Gu, YuanTong (2008) *An elasto-plastic analysis of solids by the local meshless method based on MLS.* International Journal of Modern Physics B, 22(31/32). pp. 5780-5786.

International Journal of Modern Physics B (2008) Vol. 22, Issue 31/32, pp.: 5780-5786

AN ELASTO-PLASTIC ANALYSIS OF SOLIDS BY THE LOCAL MESHLESS METHOD BASED ON MLS

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A pseudo-elastic local meshless formulation is developed in this paper for elasto-plastic analysis of solids. The moving least square (MLS) is used to construct the meshless shape functions, and the weighted local weak-form is employed to derive the system of equations. Hencky's total deformation theory is applied to define the effective Young's modulus and Poisson's ratio in the nonlinear analysis, which are obtained in an iterative manner using the strain controlled projection method. Numerical studies are presented for the elasto-plastic analysis of solids by the newly developed meshless formulation. It has demonstrated that the present pseudo-elastic local meshless approach is very effective for the elasto-plastic analysis of solids.

Keywords: Material nonlinearity, Elasto-plastic analysis, Meshless method, Moving Least Squares.

1. General Appearance

The finite element method (FEM) is currently the dominated numerical simulation tool for the analysis of material behaviors in elastic and elasto-plastic ranges. A group of numerical techniques based on FEM have been developed so far to solve the elasto-plastic problems. Owen and Hinton[1] provided finite element computer implementation of elasto-plastic problems based on incremental theory. Seshadri[2] developed a GLOSS method based on two-linear elastic finite element analysis which is used to evaluate the approximate plastic strains at certain local regions. Desikn and Sethuraman[3] developed a so-called pseudo-elastic method for the determination of inelastic material parameters.

However, the nonlinear stress-strain relationship and the loading path dependency in the plastic range make the analysis tedious. On the other hand, some shortcomings of FEM are revealed in the elasto-plastic analysis including difficulty for adaptive analysis and poor accuracy of the stress field. These shortcomings are inherent of numerical methods formulated based on predefined meshes or elements. In recent years, some meshless (or meshfree) methods have been proposed to overcome the shortcomings of FEM. According to the classifications of Liu and Gu[4], meshless methods can be largely grouped into three different categories: the meshless method based on strong-forms, the meshless method based on weak-forms, and the meshless method based on the combination of weak- and strong-forms. The famous smooth particle hydrodynamics (SPH)[5] belongs to the first category, and the element-free Galerkin (EFG) method[6] and the point interpolation method (PIM)[7] belong to the second category. The advantages of the meshless method include: 1) no mesh used; 2) high accuracy, and 3) good performance for adaptive analysis. Therefore, the meshless technique provides a big freedom for numerical modeling and simulation. However, the freedom does not present without cost (e.g., some undetermined parameters and worse computational efficiency).

In the family of meshless methods, the meshless method based on the local weak-form is a well-developed technique including the meshless local Petrov-Galerkin (MLPG) method[8,9] and the local radial point interpolation method (LRPIM)[10]. Due to the stability and accuracy of MLS, the local meshless method based on the moving least squares (MLS) is becoming a robust numerical tool for the practical analysis. However, almost all current researches and applications of this type of meshless methods (based on MLS) for solids are limited to linear elasticity, and few research for the material nonlinear analysis is reported. In this paper, a pseudo-elastic local meshless formulation is developed to solve elasto-plastic problems in solids. The locally weighted residual method is used to derive the meshless system of equations and MLS is applied to construct the meshless shape functions. The Hencky's total deformation theory with the iterative manner is used to define effective material parameters. Numerical examples are studied to demonstrate the effectivity of the newly developed pseudo-elastic local meshless formulation for the elasto-plastic analysis.

2. The local meshless formulation

Consider the following two-dimensional solid problem:

$$\sigma_{ij,j} + b_i = 0 \quad \text{in} \quad \Omega \tag{1}$$

The corresponding boundary conditions are

$$u_i = \bar{u}_i\,; \quad t_i = \sigma_{ij} n_j = \bar{t}_i \tag{2}$$

In the local meshless method, a local weak-form is constructed over a sub-domain Ω_s bounded by Γ_s. Using the locally weighted residual method, the generalized local weak-form of Eqs. (1) and (2) for a field node, I, can be written as

$$\int_{\Gamma_{si}} w_I t_i d\Gamma + \int_{\Gamma_{su}} w_I t_i d\Gamma + \int_{\Gamma_{st}} w_I \bar{t}_i d\Gamma - \int_{\Omega_s} (w_{I,j}\sigma_{ij} - w_I b_i) d\Omega = 0 \tag{3}$$

where w_I is the weight function, which is constructed based on the node I.

