of about $100km$ (Ip et al., 1991), and a hundredth of a second at a resolution of about $10\,km$, which we anticipate in our application. This consideration led to the adoption of semi-implicit discretization schemes so that the equations could be integrated with a much less stringent time step. The solution methods currently in use are typically iterative relaxation methods (Hibler, 1979; Oberhuber, 1993a) whose rates of convergence scale asymptotically as $\left(1 - \alpha\Delta x^2\right)^k$ for simple test problems, where α is a positive constant and k is the number of iterations (Elman, 1994). Furthermore, iterative methods are usually recursive and therefore difficult to adapt to parallel machines. The conjugate gradient method, on the other hand, can be used successfully on parallel machines (Smith et al., 1992), and its convergence rate is linear with resolution (Elman, 1994). Use of a preconditioner further improves this method, but good preconditioners that are usable on parallel machines are hard to find.

Now consider a hyperbolic equation of the form

$$m\frac{\partial^2\vec{u}}{\partial t^2} = E\left(\nabla^2\vec{u} + \frac{\vec{R}}{\eta}\right) + damping. \tag{14}$$

We have constructed (14) so that it converges to the same steady state solution as (12), but by means of an adjustment process involving damped elastic waves. E is a parameter analogous to Young's modulus. The one-dimensional stability bound for an explicit discretization of (14) is

$$\Delta t \le \sqrt{\frac{m}{E}}\Delta x, \tag{15}$$

and it is possible to arrange the stability restriction due to damping to be subsumed in (15). Thus, one might expect such an explicit scheme to converge to the steady state with a convergence rate proportional to $(1-\alpha\Delta x)^k$, similar to that of optimum methods for parabolic equations. This provides a rationale for considering an elastic wave adjustment process while retaining the same steady or quasi-steady solution as in (12).

An elastic-plastic model has been proposed previously for sea ice on physical grounds, but it is not used in practice because of theoretical (Pritchard, 1975) and numerical (Colony and Pritchard, 1975) difficulties. The present model, on the other hand, uses an elastic-wave mechanism as a numerical artifice or a regularization method to overcome the stiffness of the viscous equations, and therefore it differs fundamentally from these previous models. In other words, we wish to retain the essence of the viscous-plastic rheology but make it much easier to solve numerically.

To construct such a model, it is usual to separate the strain rate into the sum of plastic and elastic contributions (Reuss, 1930). The plastic part has already been given by (6), and the elastic part is approximated by

$$\frac{1}{E}\frac{\partial \sigma_{ij}}{\partial t} = \dot{\epsilon}_{ij}, \qquad (16)$$

where E, as before, corresponds to Young's modulus. Consistent with (1), we have neglected nonlinear advection terms. Adding the elastic and plastic contributions, we obtain

$$\frac{1}{E}\frac{\partial \sigma_{ij}}{\partial t} + \frac{1}{2\eta}\sigma_{ij} + \frac{\eta-\zeta}{4\eta\zeta}\sigma_{kk}\delta_{ij} + \frac{P}{4\zeta}\delta_{ij} = \dot{\epsilon}_{ij}. \qquad (17)$$

Note that the VP rheology (6) is obtained as the steady-state limit of (17) or alternatively in the limit $E \to \infty$, while in the limit $\eta, \zeta \to \infty$ we recover the elastic equation (16). Equations (1) and (17) constitute the EVP model. These prognostic equations for the velocity and stress components, u_i and σ_{ij} respectively, are discretized explicitly, as described in the following section. The characteristics of this discretization of the model are analyzed in Section 4, where we obtain the appropriate choice of E and Δt to permit efficient integration while maintaining viscous-plastic balance at slow time scales.

3. Numerical formulations

In this section we outline our numerical techniques for both the preconditioned conjugate gradient method and the explicit elastic-viscous-plastic method. The spatial discretization is specialized for

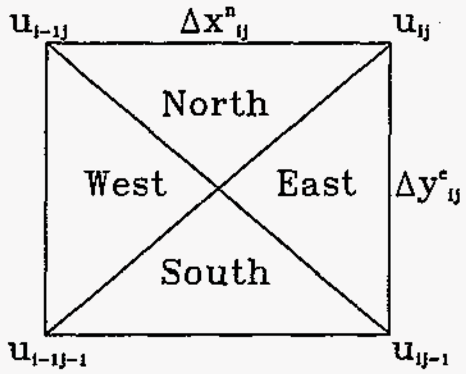

Figure 1. Triangular regions of a grid cell. Velocity components for cell (i,j) are in the upper right corner.

a generalized orthogonal B-grid as in Smith et al. (1996) or Murray (1996), and each logically rectangular grid cell is divided into four triangles, as illustrated in Figure 1. All of the thermodynamic and transport variables are given at the center of the cell, velocity is defined at the corners, and the stress tensor is constant across each triangle. We assume contravariant velocity components (velocity components aligned along grid lines). Here, σ_{ij} may take on four different values within a grid cell. This tends to avoid the grid decoupling problems associated with the B-grid. Note that the rates of strain $\dot{\epsilon}_{ij}$, and therefore the viscosities η and ζ, are also defined in each triangle. A land mask M_h is specified in the cell centers, with 0 representing land and 1 representing oceanic cells. A corresponding mask M_u for velocity and other corner quantities is given by $M_u(i,j) = \{\min M_h(l), l = (i,j), (i+1,j), (i,j+1), (i+1,j+1)\}$.

The velocity component equations (see (1), (5), (17), or (10), (11)) are coupled through the strain rate $\dot{\epsilon}_{ij}$, the viscosities, and the ocean stress $\vec{\tau}_w$. We lag the viscosities and c' to obtain a linear system, but leave the equations otherwise coupled.

3.1. Conjugate gradient solution of the viscous-plastic model

Equations (10) and (11) are discretized semi-implicitly as follows: if n indicates the previous time step, then

$$\begin{aligned} \alpha u^{n+1} - \partial_x\left[(\eta+\zeta)\,\partial_x u^{n+1}\right] - \partial_y\left(\eta\partial_y u^{n+1}\right) \\ - \partial_x\left[(\zeta-\eta)\,\partial_y v^{n+1}\right] - \partial_y\left(\eta\partial_x v^{n+1}\right) \\ = \frac{m}{\Delta t_v}u^n + \beta v^n + \tau_x - \partial_x P/2, \qquad (18) \end{aligned}$$

$$\alpha v^{n+1} - \partial_y\left[(\eta+\zeta)\,\partial_y v^{n+1}\right] - \partial_x\left(\eta\partial_x v^{n+1}\right)$$

$$- \partial_y \left[(\zeta - \eta) \partial_x u^{n+1} \right] - \partial_x \left(\eta \partial_y u^{n+1} \right) = \frac{m}{\Delta t_v} v^n - \beta u^n + \tau_y - \partial_y P/2. \quad (19)$$

Here,

$$\begin{aligned} \alpha &= \frac{m}{\Delta t_v} + c' \cos\theta \\ \beta &= mf + c' \sin\theta \\ \tau_x &= \tau_{ai} + c' \left(U_w \cos\theta - V_w \sin\theta \right) - mg \frac{\partial H_o}{\partial x} \\ \tau_y &= \tau_{aj} + c' \left(V_w \cos\theta + U_w \sin\theta \right) - mg \frac{\partial H_o}{\partial y} \\ c' &= \rho_w C_w \left| \vec{U}_w - \vec{u}^n \right|. \end{aligned}$$

All coefficients, including ζ and η, are evaluated at time level n. The viscous-plastic time step, Δt_v, is typically on the order of hours.

At time level $n+1$, spatial discretization of (18) and (19) produces a system of simultaneous equations that must be solved iteratively for the values of u^{n+1} and v^{n+1} at each grid point. The viscous-plastic rheology operator $\partial \sigma_{ij} / \partial x_j$ arises from a variational principle with the functional

$$I(u,v) = -\frac{1}{2} \int\int \left[\eta (\partial_y u + \partial_x v)^2 + \eta (\partial_x u - \partial_y v)^2 + \zeta (\partial_x u + \partial_y v)^2 \right] dx\, dy, \quad (20)$$

where η and ζ are assumed constant for the purpose of the variation in u and v, and we have temporarily ignored the pressure term. Formulas for $\partial u_i / \partial x_j$ are provided in Appendix A. We discretize I, then take its variation with respect to u and v discretely to obtain the second order derivative terms in (10) and (11). Thus, the coefficients of all "$n+1$ terms" in (18) and (19) translate into a banded matrix which may be represented by the symmetric operator

$$\begin{bmatrix} A^T A & B \\ B^T & C^T C \end{bmatrix},$$

where

$$\begin{aligned} A^T A &= -\partial_x (\zeta + \eta) \partial_x - \partial_y \eta \partial_y + \alpha \\ B &= -\partial_y \eta \partial_x - \partial_x (\zeta - \eta) \partial_y \\ B^T &= -\partial_x \eta \partial_y - \partial_y (\zeta - \eta) \partial_x \\ C^T C &= -\partial_x \eta \partial_x - \partial_y (\zeta + \eta) \partial_y + \alpha. \end{aligned}$$

The resulting matrix equation is solved iteratively with a preconditioned conjugate gradient method (Elman, 1994). The preconditioning matrix is given by

$$\begin{bmatrix} A' & 0 \\ 0 & C' \end{bmatrix},$$

where A' is the tridiagonal matrix extracted from the coefficients of $A^T A$ which couples the u-velocity components along a line of constant j, and C' is the corresponding tridiagonal matrix extracted from $C^T C$ which couples v-velocity components along a line of constant i.

