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AN ELEMENTARY APPROACH TO THE MULTIPLICITY THEORY OF MULTIPLICATION OPERATORS

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1. Introduction. The spectral theory of normal operators comprises one of the prettier and more complete chapters in the literature of operators on Hilbert space. The high point of this development is the multiplicity theory, due in original form to Hellinger and subsequently cast in the language of Lebesgue integration by Hahn [**18**], which answers for normal operators one of the favorite questions of any mathematician: how do you tell when two of these objects are "the same", that is, equivalent in the appropriate sense.

Present day multiplicity theory comes in several levels of abstraction, generality and sophistication through the efforts of a number of mathematicians; see [**7**, **10**, **11**, **12**, **16**] and the further references there. The most straightforward approach to the separable theory can easily make make an appearance (and often does) in a first graduate course in functional analysis and operator theory, say just after the spectral theorem. The kind of treatment I have in mind can be found in an eminently readable book by Conway [**9**]. Of course, every good theory deserves some meaty but accestible examples, and there's the rub, pedagogically speaking: the multiplicity theory of concrete normal operators is often either trivial or quite hard to compute. The class of examples that surely comes first to mind, multiplication operators on L^2 spaces, was worked out only relatively recently [**2**]; the treatment given there uses in a crucial way one first of the more highbrow versions of multiplicity theory (direct integrals, replete with measurable fields of Hilbert spaces) plus a theorem on disintegration of measures. It is the purpose of this note to present a straightforward, rather bare-hands calculation of the spectral multiplicity function of a multiplication operator on $L^2(0, 1)$ which is compatible with the more elementary general theory and which, I hope, will be accessible to students seeing these ideas for the first time. The train of thought here has also been used by the author [**15**] and Ball [**5**] to analyze more complicated operators.

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The work of Hellinger and Hahn actually treated self-adjoint operators ($A^* = A$), though the same ideas, with a little modification, work for the larger class of normal operators. The reader who feels more comfortable with the smaller class should not hesitate to substitute "self-adjoint" for "normal" throughout this paper; in this case the measures E, ν and $\nu_{x,y}$ appearing below are supported on the real line rather than on the complex plane.

2. Multiplicity theory for normal operators. Let us deal only with Hilbert spaces that are complex and separable and operators defined on them that are linear and bounded. Two operators A_1 and A_2, acting respectively on Hilbert spaces K_1 and K_2, are unitarily equivalent if there is a unitary operator U from K_1 to K_2 satisfying $A_2 = UA_1U^{-1}$. For an operator theorist unitary equivalence is the appropriate version of being "the same", for unitary operators preserve all Hilbert space structure.

Multiplicity theory gives a beautiful solution to the problem of unitary equivalence for the class of operators A which are normal, that is for which $AA^* = A^*A$. The first major fact about normal operators is the spectral theorem, which asserts the existence of a spectral measure E for A, defined on Borel sets of the complex plane, such that

$$A = \int \lambda \, dE(\lambda);$$

see [12, 13]. The spectrum $\sigma(A)$ of A is precisely the closed support of E. Given the spectral theorem, the solution to the unitary equivalence problem comes in two pieces; they both involve what appears to be a special kind of normal operator which we now consider.

Let H_∞ denote the Hilbert space of all infinite complex sequences $c = (c_1, c_2, \ldots)$ with

$$\sum_{j=1}^{\infty} |c_j|^2 < \infty;$$

the norm $\|c\|$ is the square root of this series. Let ν be a finite Borel measure with compact support in the complex plane $\mathbf{C}$ (henceforth called simply a measure). We denote by $L^2(\nu, H_\infty)$ the collection of all functions f: $\mathbf{C} \to H_\infty$ (so that $f(\lambda) = (f_1(\lambda), f_2(\lambda), \cdots)$ for each λ) such that each component function f_j is Borel measurable and such that

$$\int \|f(\lambda)\|^2 \, d\nu(\lambda) < \infty.$$

Just as with the scalar space $L^2(\nu)$, we identify two elements in $L^2(\nu, H_\infty)$ which agree ν-a.e. With this understanding we can make $L^2(\nu, H_\infty)$ into a Hilbert space with inner product

$$\langle f, g\rangle = \int \langle f(\lambda), g(\lambda)\rangle \, d\nu(\lambda).$$