The problem domain and boundaries are discretized by arbitrarily distributed field nodes. To approximate the displacement function $u(\mathbf{x})$ in Ω_s, a finite set of $\mathbf{p}(\mathbf{x})$ called basis functions is considered in the space coordinates $\mathbf{x}^T=[x, y]$. The moving least squares (MLS) interpolant $u^h(\mathbf{x})$ is defined in the domain Ω_s by

$$u^h(\mathbf{x}) = \sum_{j=1}^{m} p_j(\mathbf{x}) a_j(\mathbf{x}) = \mathbf{p}^{\mathrm{T}}(\mathbf{x})\mathbf{a}(\mathbf{x}) \tag{4}$$

where m is the number of basis functions, and the coefficient $a_j(\mathbf{x})$ is also a functions of $\mathbf{x}$. $\mathbf{a}(\mathbf{x})$ is obtained at a point $\mathbf{x}$ by minimizing a weighted discrete L_2 norm of:

$$J = \sum_{i=1}^{n} w(\mathbf{x} - \mathbf{x}_i)[\mathbf{p}^{\mathrm{T}}(\mathbf{x}_i)\mathbf{a}(\mathbf{x}) - u_i]^2 \tag{5}$$

where n is the number of nodes in the neighborhood of $\mathbf{x}$ for which the weight function $w(\mathbf{x}-\mathbf{x}_i)\neq 0$, and u_i is the nodal value of u at $\mathbf{x}=\mathbf{x}_i$. The stationarity of J with respect to $\mathbf{a}(\mathbf{x})$ leads to the following linear relation between $\mathbf{a}(\mathbf{x})$ and $\mathbf{u}$:

$$\mathbf{A}(\mathbf{x})\mathbf{a}(\mathbf{x}) = \mathbf{B}(\mathbf{x})\mathbf{u} \tag{6}$$

In which $\mathbf{A}(\mathbf{x})$ and $\mathbf{B}(\mathbf{x})$ are the interpolation matrices defined by coordinates and the weight functions. Solving $\mathbf{a}(\mathbf{x})$ from Eq. (6) and substituting it into Eq. (4), we have

$$u^h(\mathbf{x}) = \sum_{i=1}^{n} \phi_i(\mathbf{x}) u \text{ , in which } \phi_i(\mathbf{x}) = \sum_{j=1}^{m} p_j(\mathbf{x})(\mathbf{A}^{-1}(\mathbf{x})\mathbf{B}(\mathbf{x})) \tag{7}$$

where $\phi_i(\mathbf{x})$ is the MLS shape function. It should be mentioned here that the MLS shape function obtained above does not have the Kronecker delta function properties[4].

Substituting the displacement expression given in Eq. (7) into the local weak- form Eq. (3) and applying this local weak-form for all filed nodes, we have the following discretized system of equations,

$$\mathbf{KU} = \mathbf{F} \tag{8}$$

where **K** is the stiffness matrix and **F** is the force vector, i.e.

$$\mathbf{K}_I = \int_{\Omega_s} \mathbf{v}_I^T \mathbf{D}_e \mathbf{B} d\Omega - \int_{\Gamma_{si}} \mathbf{w}_I \mathbf{N} \mathbf{D}_e \mathbf{B} d\Gamma - \int_{\Gamma_{su}} \mathbf{w}_I \mathbf{N} \mathbf{D}_e \mathbf{B} d\Gamma \tag{9}$$

$$\mathbf{F}_I = \int_{\Omega_s} \mathbf{w}_I \mathbf{b} d\Omega + \int_{\Gamma_{st}} \mathbf{w}_I \bar{\mathbf{t}} \, \mathrm{d}\Gamma \tag{10}$$

In Eq. (9), $\mathbf{D}_e$ is the effective material matrix that is obtained from the effective constitutive equation, i.e.:

$$\mathbf{D}_e(\mathbf{x}_Q) = \frac{E_e(\mathbf{x}_Q)}{1-\nu_e^2(\mathbf{x}_Q)} \begin{bmatrix} 1 & \nu_e(\mathbf{x}_Q) & 0 \\ \nu_e(\mathbf{x}_Q) & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu_e^2(\mathbf{x}_Q)}{2} \end{bmatrix} \tag{11}$$

where E_e and ν_e are effective Young's modulus and Poisson's ratio, which will be discussed in the following section.