Success of the method hinges on symmetry of the iterating and preconditioning matrices; for this reason we lag the terms $\pm\beta\vec{u}$ during the solution of (18) and (19). This treatment of the Coriolis term, which restricts the time step to about 2 hours for accuracy, might be remedied by applying a predictor-corrector method to these terms as in Zhang and Hibler (1994). This and other improvements to the VP time stepping scheme are reserved for future work.

We have employed a simple linearized Backward-Euler time discretization scheme for (18) and (19). Other methods for dealing with the nonlinearity, such as those employed by Hibler (1979) and Zhang and Hibler (1994), are somewhat more accurate but have their own difficulties. The numerical method of Hibler (1979), which we will refer to as "point relaxation," iteratively solves the system (10) and (11) at each time step with successive overrelaxation, utilizing a predictor-corrector method to march the equations in time. Specifically, predicted velocities at time level $n + \frac{1}{2}$ are used to compute the coefficients of the linearized terms (namely, ζ, η, α and β) before advancing to the next time level. Hibler and Ackley (1983) found a splitting problem with this procedure in cases of small nonlinear viscosities (free drift) which was corrected by a modified averaging procedure.

As with the predictor-corrector scheme, problems also arise in methods which use numerical spatial splitting, and in particular, in those methods which do not treat the entire strain rate tensor implicitly. For example, Zhang and Hibler (1994) also use successive overrelaxation to solve (10) and (11), along with a predictor-corrector time discretization scheme similar to that of Hibler (1979). In this case, however, the cross derivative terms are treated at time level n instead of $n+1$, and the equations decouple. Then the equations for u_{ij} are solved iteratively along an entire row (i.e., constant j) before continuing to the next row, and the equations for v_{ij} are solved similarly along columns. We will refer to this method as "line relaxation." Stössel et al. (1994) found that treating the diagonal part of the strain rate tensor implicitly and the off-diagonal terms explicitly produced anomalous ice drifts of $6\,cm/s$. For the conjugate gradient method described above, the strain rate tensor remains unsplit.

3.2. The elastic-viscous-plastic model

Discretization in time of the momentum equation (1) is analogous to that of (18) and (19), except that the stress tensor is determined prognostically, and both (1) and (17) are subcycled with an effective EVP time step of length $\Delta t_e = \Delta t_\zeta / N$ for some integer $N > 1$ and time interval Δt_ζ. That is, N smaller timesteps are taken with (1) and (17), holding η and ζ constant, for each time interval $[t_n, t_n + \Delta t_\zeta]$. Typically, $\Delta t_\zeta = \Delta t_v$, so that Δt_v is often both the viscous-plastic implicit time step and the interval at which viscosity is updated in the EVP model. Subcycling maintains the time scale on which the viscous-plastic material characteristics are changing, ensuring that the VP and EVP formulations are equivalent in the limit $\Delta t_e \to 0$.

Denoting the subcycling with the index k, we timestep (17) as follows, holding the viscosities constant at time level n:

$$\frac{1}{\Delta t_e}\left(\sigma_{ij}^{k+1} - \sigma_{ij}^{k}\right) + \frac{E}{2\eta}\sigma_{ij}^{k+1} + E\frac{(\eta - \zeta)}{4\eta\zeta}\sigma_{ll}^{k+1}\delta_{ij} + \frac{EP}{4\zeta}\delta_{ij} = E\dot{\epsilon}_{ij}^{k}. \qquad (21)$$

(Since they both depend on the thickness variables, E and P also change on the Δt_ζ time scale, as will be seen.) This is a simultaneous equation for the three distinct stress tensor components, σ_{11}, σ_{12} and σ_{22}, which may be inverted directly. Incidentally, we found that computing σ_{11} and σ_{22} from formulas that have the same form is important for maintaining symmetry of the numerical solutions in the x and y directions, even at some computational expense.

Given the updated stress tensor σ_{ij}^{k+1}, the momentum equation (1) is marched as follows.

$$\frac{m}{\Delta t_e}\left(u_i^{k+1} - u_i^{k}\right) = \frac{\partial \sigma_{ij}^{k+1}}{\partial x_j} + \tau_{ai} + c'\left[\left(U_{wi} - u_i^{k+1}\right)\cos\theta - \epsilon_{ij3}\left(U_{wj} - u_j^{k+1}\right)\sin\theta\right] + \epsilon_{ij3} m f u_j^{k+1} - mg\frac{\partial H_o}{\partial x_i} \qquad (22)$$

where $c' = \rho_w C_w \left|\vec{U}_w - \vec{u}^{\,k}\right|$. This equation may be solved for the velocity components as follows:

$$\left(\alpha^2 + \beta^2\right) u^{k+1} = \frac{m}{\Delta t_e}\left(\alpha u^k + \beta v^k\right) + \alpha\left(\frac{\partial \sigma_{1j}^{k+1}}{\partial x_j} + \tau_x\right) + \beta\left(\frac{\partial \sigma_{2j}^{k+1}}{\partial x_j} + \tau_y\right)$$

$$\left(\alpha^2 + \beta^2\right) v^{k+1} = \frac{m}{\Delta t_e}\left(\alpha v^k - \beta u^k\right) + \alpha\left(\frac{\partial \sigma_{2j}^{k+1}}{\partial x_j} + \tau_y\right) + \beta\left(\frac{\partial \sigma_{1j}^{k+1}}{\partial x_j} + \tau_x\right),$$

where $\alpha = m/\Delta t_e + c'\cos\theta$ and β, τ_x and τ_y are defined in Section 3.1.

The proper spatial discretization of σ_{ij} is determined analogously to the variational principle method of (20). Given formulas for $\partial u_i/\partial x_j$ provided in Appendix A, we demand that in each triangle,

$$\iint \left(\sigma_{ij}\frac{\partial u_i}{\partial x_j} + u_i\frac{\partial \sigma_{ij}}{\partial x_j}\right) dA = 0. \qquad (23)$$

Taking the variation of (23) with respect to u_i yields formulas for the spatial derivatives of σ_{ij}. This is equivalent to the formalism used in the conjugate gradient solution of the VP model. Note that this spatial discretization is different from that in Hibler (1979).

4. Heuristic analysis of the elastic-viscous-plastic model

4.1. Simplified model description

The sea ice model equations are strongly nonlinear and very difficult to analyze. In this section we will consider a simplified one-dimensional version of the equations describing the EVP model in order to better understand the behavior of the model and as an aid in the selection of parameters. The simplified model assumes all spatial variation and motion occurs only in the x direction, all coefficients are constant, all forcing is absorbed into a single term τ, the constant term $P/4\zeta$ is absorbed into $\sigma = \sigma_{11}$, and $\sigma_{12} = \sigma_{22} = 0$. The model therefore is not an exact representation of the EVP model but is sufficiently similar to be useful for a heuristic analysis. The resulting equations are

$$\frac{1}{E}\frac{\partial \sigma}{\partial t} + \frac{\sigma}{\zeta} = \frac{\partial u}{\partial x}, \qquad (24)$$

$$m\frac{\partial u}{\partial t} = \frac{\partial \sigma}{\partial x} + \tau, \qquad (25)$$

where ζ is taken to be an effective constant viscosity. The VP model is recovered in the limit $E \to \infty$:

$$m\frac{\partial u}{\partial t} = \zeta\frac{\partial^2 u}{\partial x^2} + \tau, \qquad (26)$$

which will be considered as the reference for later comparisons. Conversely, in the limit $\zeta \to \infty$, (24)

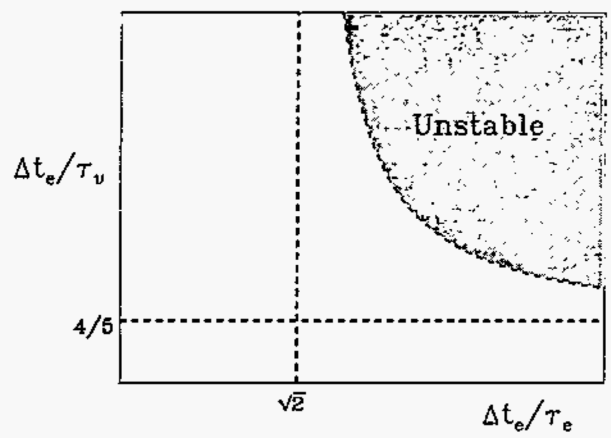

Figure 2. Stability diagram for the 2D dynamics equations (1) and (17). Given Δx, we choose E and Δt_e so that $\Delta t_e/\tau_e$ lies to the left of the vertical asymptote. In this region the viscous-plastic timescale τ_v is irrelevant.

and (25) reduce to a purely elastic model which supports undamped elastic waves,

$$\frac{\partial^2 u}{\partial t^2} = c_e^2 \frac{\partial^2 u}{\partial x^2} + \frac{1}{m}\frac{\partial \tau}{\partial t}, \tag{27}$$

where $c_e = \sqrt{E/m}$ is the elastic wave speed. It is convenient to introduce a viscous time scale

$$\tau_v = \frac{m}{\zeta}\,\Delta x^2, \tag{28}$$

and an elastic time scale,

$$\tau_e = \sqrt{\frac{m}{E}}\,\Delta x = \frac{\Delta x}{c_e}. \tag{29}$$

As discussed in Section 2, τ_v is on the order of a hundredth of a second for resolutions of 10 *km*. In contrast, as we will see shortly, we may be allowed to choose τ_e to be on the order of an hour.

