AN ELEGANT 3-BASIS FOR INVERSE SEMIGROUPS

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ABSTRACT. It is well known that in every inverse semigroup the binary operation and the unary operation of inversion satisfy the following three identities:

$$x = (xx')x \qquad (xx')(y'y) = (y'y)(xx') \qquad (xy)z = x(yz'')\,.$$

The goal of this note is to prove the converse, that is, we prove that an algebra of type $\langle 2, 1\rangle$ satisfying these three identities is an inverse semigroup and the unary operation coincides with the usual inversion on such semigroups.

1. Introduction

In the language of a binary operation $\cdot$ and a unary operation $'$, a set of n independent identities is an n-basis for inverse semigroups, if those identities define the variety of inverse semigroups considered as algebras $(S, \cdot, ')$ of type $\langle 2, 1\rangle$, where the unary operation coincides with the natural inversion. Denoting by x' the inverse of an element x in an inverse semigroup, we then have $x = (xx')x$ (as inverse semigroups are regular semigroups) and $(xx')(y'y) = (y'y)(xx')$ (as both xx' and $y'y$ are idempotents, and idempotents commute in inverse semigroups). Thus we might be tempted to think that the following identities provide a 3-basis for inverse semigroups:

$$x = (xx')x, \qquad (xx')(y'y) = (y'y)(xx') \qquad \text{and} \qquad (xy)z = x(yz)\,. \tag{1.1}$$

However, for $S = \{0, 1\}$ with $xy = 0$, except for $11 = 1$, and defining $x' = 1$, we have the previous identities satisfied, but $0' \neq 0'00'$ and hence $'$ does not coincide with the natural inversion in $(S, \cdot)$.

B.M. Schein [4] repaired the *defect* of (1.1) by adjoining two additional identities: $x'' = x$ and $(xy)' = y'x'$. The resulting set of five identities indeed provides a 4-basis for inverse semigroups. (The identity $(xy)' = y'x'$ is dependent upon the others, and hence can be discarded. However it is worth observing that in the same paper Schein also provided a 5-basis using $xx'x'x = x'xxx'$ instead of $xx'y'y = y'yxx'$; see [4, Theorem 1.6] and [2, p. 15, Ex. 20(b)].) Therefore the natural question to ask would be: *is it possible to find a 3-basis for inverse semigroups?* This question was first answered in the affirmative in [1], but the 3-basis given there requires an extremely complicated proof (it is still an open problem to provide a reasonable proof for that result).

The aim of this note is to repair (1.1) by providing an easy, transparent and *elegant* 3-basis for inverse semigroups.

Theorem. *Let $(S, *, ')$ be an algebra of type $\langle 2, 1\rangle$. Then this algebra is an inverse semigroup and the unary operation coincides with the usual inversion on such semigroups if and only if*

$$(\mathbf{E}_1)\quad x = (xx')x, \qquad (\mathbf{E}_2)\quad (xx')(y'y) = (y'y)(xx'), \qquad (\mathbf{E}_3)\quad (xy)z = x(yz'')\,.$$

2. Proof of the Theorem

In this section we prove that the identities ($\mathbf{E}_1$)–($\mathbf{E}_3$) imply Schein's 4-basis for inverse semigroups. As the converse is obvious, the equivalence of the two bases will follow.

Throughout this section let $(S, \cdot, ')$ be an algebra of type $\langle 2, 1\rangle$ satisfying ($\mathbf{E}_1$)–($\mathbf{E}_3$). We start by proving a few handy identities.

Lemma 1. *The following identities hold.*

$$x'x'' = x'x \tag{2.1}$$

$$(xy')y = x(y'y) \tag{2.2}$$

$$x = x(x'x) \tag{2.3}$$

$$x'' = (x''x')x = x''(x'x) \tag{2.4}$$

$$x'''x = x'''x'' = x'''x^{(4)} \tag{2.5}$$

Proof. Firstly, for (2.1), we have

$$x'x'' \overset{(\mathbf{E}_1)}{=} x'[(x''x''')x''] \overset{(\mathbf{E}_3)}{=} [x'(x''x''')]x \overset{(\mathbf{E}_3)}{=} [(x'x'')x']x \overset{(\mathbf{E}_1)}{=} x'x\,.$$

Next, for (2.2), we compute $(xy')y \overset{(\mathbf{E}_3)}{=} x(y'y'') \overset{(2.1)}{=} x(y'y)$.

Regarding (2.3), we have $x(x'x) \overset{(2.2)}{=} (xx')x \overset{(\mathbf{E}_1)}{=} x$.

Then for (2.4), we compute $x'' \overset{(2.3)}{=} x''(x'''x'') \overset{(\mathbf{E}_3)}{=} (x''x''')x \overset{(2.1)}{=} (x''x')x \overset{(2.2)}{=} x''(x'x)$.

Finally, for (2.5), we have

$$x'''x \overset{(2.3)}{=} [x'''(x''''x''')]x \overset{(\mathbf{E}_3)}{=} x'''[(x''''x''')x''] \overset{(2.4)}{=} x'''x'''' \overset{(2.1)}{=} x'''x''\,.$$

∎

The next two lemmas are the key tools in the proof that the identities ($\mathbf{E}_1$)–($\mathbf{E}_3$) imply $x'' = x$.

Lemma 2. $(x'x)x''' = x'''$.