It should be mentioned here that to get the matrix **K** in Eq. (9), Gauss quadrature is used, and it means that **K** is obtained based on all quadrature points. Hence, $\mathbf{D}_e$ is the material parameter matrix at the Gaussian quadrature point $\mathbf{x}_Q$.

3. Effective material parameters

The strain-stress relationship can be taken in the form[3] of $\varepsilon_{ij} = f(\sigma_{ij})$. ε_{ij} is the total strain tensor which is the summation of conservative elastic ε_{ij}^e and nonconservative plastic part ε_{ij}^p, i.e.,

$$\varepsilon_{ij} = \varepsilon_{ij}^e + \varepsilon_{ij}^p \tag{12}$$

The elastic strain tensor is related to the stress tensor and is given by Hooke's law[11] for isotropic material. The plastic strain tensor is related to the deviatoric part of stress tensor and is given by Hencky's total deformation relation of $\varepsilon_{ij}^p = \Psi S_{ij}$, where Ψ is a scalar valued function, given by

$$\Psi = \frac{3\varepsilon^p_{\text{equivalent}}}{2\sigma_{\text{equivalent}}} = \frac{3}{2}\frac{\sqrt{2\varepsilon_{ij}^p\varepsilon_{ij}^p/3}}{\sqrt{3S_{ij}S_{ij}/2}}; \quad S_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij} \tag{13}$$

Hence, from Equation (12), we can get

$$\begin{aligned} \varepsilon_{ij} &= \varepsilon_{ij}^e + \varepsilon_{ij}^p = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij} + \Psi S_{ij} \\ &= \left(\frac{1+\nu}{E} + \Psi\right)\sigma_{ij} - \left(\frac{\nu}{E} + \frac{1}{3}\Psi\right)\sigma_{kk}\delta_{ij} \end{aligned} \tag{14}$$

The above equation can be re-written as

$$\varepsilon_{ij} = \left(\frac{1+\nu_e}{E_e}\right)\sigma_{ij} - \left(\frac{\nu_e}{E_e}\right)\sigma_{kk}\delta_{ij} \tag{15}$$

where E_e and ν_e are the equivalent Young's modulus and Poisson's ratio, which are given by

$$E_e = 1\Big/\left(\frac{1}{E} + \frac{2\Psi}{3}\right); \quad \nu_e = \left(\frac{\nu}{E} + \frac{\Psi}{3}\right)\Big/\left(\frac{1}{E} + \frac{2\Psi}{3}\right) \tag{16}$$

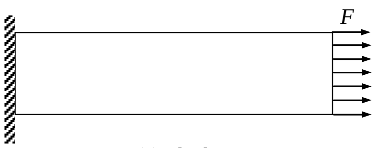
Eq. (15) is the effective constitutive equation for the analysis of material nonlinearity. It should be mentioned here that the effective material parameters are functions of the final state of stress fields, which are usually unknown. Because the system of equations is constructed based on the Gauss quadrature points, the effective material parameters should be also calculated for Gauss quadrature points. In addition, the direct method is unable to lead to the final solution, and the following iteration method based on the projection technique[3] is used.

A linear elastic analysis is firstly carried out to get the initial stress field. To determine whether a material enters the plastic range, the Von Mises yield criterion, which compares the equivalent stress with the yield stress, is used. If the equivalent stress calculated from linear analysis is smaller than the yield stress, σ_0, the computing is finished because the material still satisfies the linear elasticity; if the equivalent stress is larger than the yield stress, it means the deformation already enters the plastic region, and the following iteration computing will be performed.

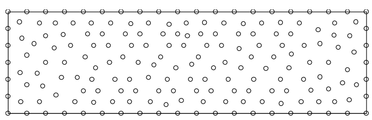
From the initial linear elastic analysis, the strain value is kept unchanged (i.e. strain controlled), and is projected on the experimental uniaixial $\sigma - \varepsilon$ curve. Based on the projection point, the effective value of Young's modulus, $E_e^{(1)}$, for the next iteration is obtained from the slope, and then the effective Poisson's ratio, $\nu_e^{(1)}$, can also be obtained from Equation (16). Using the new effective materials parameters $E_e^{(1)}$ and $\nu_e^{(1)}$, the next linear elastic meshless analysis is carried out to get the next point, its projection point, and further to obtain $E_e^{(2)}$ and $\nu_e^{(2)}$, similarly. This iterative procedure is repeated until all the effective material parameters converge and equivalent stress falls on the experimental uniaxial stress-strain curve. However, if the applied loading is too big, the computing may not converge, and it means that the material is already failure, and this certain loading is called the critical failure loading which is also an important parameter for solids and structures.