Equations (24)–(26) may be discretized analogously to the full set of equations given in Section 3. Since the equations are linear with constant coefficients, a von Neumann stability analysis may be performed; it is outlined in Appendix B for the one-dimensional EVP case described in Section 5. A two-dimensional stability analysis for the complete set of equations, analogous to that given in Appendix B and assuming $\Delta x = \Delta y$, is summarized in Figure 2 in terms of the time scales (28) and (29). It is remarkable that, provided $\Delta t \le \tau_e\sqrt{2}$, the numerical scheme is stable irrespective of the value of the viscous time scale τ_v. Had (26) been discretized explicitly, the stability limit would have been $\Delta t \le \tau_v/2$, implying a prohibitively small value of the time step. We will thus be able to integrate the EVP model with a time step

$$\Delta t_e = \tau_e\sqrt{2}, \tag{30}$$

which is much larger than the shortest viscous time scale τ_v, without resorting to implicit discretization. The time discretization of the EVP model therefore subsumes the viscous stability limit as mentioned in Section 2.

In what follows it is essential to understand the effect of the time discretization and so, in the interest of simplicity, we will consider a continuous spatial representation, keeping in mind that only wavenumbers k which satisfy $k^2\Delta x^2 \le 1$ are meaningful on a grid. The time discretization of the implicit VP model is

$$\frac{m}{\Delta t_v}\left(u^{n+1} - u^n\right) = \zeta \frac{\partial^2 u}{\partial x^2}^{n+1} + \tau^n, \tag{31}$$

where u^n is the value of u at time level n, and the corresponding discretization of the EVP model is

$$\frac{1}{E\Delta t_e}\left(\sigma^{n+1} - \sigma^n\right) + \frac{\sigma^{n+1}}{\zeta} = \frac{\partial u}{\partial x}^n, \tag{32}$$

$$\frac{m}{\Delta t_e}\left(u^{n+1} - u^n\right) = \frac{\partial \sigma}{\partial x}^{n+1} + \tau^n, \tag{33}$$

where Δt_e is given by (30), and

$$\frac{\Delta t_v}{\Delta t_e} = N \gg 1$$

is the number of steps, or subcycles, that the EVP model takes for each step of the VP model. One of the objectives in this section is to estimate a suitable value for N, or in other words, to define an EVP time step Δt_e such that $\min(\tau_v) \ll \Delta t_e \ll \Delta t_v$. In this section, $\Delta t_\zeta = \Delta t_v$.

4.2. Forced response

Since (31)–(33) are linear with constant coefficients, it is possible to do a rather complete analysis. However, it is sufficient for our purposes, since the ice is forced primarily by time-varying winds, to focus on the amplitude response to periodic forcing for solutions in the form of plane waves, i.e., $(u,\tau) = (\hat{u},\hat{\tau})\,e^{i(kx-\omega t)}$, where ω is the angular frequency of the forcing. We characterize the response by a nondimensional parameter F,

$$F = k^2\Delta x^2 \frac{m\hat{u}}{\tau_v\hat{\tau}},$$

utilizing the damping factor

$$\delta = e^{-i\omega\Delta t_v} = e^{-i\omega N\Delta t_e} = \delta_e^N.$$

Here, $\delta_e = e^{-i\omega\Delta t_e}$ is the damping factor for an elastic time step Δt_e. For convenience, define $\chi = k^2\Delta x^2\Delta t_v/\tau_v$. Substituting plane wave solutions into the "exact" equation (26) and discretizations (31)–(33), we obtain the following response in the three cases:

Exact:

$$F = \frac{1}{1 - i\omega\Delta t_v/\chi},$$

Viscous-Plastic:

$$F = \frac{1}{1 + (\delta - 1)(1 + 1/\chi)},$$

Elastic-Viscous-Plastic:

$$F = \frac{1}{1 + (\delta_e - 1)\left(\frac{N}{\chi} - \frac{\tau_e/\tau_v}{\delta_e + (\delta_e - 1)\tau_e/\tau_v}\right)}.$$

Assuming $\omega\Delta t_e \ll 1$, the EVP response parameter becomes

$$F = \frac{1}{1 + (\delta_e - 1)\left(\frac{N}{\chi} - \frac{1}{\tau_v/\tau_e - i\omega\Delta t_e}\right)}. \quad (34)$$

We can deduce by inspection that the response in the viscous-plastic case is accurate (i.e., approximates the exact response) whenever $\delta \approx 1$, $\chi \ll 1$, and therefore the conditions for accuracy are

$$\omega\Delta t_v \ll 1, \quad (35)$$

$$k^2\Delta x^2\frac{\Delta t_v}{\tau_v} \ll 1. \quad (36)$$

In the elastic-viscous-plastic case, there are two possibilities:

$$\omega\Delta t_e \ll \frac{\tau_v}{\tau_e} \ll 1, \quad (37)$$

$$k^2\Delta x^2\frac{\Delta t_e}{\tau_v}\frac{\tau_e}{\tau_v} \ll 1, \quad (38)$$

and

$$\frac{\tau_v}{\tau_e} \ll \omega\Delta t_e \ll 1, \quad (39)$$

$$\frac{k^2\Delta x^2}{\omega\tau_v} \ll 1, \quad (40)$$

These are consistent with the assumption made to obtain (34).

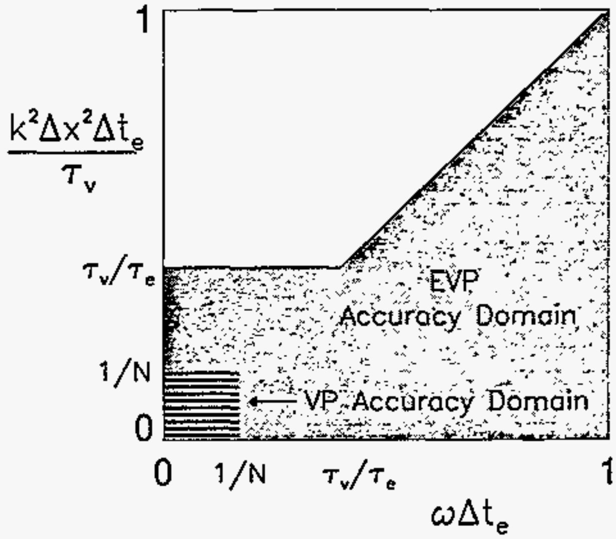

Figure 3. Domains of accuracy of the elastic-viscous-plastic and viscous-plastic models under imposed forcing. We choose numerical parameters so that the domains are as common as possible.

4.3. Choosing appropriate parameters

Conditions (35)–(40) may be more easily understood graphically. Figure 3 illustrates the domains of accuracy for the two models, using $k^2\Delta x^2\Delta t_e/\tau_v$ as the ordinate and $\omega\Delta t_e$ as the abscissa. In general, the EVP domain is larger than the VP domain: if $1/N < \tau_v/\tau_e$ then the VP domain is entirely contained within the EVP domain.

It is reasonable to choose parameters so that the domains of accuracy of the EVP model and the VP model are as common as possible. It is therefore desirable to have

$$\frac{\Delta t_e}{\Delta t_v} = \frac{1}{N} \approx \frac{\tau_v}{\tau_e}. \quad (41)$$

In view of (30), this is equivalent to

$$\Delta t_e^2 = \tau_v\Delta t_v\sqrt{2}, \quad (42)$$

and therefore E may be determined by means of (29):

$$E = \frac{\sqrt{2}m\Delta x^2}{\tau_v\Delta t_v}. \quad (43)$$

It is interesting that (42) implies that a suitable subcycling time step Δt_e is proportional to the harmonic average of Δt_v and the viscous time scale. This result highlights the benefits of the EVP model: the

EVP time step may be orders of magnitude larger than the explicit VP time step. Since there is an entire range of viscous time scales associated with the very large range of effective viscosity coefficients, it may be appropriate to choose some intermediate value of the viscous time scale to use in estimating E. The larger the value of E one chooses, the larger the role of the elastic versus the viscous-plastic strain rates and the shorter the time step Δt_e must be.

The parameter E cannot be considered a constant since then the EVP model would have dynamical effects even under free-drift conditions ($cH \to 0$) when the ice rheology should play no role. To avoid this problem, it is sufficient to assume that E has the form $E = cHE_o$ for some constant E_o. Given suitable values of τ_v and Δt_v (Δt_ζ), we calculate Δt_e by (42), then calculate E for the two-dimensional problem as

$$E = \frac{2E_o \rho_i cH}{\Delta t_e^2} \min\left(\Delta x^2, \Delta y^2\right), \qquad (44)$$

where $0 < E_o < 1$.

5. A one-dimensional test problem

In this section we further compare the behavior of the elastic-viscous-plastic and viscous-plastic models for an essentially 1D test problem, but one which now includes nonlinear effects. Consider the more complete one-dimensional form of (1) and (17):

$$\frac{1}{E}\frac{\partial \sigma_{11}}{\partial t} + \frac{\sigma_{11}}{2\eta} + \frac{\eta - \zeta}{4\eta\zeta}\left(\sigma_{11} + \sigma_{22}\right) + \frac{P}{4\zeta} = \frac{\partial u}{\partial x}, \qquad (45)$$

$$\frac{1}{E}\frac{\partial \sigma_{12}}{\partial t} + \frac{\sigma_{12}}{2\eta} = 0, \qquad (46)$$

$$\frac{1}{E}\frac{\partial \sigma_{22}}{\partial t} + \frac{\sigma_{22}}{2\eta} + \frac{\eta - \zeta}{4\eta\zeta}\left(\sigma_{11} + \sigma_{22}\right) + \frac{P}{4\zeta} = 0, \qquad (47)$$

$$m\frac{\partial u}{\partial t} = \frac{\partial \sigma_{11}}{\partial x} + \tau, \qquad (48)$$

where, as in Section 4, we have lumped all forcing into τ and assumed that all motion and spatial variation occurs only in the x-direction.