For each $i = 1, 2, 3, \ldots$ let H_i be the subspace of H_∞ consisting of all sequences c with $c_k = 0$ whenever $k > i$. Let n be a Borel function on $\mathbf{C}$ with values in $\{1, 2, 3, \cdots\} \cup \{\infty\}$; we call such an n a multiplicity function. Let $\mathscr{D}$ denote the subspace of $L^2(\nu, H_\infty)$ consisting of all f with $f(\lambda) \in H_{n(\lambda)}$ ν-a.e. We are interested in the normal operator M on the Hilbert space $\mathscr{D}$ given by $(Mf)(\lambda) = \lambda f(\lambda), f \in \mathscr{D}$. We will say that $\mathscr{D}$ and M are associated with the pair (ν, n). The first part of the Hellinger-Hahn solution is provided by the following theorem; see Theorem 9.18 and the succeeding remarks in [9].

Representation Theorem. *If A is a normal operator, there exists a measure ν and a multiplicity function n such that the associated operator M is unitarily equivalent to A.*

The main reason that this is so interesting is the second piece of the puzzle; see Corollary 9.12b in [9].

Invariance Theorem. *Suppose that ν_1 and ν_2 are measures and that n_1 and n_2 are multiplicity functions. Let M_1 and M_2 be associated with (ν_1, n_1) and (ν_2, n_2) respectively. Then M_1 and M_2 are unitarily equivalent if and only if ν_1 and ν_2 have the same null sets and $n_1 = n_2$* ν_1 (or ν_2)-a.e.

Given a normal operator A acting on the Hilbert space K, the representation theorem produces ν and n; the invariance theorem tells us that these two objects contain all the information about A that we could want. The customary terminology is that ν is a scalar spectral measure and n is a multiplicity function for A.

From the representation theorem we know there is a unitary operator $U\colon K \to \mathscr{D}$ with $UAU^{-1} = M$. One easily sees that for each Borel set G, $UE(G)U^{-1}$ is equal to M_G, the operator on $\mathscr{D}$ of multiplication by $\mathscr{X}_G$, the characteristic function of G. From this it is clear that E and ν have the same null sets; by the invariance theorem this requirement precisely describes, in terms of E, the possible scalar spectral measures for A.

How do we find n? We present a simple general principle for doing so. First observe that it is enough to determine $n(\lambda)$ ν-a.e. on $\sigma(A)$ since $\mathbf{C}\backslash \sigma(A)$ has ν-measure zero. Let $\mathscr{S}$ be a countable spanning subset of K. Clearly $\{Ux\colon x \in \mathscr{S}\}$ spans $\mathscr{D}$.

Lemma 1. *For ν-almost every λ, $\{(Ux)(\lambda)\colon x \in \mathscr{S}\}$ spans $H_{n(\lambda)}$.*

Proof. Fix k in $\{1, 2, \cdots\} \cup \{\infty\}$ and put $G_k = \{\lambda\colon n(\lambda) = k\}$. Let us pick a countable dense subset $\mathscr{E}$ of H_k and choose c in $\mathscr{E}$. Define a function f by $f(\lambda) = c$ for λ in G_k and $f(\lambda) = 0$ otherwise. Then f is in $\mathscr{D}$

so there exists a sequence $\{f_n\}$ of functions in $\mathscr{D}$, each of which is a finite linear combination of $\{Ux: x \in \mathscr{S}\}$, with

$$\int \|f_n(\lambda) - f(\lambda)\|^2 \, d\nu(\lambda) = \|f_n - f\|^2 \to 0$$

as $n \to \infty$. There is a subsequence $\{f_{n_j}\}$ of $\{f_n\}$ and a set Y_c of ν-measure zero such that $\|f_{n_j}(\lambda) - f(\lambda)\| \to 0$ as $j \to \infty$ whenever $\lambda \notin Y_c$. But $f_{n_j}(\lambda)$ is in the closed linear span of $\{(Ux)(\lambda): x \in \mathscr{S}\}$ and therefore so is c. It follows that this linear span contains $\mathscr{E}$ whenever $\lambda \in G_k \setminus \bigcup \{Y_c: c \in \mathscr{E}\}$. The union $\bigcup \{Y_c: c \in \mathscr{E}\}$ has ν-measure zero and thus we have shown that

$$\text{span } \{(Ux)(\lambda): x \in \mathscr{S}\} = H_{n(\lambda)}$$

ν-a.e. on G_k. Since k is arbitrary, we are done.