4. Numerical example

To illustrate the effectiveness of the presented pseudo-elastic local meshless formulation for the material nonlinear problems, several cases of elasto-plastic analyses have been studied, and good results have been obtained. Following, the results for the uniaxial tension of a bar is presented and discussed in details.



(a) The beam



(b) The nodal distribution

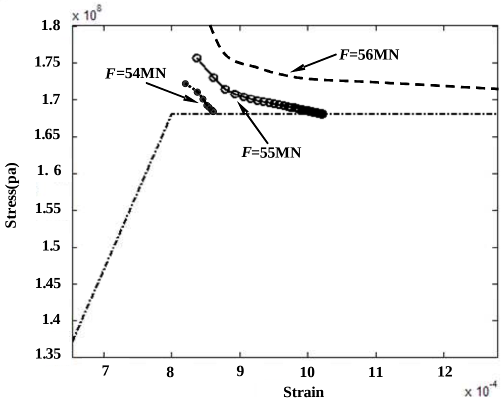


Fig. 1 A cantilever bar under uniaxial tension

Fig. 2 The convergence path for the bar with the elastic-perfectly plastic material (*F*=54MN, 55MN and 56MN)

As shown in Fig. 1, a cantilever bar with length 3m and height 0.3m is subjected to a uniform tensile pressure (the resultant force is *F*). The Young's modulus is assumed as $E = 2.1\times10^{11}$ Pa, the Poisson's ratio is $\nu = 0.3$, and the yield stress is $\sigma_0 = 1.68\times10^8$ Pa. The bar is assumed as being in a plane stress state. As shown in Fig. 1, 163 irregularly distributed field nodes are used to discretize the problem domain and boundaries.

The material is initially considered as elastic-perfectly plastic. The new developed pseudo-elastic local meshless method and the iterative projection technique are applied to get the results. Fig. 2 shows the convergence paths for different *F*. It can be seen that the present method using the projection technique can quickly produce convergent results. However, when *F*=56MN, the result is not convergent. It has been found that when *F* is larger than a certain value, the results will become non-convergent, and the structure fails. This value is called the critical failure load, and it is *F*=55.5 MN for this problem. Comparing with the FEM and other method results[10,12], it can be seen that the results obtained by the present method are in good agreement with those obtained by other methods. It should be mentioned that the present method needs much less iteration steps than FEM. Therefore, it is more efficient than FEM.

A work-hardening material is also considered, as shown in Fig. 3. It clearly shows that the pseudo-elastic local meshless method gives convergent results for *F*=80 MN. It is reasonable that a work-hardening material has a much higher critical failure load than an elastic-perfectly plastic material.

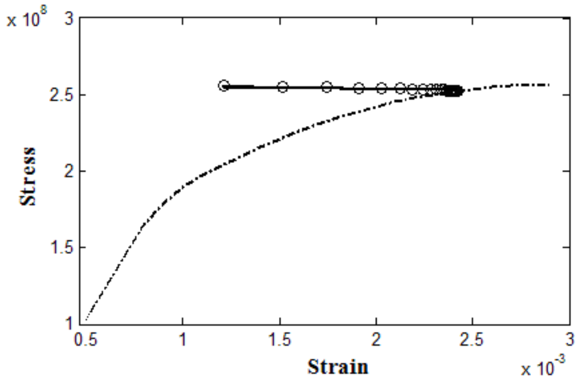


Fig.3 The convergence path for the bar with the work-hardening material (the unit of stress is Pa and *F*=80MN)

5. Conclusions

A pseudo-elastic local meshless formulation is developed for solving elasto-plastic problems in solids. The moving least squares (MLS) is used to construct meshless shape function and the weighted local weak-form is used to derive the meshless system of equations. The Hencky's total deformation theory is utilized to define the effective material parameters, which are obtained in an iterative manner using strain controlled projection method. Numerical studies are presented for the elasto-plastic analysis of solids by the newly developed meshless formulation. It has demonstrated that the present pseudo-elastic local meshless approach is very effective for the elasto-plastic analysis of solids.

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