5.1. Steady state

We now consider the associated steady-state problem, which we can solve analytically with constant τ and boundary conditions $u = 0$ on the domain $0 \leq x \leq L$. At steady state, the stress tensor components are obtained from (45)–(47):

$$\sigma_{11} = (\zeta + \eta)\frac{\partial u}{\partial x} - \frac{P}{2}, \qquad (49)$$

$$\sigma_{12} = 0, \qquad (50)$$

$$\sigma_{22} = (\zeta - \eta)\frac{\partial u}{\partial x} - \frac{P}{2}. \qquad (51)$$

Noting that $\eta = \zeta/4$ for $e = 2$ in (49), we have

$$\sigma_{11} = \frac{5\zeta}{4}\frac{\partial u}{\partial x} - \frac{P}{2}. \qquad (52)$$

Equation (48) states that

$$\frac{\partial \sigma_{11}}{\partial x} = -\tau. \qquad (53)$$

Combining (52) and (53), assuming P is constant, we have

$$\frac{\partial}{\partial x}\left(\frac{5\zeta}{4}\frac{\partial u}{\partial x} + \tau x\right) = 0. \qquad (54)$$

Recalling (7) and (9) and noting that the strain rate has only one component, $\dot\epsilon_{11} = \partial u/\partial x$, we obtain $\Delta = \sqrt{5/4}\,|\partial u/\partial x|$. Therefore, the viscosity ζ can have one of three possible values: $\zeta_{\min}$, $\zeta_{\max}$, and

$$\frac{P}{\sqrt{5}\left|\frac{\partial u}{\partial x}\right|}. \qquad (55)$$

Now, $\partial\sigma_{11}/\partial x$ cannot simultaneously be both a constant, as required by (53), and a delta function, as implied by (52) and (55): hence the solution must be composed of segments characterized by $\zeta_{\min}$ and $\zeta_{\max}$. Each segment is of the form

$$\frac{5}{4}\zeta u + \frac{1}{2}\tau x^2 + c_1 x + c_2 = 0, \qquad (56)$$

where c_1, c_2 are constants. There will be three segments: two boundary segments characterized by $\zeta_{\min}$, and one in the middle characterized by $\zeta_{\max}$. We therefore have a total of six undetermined constants, plus the location of the interior break-points. Boundary conditions and continuity of the solution at the two interior break-points provide four constraints. Because of the lack of slope continuity at the break-points, integral moments of (54) provide the additional constraints:

$$\int_0^L u^n \frac{\partial}{\partial x}\left(\frac{5\zeta}{4}\frac{\partial u}{\partial x} + \tau x\right) dx = 0,$$

where $n = 0, 1, 2, \ldots$ This closes the system.

This solution is confirmed by numerical results, to be shown shortly. We now present a series of simulations that explore and compare the behavior of the EVP and VP models. Unless otherwise noted, parameter values for the simulations in this section are those

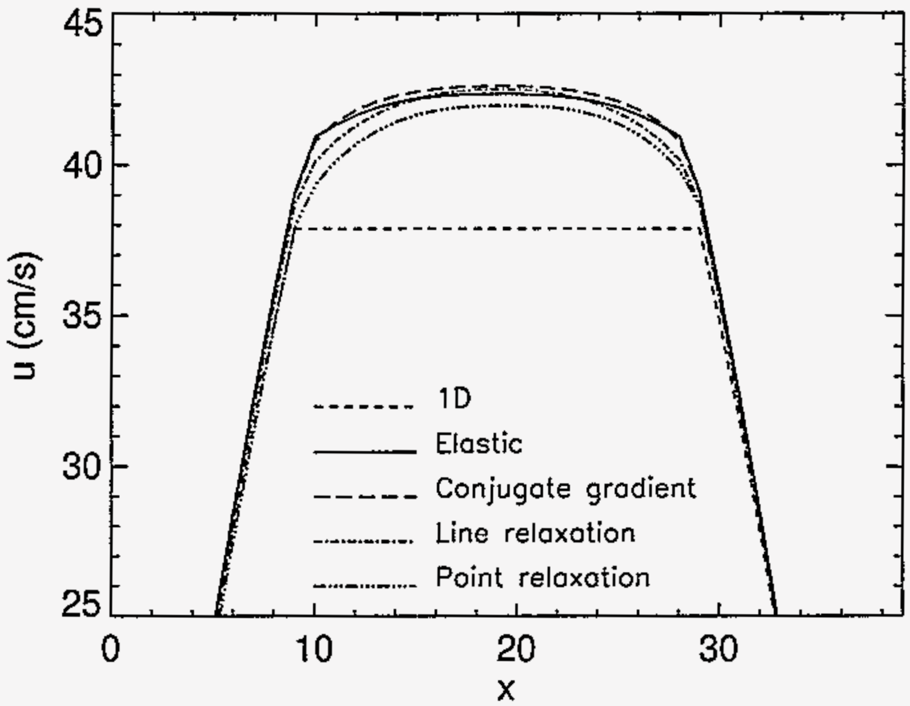

Figure 4. Cross-sections of the velocity component u produced by the (2D) viscous-plastic and elastic-viscous-plastic codes as solutions of the 1D test problem, and the 1D numerical solution.

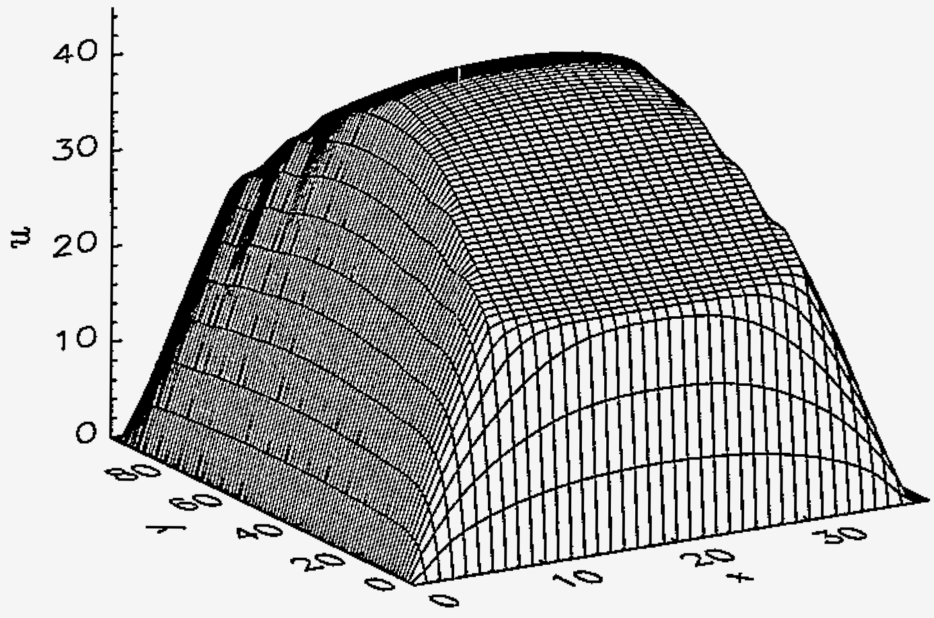

Figure 5. Steady state velocity component u for the 1D problem, produced by the conjugate gradient VP model. The solution is inherently two-dimensional due to the boundary conditions. The cross-section shown in Figure 4 lies at $j = 50$, halfway along the y axis.

Table 3. Initial values and parameters for the tests shown in Figure 4. The error tolerance on the residual for the VP implicit schemes is given by *err*.

Initial Values	Parameters
$c = 0.9$	$E_o = 0.25$
$H = 60.0\,cm$	$\Delta t_v = 21600\,s$
$h = 10.0\,cm$	$\Delta t_\zeta = 21600\,s$
$H_s = 10.0\,cm$	$\Delta t_e = 300\,s$
$h_s = 1.0\,cm$	$\tau = 0.09\,g/\left(cm\,s^2\right)$
$\vec{u} = 0\,cm/s$	$err = 10^{-5}$

given in Table 3. The 1D solution was obtained with a simple numerical code that integrates (45)–(48) to steady state.

As predicted by our analysis, the steady state solution is composed of line segments, illustrated in Figure 4 (labeled "1D"). Figure 4 also presents corresponding numerical solutions of this problem from the 2D models. Due to the imposed land mask, the numerical solutions remain fundamentally two dimensional, as illustrated in Figure 5, and therefore not exactly comparable to the 1D solution. Implementing Neumann or periodic boundary conditions in the SOR viscous-plastic codes in order to make the solution more one-dimensional would have been time consuming and not necessary for our purposes. The four 2D models produce remarkably similar steady state solutions.

The 2D equations were solved on a 40×100 grid of square cells ($\Delta x = \Delta y = 12.7\,km$), and the cross-sections shown are centrally located in the y-direction ($j = 50$). The integration began with a uniform ice field at rest, no-slip conditions were maintained along all four boundaries, and all of the forcing terms were replaced by a single stress $\vec{\tau} = (\tau, 0)$. The CPU times shown in Table 4 represent the time used for the dynamics calculation alone; for each case, $31\,s$ were spent in other sections of the calculation and are not included in the table. These calculations were performed by a CRAY Y-MP8/8128 supercomputer. The models were integrated for 2700 simulated hours, taking 450 time steps with $\Delta t_\zeta = \Delta t_v = 21600\,s$. The EVP dynamics were subcycled 72 times for each viscous-plastic time step, thus taking an effective EVP time step of length $\Delta t_e = 300\,s$. The EVP numerical model is nearly 40 times more efficient than the original VP code on this test problem.