Proof. We start with two observations. Firstly, as

$$[x(y'''y)]y' \overset{(\mathbf{E}_3)}{=} x[(y'''y)y'''] \overset{(2.5)}{=} x[(y'''y'''')y'''] \overset{(\mathbf{E}_1)}{=} xy'''\,,$$

we have

$$(x(y'''y))y' = xy'''\,. \tag{2.6}$$

Secondly,

$$(x'x)(x'''x) \overset{(2.5)}{=} (x'x)(x'''x'''') \overset{(\mathbf{E}_2)}{=} (x'''x'''')(x'x) \overset{(2.5)}{=} (x'''x'')(x'x) \overset{(2.2)}{=} [(x'''x'')x']x \overset{(2.4)}{=} x'''x\,,$$

so that

$$(x'x)(x'''x) = x'''x. \tag{2.7}$$

Now we have all we need to prove the lemma.

$$x''' \overset{(2.4)}{=} (x'''x'')x' \overset{(2.5)}{=} (x'''x)x' \overset{(2.7)}{=} [(x'x)(x'''x)]x' \overset{(2.6)}{=} (x'x)x'''.$$

∎

Lemma 3. $(xy)z' = x(yz')$.

Proof. We start by proving that

$$x''' = x' \,. \tag{2.8}$$

In fact we have $xx' \overset{(2.3)}{=} [x(x'x)]x' \overset{(\mathbf{E}_3)}{=} x[(x'x)x'''] = xx'''$, using Lemma 2 in the last equality. Thus

$$xx''' = xx' \,. \tag{2.9}$$

Now, by Lemma 2,

$$x''' = (x'x)x''' \overset{(2.1)}{=} (x'x'')x''' \overset{(\mathbf{E}_3)}{=} x'(x''x^{(5)}) \overset{(2.9)}{=} x'(x''x''') \overset{(\mathbf{E}_3)}{=} (x'x'')x' \overset{(\mathbf{E}_1)}{=} x' \,.$$

Replacing z by z' in $(\mathbf{E}_3)$, we get

$$(xy)z' = x(yz''') = x(yz') \,,$$

where the last equality follows from (2.8). The lemma is proved. □

We have everything we need to prove our main result.

Theorem 1. *The identities* $(\mathbf{E}_1)$–$(\mathbf{E}_3)$ *imply* $x'' = x$ *and the associative law.*

Proof. First, we have

$$\begin{aligned} x''x' &\overset{(2.4)}{=} [(x''x')x]x' = (x''x')(xx') \overset{(\mathbf{E}_2)}{=} (xx')(x''x') \\ &= [(xx')x'']x' = [x(x'x'')]x' = x[(x'x'')x'] \overset{(\mathbf{E}_1)}{=} xx' \,, \end{aligned}$$

where we have used Lemma 3 in the unlabeled equalities. Thus

$$x''x' = xx' \,. \tag{2.10}$$

Now $x'' \overset{(2.4)}{=} (x''x')x \overset{(2.10)}{=} (xx')x \overset{(\mathbf{E}_1)}{=} x$, as claimed.

Associativity now follows easily: $(xy)z \overset{(\mathbf{E}_1)}{=} x(yz'') = x(yz)$. □

3. Other Sets of Axioms

It is natural to ask how sensitive the axioms $(\mathbf{E}_1)$–$(\mathbf{E}_3)$ are to certain modifications, such as shifting the parentheses in $(\mathbf{E}_1)$ or changing the placement of the double inverse in $(\mathbf{E}_3)$.

If, for instance, we leave $(\mathbf{E}_2)$ intact, replace $(\mathbf{E}_1)$ with $x(x'x) = x$ and replace $(\mathbf{E}_3)$ with $(x''y)z = x(yz)$, then we obtain a set of identities which are dual to $(\mathbf{E}_1)$–$(\mathbf{E}_3)$. By an argument dual to that in §2, this set of identities is another 3-basis for inverse semigroups.

Thus to dispense with these sorts of obvious dualities, we will assume that both $(\mathbf{E}_1)$ and $(\mathbf{E}_2)$ are left intact, and consider only alternative placement of the double inverse in $(\mathbf{E}_3)$. Using Prover9, we found that each of the following identities can substitute for $(\mathbf{E}_3)$ to give another 3-basis for inverse semigroups:

$$\begin{array}{ll} (xy)z = x''(yz) & \qquad (xy)z = x(y''z) \\ x(yz) = (xy'')z & \qquad x(yz) = (xy)z''. \end{array}$$

The remaining possibility, $x(yz) = (x''y)z$, does not work. Using Mace4, we found the counterexample given by the following tables. It satisfies $(\mathbf{E}_1)$, $(\mathbf{E}_2)$ and $x(yz) = (x''y)z$, but the binary operation is not associative ($(0 \cdot 0) \cdot 0 = 1 \cdot 0 = 7 \neq 6 = 0 \cdot 1 = 0 \cdot (0 \cdot 0)$), and the unary operation clearly fails to satisfy $x'' = x$.

$\cdot$	0	1	2	3	4	5	6	7	8	9	10	11
0	1	6	5	7	3	8	4	2	0	4	4	4
1	7	2	6	0	8	4	5	1	3	5	5	5
2	5	8	3	6	1	7	0	4	2	0	0	0
3	8	0	7	4	6	2	1	3	5	1	1	1
4	3	7	1	8	5	6	2	0	4	2	2	2
5	6	4	8	2	7	0	3	5	1	3	3	3
6	0	1	2	3	4	5	6	7	8	6	6	6
7	4	3	0	5	2	1	7	8	6	7	7	7
8	2	5	4	1	0	3	8	6	7	8	8	8
9	0	1	2	3	4	5	6	7	8	9	10	6
10	0	1	2	3	4	5	6	7	8	10	9	6
11	0	1	2	3	4	5	6	7	8	6	6	11

$'$	0	1	2	3	4	5	6	7	8	9	10	11
	1	2	3	4	5	0	6	8	7	9	10	11

4. Problem

Does there exist a 2-basis for inverse semigroups?

We guess that the answer is no.

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