Now $n(\lambda)$, the object of our quest, is just the dimension of $H_{n(\lambda)}$. By Lemma 1, this is ν-a.e. equal to the supremum of those natural numbers k for which there exist $x_1, \cdots, x_k$ in $\mathscr{S}$ with $(Ux_1)(\lambda)$, $(Ux_2)(\lambda)$, $\cdots$, $(Ux_k)(\lambda)$ linearly independent. Note that these vectors are linearly independent precisely when the Gram matrix

$$[\langle (Ux_i)(\lambda), (Ux_j)(\lambda)\rangle]_{i,j=1}^k$$

has non-zero determinant; we will be in business if we can calculate this Grammian in terms of E. We choose vectors x and y in K and define a complex measure $\nu_{x,y}$ by

$$\nu_{x,y}(G) = \langle E(G)x, y\rangle$$

for G a Borel set in the plane. Then

$$\begin{aligned} \nu_{x,y}(G) &= \langle UE(G)x, Uy\rangle \\ &= \langle M_G Ux, Uy\rangle \\ &= \int_G \langle (Ux)(\lambda), (Uy)(\lambda)\rangle \, d\nu(\lambda). \end{aligned}$$

Let $d\nu_{x,y}/d\nu$ be some choice of a Radon-Nikodym derivative of $\nu_{x,y}$ with respect to ν. Since $\mathscr{S}$ is a countable collection of vectors, it must be the case that for ν-almost every λ,

$$\frac{d\nu_{x,y}}{d\nu}(\lambda) = \langle (Ux)(\lambda), (Uy)(\lambda)\rangle$$

for all x, y in $\mathscr{S}$. Thus we have the following

GENERAL PRINCIPLE. *We may take $n(\lambda)$ to be the supremum of those natural numbers k for which there exist $x_1, \cdots, x_k$ in $\mathscr{S}$ such that*

$$\det\left[\frac{d\nu_{x_i,x_j}}{d\nu}(\lambda)\right]_{i,j=1}^{k} \neq 0.$$

This makes sense ν-a.e. and thus gives a ν-a.e. determination of $n(\lambda)$.

3. Multiplication operators. Let a be a bounded, complex-valued Borel function on $[0, 1]$ and consider the normal operator A on $K = L^2(0, 1)$ given by $(Af)(x) = a(x)f(x)$. The reader who wishes to consider only self-adjoint operators should take a to be real valued. We know that $\|A\|$ is equal to $\|a\|_\infty$, the essential sup norm of a with respect to Lebesgue measure m on $[0, 1]$, and $\sigma(A)$ coincides with $R_e(a)$, the essential range of a with respect to m. The spectral measure E of A is easily seen to be given by $E(G)f = \mathscr{X}_{a^{-1}(G)} f$ and thus we may take our scalar spectral measure ν to satisfy $\nu(G) = m(a^{-1}(G))$. The closed supports of ν and E coincide with $\sigma(A) = R_e(a)$.

The multiplicity function n for A is more interesting. First note that we only need to determine n ν-a.e. on $R_e(a)$. Let us write $\# F$ for the cardinality of a finite set F and put $\# F = \infty$ if F is infinite. According to folk wisdom, $n(\lambda) = \# a^{-1}(\{\lambda\})$. While this is false in general, it can be made correct by replacing the set-theoretic pre-image $a^{-1}(\{\lambda\})$ by a certain measure-theoretic analogue. For each complex λ let $B_\delta(\lambda)$ denote the closed disk of radius δ centered at λ. If λ is in $R_e(a)$ and S a Borel set in $[0, 1]$, we write

$$D(S, \lambda) = \liminf_{\delta \downarrow 0} \frac{m(S \cap a^{-1}(B_\delta(\lambda)))}{m(a^{-1}(B_\delta(\lambda)))}.$$

For each λ in $R_e(a)$, we define the essential pre-image of λ, denoted $a_e^{-1}(\lambda)$, to be the set of those points x in $[0, 1]$ such that $D(U, \lambda) > 0$ whenever U is a (relatively) open subset of $[0, 1]$ containing x. Note that this makes sense inasmuch as $m(a^{-1}(B_\delta(\lambda))) > 0$ whenever $\lambda \in R_e(a)$. We define $a_e^{-1}(\lambda)$ to be empty if $\lambda \notin R_e(a)$. Heuristically, we can think of $a_e^{-1}(\lambda)$ as the set of those points x such that for every open set U containing x, there is a positive probability that a solution t of $a(t) = \lambda$ lies in U. The set $a_e^{-1}(\lambda)$ is always closed, and if a is continuous, then $a_e^{-1}(\lambda)$ is a subset of $a^{-1}(\{\lambda\})$. For examples and further properties the reader should consult both **[2]**, where the following determination of n was made, and the fine exposition by Abrahamse of the direct integral point of view **[1]**. For related work and further progress see **[3, 4, 5, 14, 15]**.