The conjugate gradient solution shown in Figures 4 and 5 is at steady state; the others are not, although the elastic model has reached a quasi-steady state. Since the corresponding EVP solution is essentially identical to Figure 5, it is not shown. Its magnitude oscillates by $\pm 2.5\%$ around the steady state solution, illustrated in Figure 6. For a given domain size, decreasing E_o decreases the oscillation frequency but increases the amplitude. Doubling the size of the domain from 40×100 to 40×200, keeping the resolution the same, removes the oscillation altogether, indicating that it is due to transverse effects. In this case the elastic velocity is about $4\,cm/s$ less than the viscous-plastic steady state velocity in the center of the domain, closely approximating the 1D numerical solution.

Table 4. Estimated total CPU times for the dynamics calculations by each of the four models and the corresponding average CPU time spent for each of the 4000 grid cells, for the tests shown in Figure 4.

Model	CPU	CPU/cell
Elastic	340 s	0.09 s
Conjugate Gradient	520 s	0.13 s
Line Relaxation	6083 s	1.52 s
Point Relaxation	12321 s	3.08 s

5.2. Transient behavior

We investigate the transient behavior of the EVP and VP models using two one-dimensional numerical codes, the 1D EVP code mentioned earlier and its viscous-plastic counterpart, which uses tridiagonal matrix inversion to implicitly solve the equation

$$m\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(\frac{5\zeta}{4}\frac{\partial u}{\partial x} - \frac{P}{2}\right) + \tau. \qquad (57)$$

Both 1D models are discretized as described in Section 3 for the 2D models.

Although the steady state solutions of the VP and EVP models are the same, their transient behavior differs for typical values of Δt_v and Δt_ζ. First, consider the VP model transient behavior for different values of Δt_v, shown in Figure 7. The very slow response of the VP model for large Δt_v is due to the linearization used in the rheology operator. If the viscosity ζ is held at time t_n during integration to time $t_{n+1} = t_n + \Delta t_v$, the linearized ice rheology operator

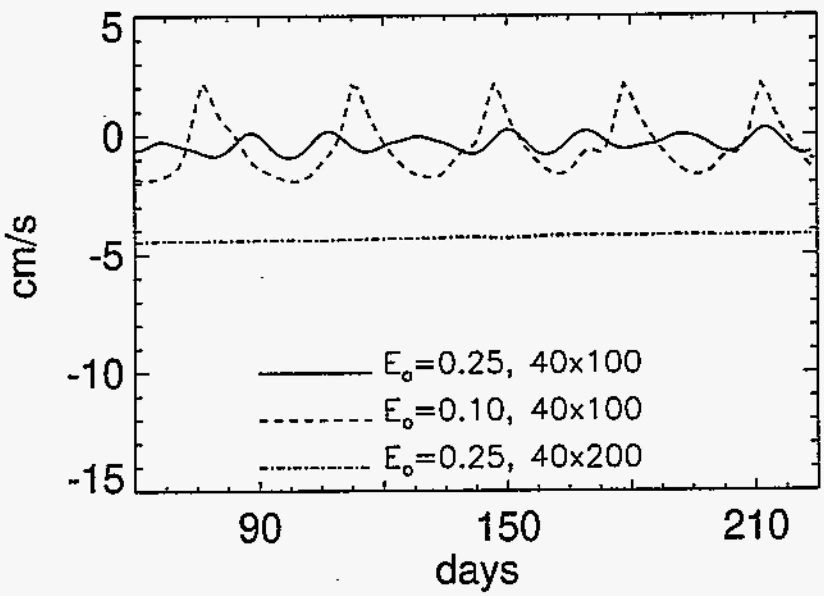

Figure 6. The difference between the elastic velocity in the center of the domain and the corresponding steady state value of the viscous-plastic velocity. The magnitude of E controls the frequency and amplitude of the oscillation, a two-dimensional effect that disappears when the grid is extended in the transverse direction. On the larger domain, the 2D elastic solution agrees well with the 1D solution, about $4\,cm/s$ below the steady state viscous-plastic value.

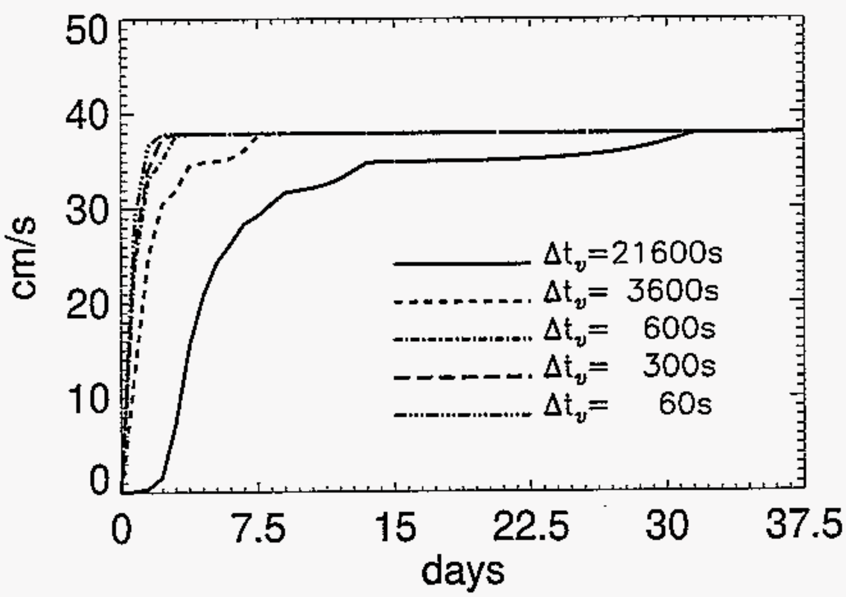

Figure 7. Transient response of the VP model to constant surface stress, for different time steps. Because of the time lag inherent in the calculation of ζ, the viscous-plastic model requires numerous time steps to reach steady state. Thus, small time steps, on the order of 10 minutes for this test problem, are necessary to obtain a converged response to an implusively applied physical stress. Velocity at the center of the domain ($i = 20$) is shown.

in the viscous-plastic case (57) takes the form

$$\frac{\partial}{\partial x}\left(\zeta^n \frac{\partial u}{\partial x}^{n+1}\right).$$

Steady state is reached when $\zeta^{n+1} = \zeta^n$. Convergence of this "outer" iteration determines the effective time response. The adjustment process is described fully for this test problem in Appendix C, where we determine the steady state solution analytically and estimate the effective transition time to steady state, about 35 days for $\Delta t_v = 6\,hr$. Decreasing Δt_v lessens the time lag between ζ^n and $\partial u^{n+1}/\partial x$; time steps on the order of a minute produce the "true inertial limited response" (Hibler, 1979, Appendix B), illustrated in Figure 7. That is, in order to respond accurately to an impulsively applied τ, the viscous-plastic numerical model must be integrated with a time step of $60\,s$ or less. We refer to this solution (obtained with $\Delta t_v = 60\,s$) as the reference solution.

Subcycling the EVP dynamics overcomes this difficulty somewhat. In this case, the ice rheology term has the form

$$\frac{\partial}{\partial x}\left(\zeta^n \frac{\partial u}{\partial x}^{k}\right),$$

where $k = 1, 2, \ldots, N$ denotes the subcycling. The improved estimates of $\partial u/\partial x$ during the VP time step improve the adjustment of the solution. When $\Delta t_\zeta = \Delta t_v$ exceeds the stability limit, as it often does, the EVP results generally lie within an envelope bounded by the viscous-plastic solutions for $\Delta t_v \to 0$ and $\Delta t_v \to \infty$, as indicated in Figure 8.

Without subcycling, $\Delta t_e = \Delta t_\zeta$ and the elastic waves do not damp out within the viscous-plastic time step. The EVP results are then quite energetic for larger time steps, as illustrated in Figure 9. As the time step approaches zero, however, the solutions converge to the reference solution. Furthermore, the two models produce identical results when Δt_v and Δt_ζ are much shorter than the viscous-plastic stability limit, regardless of subcycling.

Poor adjustment of the VP model has been noticed previously. Hibler (1979) remarks that the viscous-plastic rheology is slow to converge to steady state and requires several time steps with constant forcing to respond accurately. Similarly, Flato and Hibler (1992) note that even the cavitating fluid model should be subcycled several times without changing the forcing. However, many numerical simulations that utilize the viscous-plastic rheology, including numerous sensitivity studies, use 1-day time steps with daily varying winds (Hibler and Walsh, 1982; Hibler and Ackley, 1983; Walsh et al., 1985; Ip et al., 1991; Riedlinger and Preller, 1991; Chapman et al., 1994, to name a few). These wind stresses may vary significantly on time scales of a day or so. For example, the wind stress imposed in this example is less than $0.1\,dyn/cm^2$. Since the initial change in wind stress occurs over the first time step ($6\,hr$), this is equivalent to a change in the applied wind stress of $0.4\,dyn/cm^2$ per day. The physical wind stress may vary as much as $5\,dyn/cm^2$ per day (Coon, 1980), an order of magnitude larger. Not surprisingly, we observe that when integrated with 1-day time steps, the VP numerical model exhibits a weak response to strongly varying winds. The improved transient behavior of the EVP model enhances its ability to capture the response of the ice to such variations in the stress. We will explore the models' responses to more realistic, time-dependent forcing in the next section.

Both the viscous-plastic transition to steady state and the magnitude of u at steady state depend on ice concentration as shown in Figure 10, since the maximum viscosity $\zeta_{\max}$ varies with compactness as $ce^{c^*(1-c)}$ through the pressure P. Because of this exponential dependence on c, P and $\zeta_{\max}$ are about two orders of magnitude less for ice concentrations of 0.8 than for 0.9, and therefore the ice rheology is immaterial for $c < 0.8$, and one cannot distinguish between elastic and viscous-plastic models.

All of the calculations reported here were done with $c = 0.9$. Holland et al. (1993) point out that shear stress becomes significant for ice concentrations greater than about 0.9. Furthermore, while open water typically exists year round throughout the Arctic, both Arctic and Antarctic ice concentrations are predominantly greater than 90% during the winter (Stössel and Claussen, 1993; Gloersen et al., 1992).

6. A two-dimensional problem

As a further test, we compare the results of the numerical formulations on a geometrically simple 2D problem in which the geostrophic ocean current and wind stress terms have physically realistic magnitudes. Table 5 contains the parameter values used here. A circular ocean current is used which has an amplitude on the order of $10\,cm/s$:

$$\begin{aligned} U_x &= +20\,(2y - L_y)\,/2L_y, \\ V_y &= -20\,(2x - L_x)\,/2L_x, \end{aligned}$$

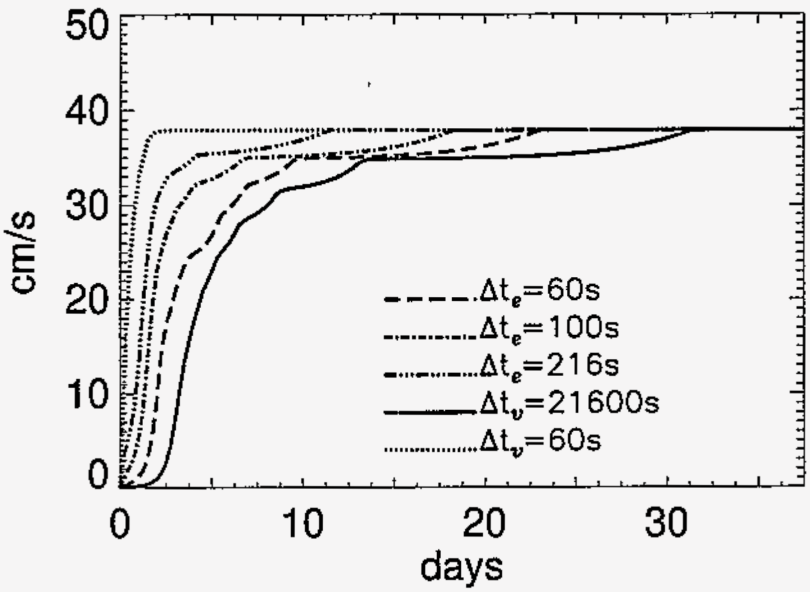

Figure 8. Response of the EVP model as a function of Δt_e, compared to the VP model response for $\Delta t_v = 60\,s$ and $21600\,s$. Elastic-viscous-plastic solutions $\Delta t_e = 60\,s$ ($N = 360$), $100\,s$ ($N = 216$), and $216\,s$ ($N = 100$), with viscosity updated every $\Delta t_\zeta = 21600\,s$, all give better transient response than the viscous-plastic solution with $\Delta t_v = 21600\,s$, but not as good as the converged viscous-plastic solutions with $\Delta t_v = 60\,s$.

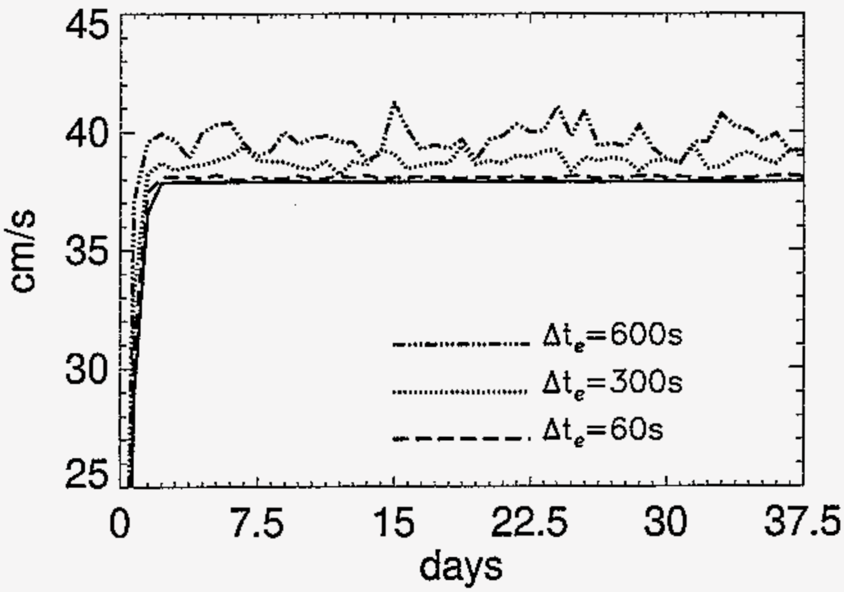

Figure 9. Response of the EVP model without subcycling (viscosity updated on every step), as a function of time step. A substantial amount of elastic energy is excited, but the solution converges to the reference solution (solid) when $\Delta t_e = 60\,s$.

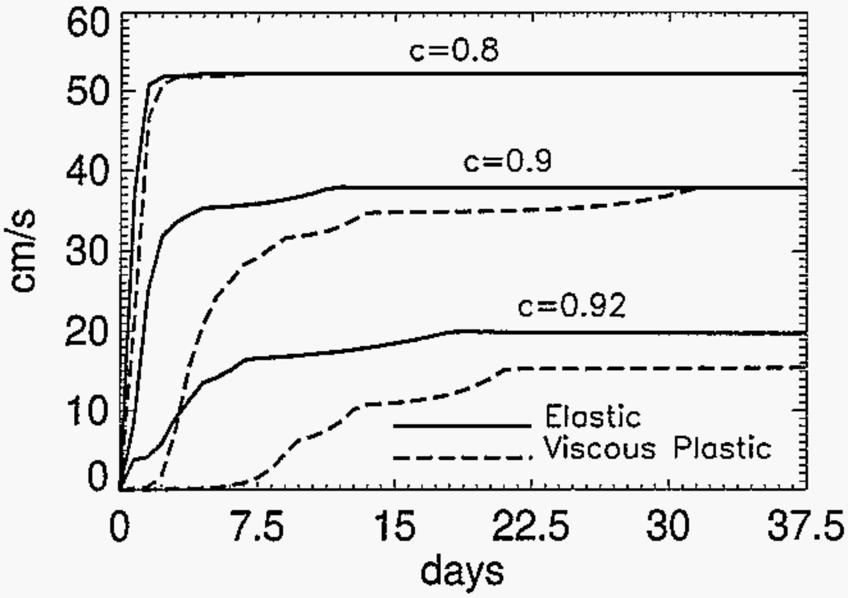

Figure 10. Response of the EVP and VP models for different ice concentrations. While both the VP and EVP solutions exhibit appropriate transient behavior for $c \leq 0.8$, when ice rheology plays no role, the VP model response deteriorates as ice concentration increases. Velocity at the center of the domain ($i = 20$) is shown; $\Delta t_\zeta = \Delta t_v = 21600\,s$ and $\Delta t_e = 216\,s$.

Table 5. Initial values and parameters for the 2D tests.

Initial Values	Parameters
$c = 0.9$	$E_o = 0.25$
$H = 200.0\,cm$	$\Delta t_v = 86400\,s$
$h = 10.0\,cm$	$\Delta t_\zeta = 21600\,s$
$H_s = 10.0\,cm$	$\Delta t_e = 60\,s$
$h_s = 1.0\,cm$	$\tau = 1.00\,g/\,(cm\,s^2)$
$\vec{u} = 0\,cm/s$	$err = 10^{-6}$

where $0 \leq x \leq L_x$ and $0 \leq y \leq L_y$. The ocean drag terms are computed as in (2). The wind stress is also specified analytically, but is based on Arctic data for the month of January, 1986, provided by the Naval Research Laboratory. Fourier analysis of data in the Greenland Sea (9.2W, 75.5N) indicates that the characteristic time scale of the wind forcing is generally between 1 and 5 days. Based on these data, we allow the wind stress to vary 33% from a divergent stress field whose average amplitude is $3\,dyne/cm^2$, with a period $T = 4\,days$:

$$\tau_i = \left[\tau \sin\left(\frac{2\pi t}{T}\right) - 3\right] \sin\left(\frac{2\pi x}{L_x}\right) \sin\left(\frac{\pi y}{L_y}\right)$$

$$\tau_j = \left[\tau \sin\left(\frac{2\pi t}{T}\right) - 3\right] \sin\left(\frac{2\pi y}{L_y}\right) \sin\left(\frac{\pi x}{L_x}\right).$$

Coriolis and ocean tilting effects have been omitted. Note that the time variation of this forcing occurs only in its magnitude. Although directional variation is not included, this (relatively quiescent) wind stress varies sufficiently to illustrate the difficulties one encounters with the VP model.

The model equations were integrated for 25 simulated days from rest with a time step $\Delta t_v = 1\,day$, on a 40×40 grid of square cells ($\Delta x = \Delta y = 12.7\,km$). Such a large time step is not feasible for the EVP dynamics model; for this case, $\Delta t_\zeta = 6\,hr$ and $\Delta t_e = 60\,s$. These time steps were chosen to illustrate the VP model's inaccuracy for conditions under which it is often used, and the improvement offered by the EVP formulation. Strictly speaking, results from the various codes are comparable only for very small Δt_v and Δt_ζ, although we observe in Figures 11 and 12 that the values of Δt_v and Δt_ζ used here are sufficiently small to produce comparable results.

These results differ slightly from a time-accurate reference solution, which we define as that produced by the conjugate gradient method with a time step of $60\,s$. In Figure 13 we present the differences of domain-averaged kinetic energies per unit mass for

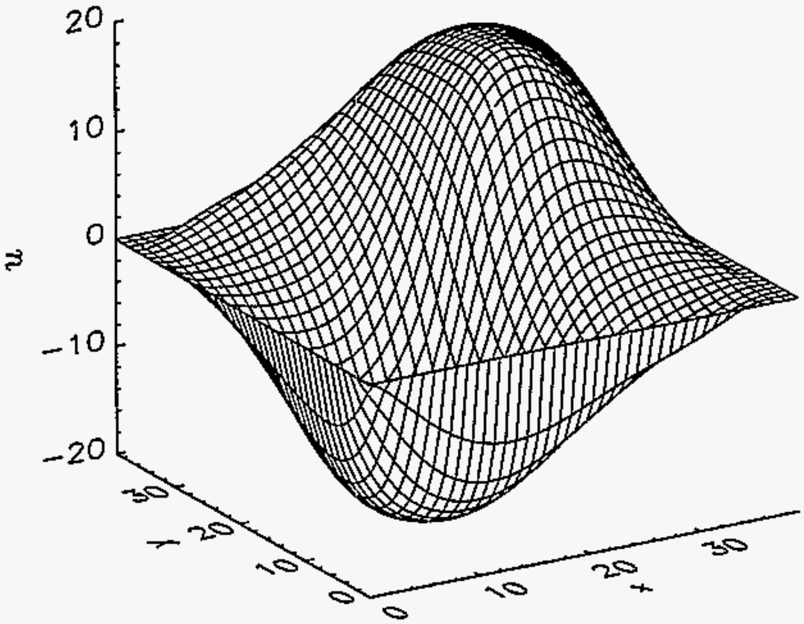

Figure 11. The VP velocity component u for the 2D test case at $t = 25\,days$, produced with the point relaxation numerical model.

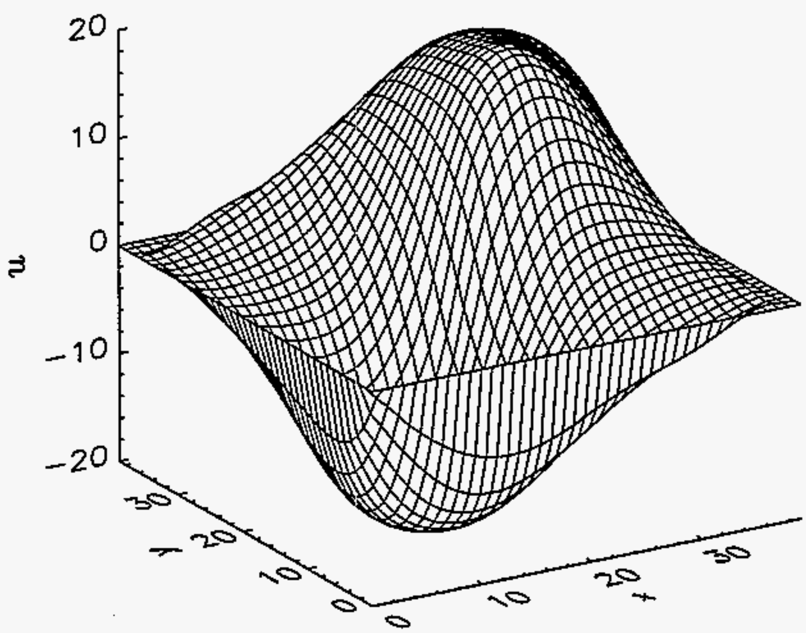

Figure 12. The EVP velocity component u for the 2D test case at $t = 25\,days$. Comparison with the corresponding VP solution in Figure 11 shows that the EVP model produces solutions equivalent to those of the VP model and validates the 1D results of Sections 4 and 5.

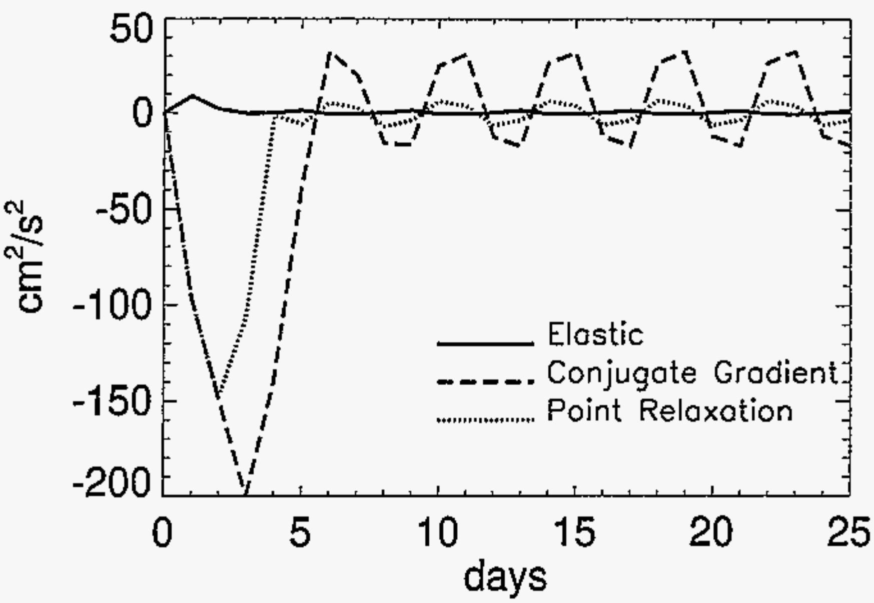

Figure 13. The difference between the domain averaged kinetic energy per unit mass of the elastic, conjugate gradient and point relaxation methods, computed with $\Delta t_v = 1\,day$, and the domain averaged kinetic energy per unit mass of the conjugate gradient solution, computed with $\Delta t_v = 60\,s$ (the reference solution). For the elastic solution, $\Delta t_\zeta = 6\,hr$. The viscous-plastic models require 4 days to reach a quasi-steady state, after which their response to the variable forcing tends to lag behind the exact response.

each of the methods with that of the reference solution. This comparison indicates that while all of the methods reach a quasi-steady state, the EVP model is much more accurate during the initial "spin up" from rest, and suggests that the EVP model will behave significantly better under the severe wind forcing conditions observed in the polar regions. For example, Arctic winds have been observed to change as much as 350% in a three day period (Reynolds, 1984), and the ice edge may move $35\,km/day$ under gale conditions (Roed and O'Brien, 1983). In general, geostrophic winds are responsible for 60–80% of the daily ice variance (Serreze et al., 1989). On these time scales, it is essential that a numerical model for ice dynamics respond accurately to the imposed forcing.

Furthermore, the magnitude of the differences between the viscous-plastic model solutions and the reference case in Figure 13 indicate that the VP models are slow to respond to more typical forcing variations. The kinetic energy of the line relaxation solution is better than the conjugate gradient solution by about a factor of two, due to effectively two iterations of the linearization being taken in the predictor-corrector method used for the time stepping. Incorporating a predictor-corrector method into the time discretization of the conjugate gradient numerical model would improve its accuracy to that of the point relaxation model, but degrade its efficiency. Regardless, neither VP model is as accurate as the EVP model.

Table 6. Estimated total CPU times for the 2D tests by each of the models and the corresponding average CPU time spent for each of the 1600 grid cells.

Model	CPU	CPU/cell
Elastic	45 s	0.028 s
Conjugate Gradient	9 s	0.006 s
Point Relaxation	107 s	0.067 s

The CPU times given in Table 6 represent the time used for the dynamics calculation alone; the $4\,s$ spent performing I/O for each case is not included in the table. Implementing a two-step time discretization scheme for the conjugate gradient VP numerical model would improve its forcing response to roughly the level of the point relaxation code and slow it down by approximately a factor of two. Note that for the $\Delta t_v = 60\,s$ calculation, the conjugate gradient dynamics used $1379\,s$ CPU. We have not made the corresponding calculation with the point relaxation method, but based on the figures in Table 6, the point relaxation numerical model would have taken about 12 times longer, or 4.5 hr, to perform this calculation. Thus, the standard VP model would require several CPU hours to reach the level of accuracy obtained with $\Delta t_v = 60\,s$, which the EVP model simulates fairly well in only 45 CPU seconds using $\Delta t_\zeta = 6\,hr$.

7. Summary

Despite its physical and computational problems, the nonlinear viscous-plastic rheology proposed by Hibler (1979) is the most widely accepted model for sea ice dynamics. In the model's physical description, the ice viscosity suffers a severe singularity: treated as a viscous fluid, rigid sea ice has infinite viscosity. Hibler (1979) regularized this problem by setting a maximum viscosity bound, thereby allowing the ice to creep slowly rather than being completely rigid. Even so, the viscosity ranges over many orders of magnitude, and integrating the implicit VP numerical model requires large computational resources, particularly for high resolution grids on parallel architectures. Using smaller maximum viscosity values increases the model's computational efficiency but produces less accurate results. Our explicit elastic-viscous-plastic model incorporates a more physical regularization, utilizing an elastic mechanism in regions of rigid ice to significantly increase the computational efficiency of the VP numerical model. For comparison purposes, we have chosen to retain the maximum viscosity bound for the results presented here. In this paper we also present a fast, though still implicit, conjugate gradient method for solving the VP equations. Although the conjugate gradient method's efficiency is comparable to the EVP method's on serial machines, the explicit EVP model will be substantially more efficient on parallel computers.

Furthermore, due to its semi-implicit treatment of the ice rheology, the standard numerical formulation of the VP model has very poor time response for time steps typically used by researchers in the field, which are often as long as a day. Our investigation of a simplified, one-dimensional version of the VP model indicates that the viscous-plastic model behavior is acceptable only for wind stresses that vary slowly. However, for wind stresses that vary significantly on time scales of a day, the viscous-plastic model response is weak.

This computational pathology may be resolved by improving the numerical method or by changing the physical parameterization in the model. The EVP model represents a combination of these approaches: its (albeit nonphysical) elastic waves enable the use of an efficient, explicit numerical method. We observe improved transient behavior of the solutions, enhancing the model's ability to capture the ice response to variations in the imposed stress. However, because the EVP model is based on the same inearized viscous-plastic rheology as the VP model, it may inherit similar problems in some parameter regimes.

We have shown that a large range of the elastic wave parameter E exists for which the EVP numerical method is both stable and efficient. In particular, this allows the elastic time step to be orders of magnitude larger than the viscous-plastic time scale in areas of rigid ice. Several considerations must be weighed when choosing the model parameters. The time scale of the external forcing places an upper bound on Δt_v or Δt_ζ. The choice of the subcycling time step Δt_e is based on considerations of efficiency and accuracy. Some guidelines for choosing Δt_e are given in Section 4. The parameter values used in this paper, namely for Δt_ζ, E, and Δt_e, are representive of suitable values that improve both the numerical efficiency and accuracy of the viscous-plastic ice model. A more complete parameter sensitivity study will be reported later.

Other numerical concerns involve maintaining spatial symmetry and energy conservation in the discretization of the stress tensor, which arises from a variational principle. Dividing the grid cells into four triangles for spatial discretizations results in higher resolution of the stress tensor and viscosity fields than of thickness and velocity. These numerical improvements, along with the formulation of the EVP model, have resulted in a fast, efficient model of sea ice dynamics that is well suited to climate studies on parallel machines.

We have coupled the EVP dynamics model to thermodynamic and transport components, and will be testing this ice model with daily atmospheric fluxes and validating it with remotely sensed and in-situ observations. More complete descriptions of the thermodynamics and transport components and results from the validation of the complete sea ice model are forthcoming.

Appendix A: Numerical formulations

Formulas for the spatial derivatives of a field A_{ij} defined at the upper right corner of the grid cell (see Figure 1). Δx^c_{ij} and Δy^c_{ij} are mid-cell lengths. Assuming the field A is linear in x and y within each triangle,

North

$$\frac{\partial A}{\partial x} = \frac{A_{ij} - A_{i-1j}}{\Delta x^n_{ij}}$$
$$\frac{\partial A}{\partial y} = \frac{A_{ij} + A_{i-1j} - A_{ij-1} - A_{i-1j-1}}{2\Delta y^c_{ij}}$$

East

$$\frac{\partial A}{\partial x} = \frac{A_{ij} + A_{ij-1} - A_{i-1j} - A_{i-1j-1}}{2\Delta x^c_{ij}}$$
$$\frac{\partial A}{\partial y} = \frac{A_{ij} - A_{ij-1}}{\Delta y^e_{ij}}$$

South

$$\frac{\partial A}{\partial x} = \frac{A_{ij-1} - A_{i-1j-1}}{\Delta x^n_{ij-1}}$$
$$\frac{\partial A}{\partial y} = \frac{A_{ij} + A_{i-1j} - A_{ij-1} - A_{i-1j-1}}{2\Delta y^c_{ij}}$$

West

$$\frac{\partial A}{\partial x} = \frac{A_{ij} + A_{ij-1} - A_{i-1j} - A_{i-1j-1}}{2\Delta x^c_{ij}}$$
$$\frac{\partial A}{\partial y} = \frac{A_{i-1j} - A_{i-1j-1}}{\Delta y^e_{i-1j}}.$$

Appendix B: Stability of the 1D equations

We perform a von Neumann stability analysis of the simplified, 1D dynamics equations (45)– (48). Discretizing time, these equations become

$$\frac{m}{\Delta t}\left(u^{n+1} - u^n\right) - \frac{\partial \sigma}{\partial x}^{n+1} = 0$$
$$\frac{1}{E\Delta t}\left(\sigma^{n+1} - \sigma^n\right) + \frac{5\sigma^{n+1}}{4\zeta} - \frac{25}{16}\frac{\partial u^n}{\partial x} = 0.$$

Assume that both u and σ have the form $a^n e^{ijk\Delta x}$, and $a^{n+1} = \lambda a^n$. Then the characteristic equation is

$$\lambda^2(1+\alpha) + \lambda\left(-2 - \alpha + k^2\beta\right) + 1 = 0,$$

where $\alpha = 5E\Delta t/4\zeta$ and $\beta = 25E\Delta t^2/16m$. Solutions are stable whenever $|\lambda| < 1$, that is for

$$\alpha > \frac{1}{2}k^2\beta - 2.$$

Let $\xi = \Delta t/\tau_e$ and $\gamma = \Delta t/\tau_v$. Then $\gamma = 4\beta/5\alpha\Delta x^2$ and the boundary of the stability region is given by the hyperbolic function

$$\gamma = \frac{10\xi^2}{\frac{25}{4}\xi^2 k^2 \Delta x^2 - 16}.$$

The stability region of the 2D equations, shown in Figure 2, is similar.

Appendix C: VP model adjustment to imposed forcing

The VP adjustment time illustrated in Figure 7 for time steps of 6 *hr* or more may be estimated as follows. For large time steps, the acceleration term may be neglected, and the transient iterates of the resulting numerical scheme approximate the transition to steady state. That is, we integrate (54) over $[x, L/2]$ and take advantage of the problem's symmetry about $x = L/2$ to produce the relation

$$\zeta\frac{\partial u}{\partial x} = \frac{4}{5}\tau\left(\frac{L}{2} - x\right).$$

The transition to steady state is then governed by the associated iterative scheme,

$$\frac{\partial u}{\partial x}^{n+1} = \begin{cases} GP'/\zeta_{\max} & \text{if } \zeta > \zeta_{\max} \\ G\left|\frac{\partial u}{\partial x}^n\right| & \text{if } \zeta_{\min} < \zeta < \zeta_{\max} \\ GP'/\zeta_{\min} & \text{if } \zeta < \zeta_{\min}. \end{cases} \quad \text{(C1)}$$

where $P' = P/\sqrt{5}$, (55) has been incorporated for ζ and

$$G = \frac{4\tau}{\sqrt{5}P}\left(\frac{L}{2} - x\right). \quad \text{(C2)}$$

Thus, the upper and lower bounds imposed on ζ now limit $\partial u/\partial x$. The iteration begins with $u = 0$ and $\zeta = \zeta_{\max}$ for all x. Recall that the steady state solution is composed of three line segments, the inner section characterized by $\zeta_{\max}$ and the two outer sections by $\zeta_{\min}$. In the inner section, ζ will remain equal to $\zeta_{\max}$, but in the outer regions, ζ, given by (55), will change from $\zeta_{\max}$ to $\zeta_{\min}$. Equivalently, $|\partial u/\partial x|$ will change from $|\partial u/\partial x|_{\max}$ to $|\partial u/\partial x|_{\min}$, under the iteration (C1). Let k be the number of iterations for this change to occur; then

$$|G|^k \left|\frac{\partial u}{\partial x}\right|_{\min} = \left|\frac{\partial u}{\partial x}\right|_{\max}.$$

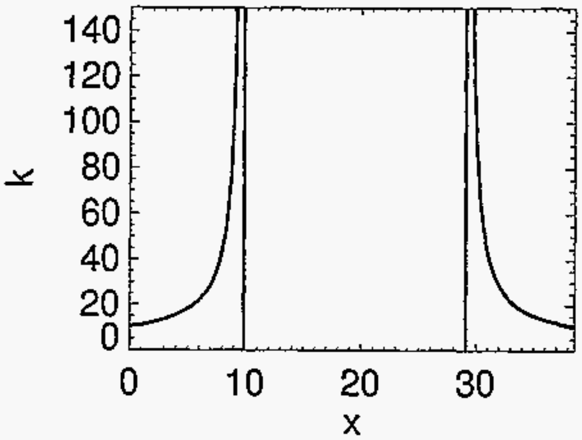

Figure C1. The number of iterations, k, needed for ζ to change from $\zeta_{\max}$ to $\zeta_{\min}$, as a function of x. The vertical lines indicate the break-points, between which $\zeta = \zeta_{\max}$. The number of steps required for the solution to reach steady state is given approximately by the largest value of k at the gridpoints nearest the break-points, about 140 in this case.

In general, k will be a function of x. For this test case, $|\partial u/\partial x|_{\max} = 2.2 \times 10^{-6}$ and $|\partial u/\partial x|_{\min} = 1.8 \times 10^{-9}$ (for $P = 2 \times 10^6$), so that

$$k = \frac{\ln 1.2 \times 10^3}{\ln |G|}, \tag{C3}$$

illustrated in Figure C1. This formula is valid only in the outer regions and fails at the break-points, where $|G| = 1$:

$$x = \frac{L}{2} \pm \frac{\sqrt{5}P}{4\tau}.$$

Here, k is largest for the gridpoints nearest the inner region; this analysis suggests that approximately 140 iterations are needed for the solution to reach steady state, in good agreement with Figure 7.

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