THEOREM. *$n(\lambda) = \# a_e^{-1}(\lambda)$ ν-a.e.*

For our proof we will need a generalization due to Besicovitch **[6]** of a standard theorem in real variables.

BESICOVITCH'S THEOREM. *Let α and β be measures with α absolutely continuous with respect to β. Then for β-almost every λ,*

$$\lim_{\delta \downarrow 0} \frac{\alpha(B_\delta(\lambda))}{\beta(B_\delta(\lambda))} \text{ exists and equals } \frac{d\alpha}{d\beta}(\lambda).$$

If β is planar Lebesgue measure (or linear Lebesgue measure if we are considering the self-adjoint case), this is just the usual differentiation theorem [17, Chap. 8]. In fact, if β is absolutely continuous with respect to Lebesgue measure, we can multiply and divide in the above quotient by the Lebesgue measure of $B_\delta(\lambda)$ and recover the Besicovitch theorem from the standard version and the chain rule.

To put this to use we fix a Borel set S in [0, 1] and define a measure μ_S on the complex plane by $\mu_S(G) = m(S \cap a^{-1}(G))$. Then set $\alpha = \mu_S$ and $\beta = \nu$; the Besicovitch theorem says that the lim inf defining $D(S, \lambda)$ is actually a limit ν-*a.e.* and furthermore, $D(S, \lambda)$ is a legitimate choice of the Radon-Nikodym derivative of μ_S with respect to ν. That is, we can take

$$\frac{d\mu_S}{d\nu}(\lambda) = D(S, \lambda), \lambda \in R_e(a). \tag{1}$$

Let $\mathscr{A}$ denote the algebra of all finite unions of intervals (open, closed, half-open, and degenerate) in [0, 1] with rational endpoints. $\mathscr{A}$ is a countable collection of sets. We write $\bar{S}$ for the closure of the set S.

LEMMA 2. *For every λ in $R_e(a)$ the set functions $S \to D(S, \lambda)$ is monotone on the class of all Borel sets in* [0, 1]. *For ν-almost every λ in $R_e(a)$, this set function is finitely additive on $\mathscr{A}$ and $D(S, \lambda) = D(\bar{S}, \lambda)$ for every S in $\mathscr{A}$.*

PROOF. The first statement is obvious. If $\delta > 0$ is fixed, the remaining conclusions of the lemma hold for the set function $S \to m(S \cap a^{-1}(B_\delta(\lambda)))$. If we divide by $m(a^{-1}(B_\delta(\lambda)))$ and let $\delta \to 0$, we see, using Besicovitch's assertion that the limit of the quotient exists for ν-almost every λ, that the lemma follows. Note that the countability of $\mathscr{A}$ and the fact that a countable union of sets of ν-measure zero has ν-measure zero are both strongly in use here.

LEMMA 3. *For ν-almost every λ in $R_e(a)$ we have $D(F, \lambda) = 0$ whenever F is a compact set disjoint from $a_e^{-1}(\lambda)$. For such λ, $a_e^{-1}(\lambda)$ is non-empty.*

PROOF. Suppose that λ is chosen so that the conclusions of Lemma 2 hold. The definition of $a_e^{-1}(\lambda)$ tells us that for each x in F there is an open interval V_x in $\mathscr{A}$ containing x with $D(V_x, \lambda) = 0$. We can select $x_1, \cdots, x_n$ in F so that $V_{x_1}, \cdots, V_{x_n}$ cover F. Then we have

$$D(F, \lambda) \leqq D\left(\bigcup_{i=1}^{n} V_{x_i}, \lambda\right)$$
$$\leqq \sum_{i=1}^{n} D(V_{x_i}, \lambda) = 0$$

as desired. If $a_e^{-1}(\lambda)$ were empty we could take $F = [0, 1]$ and contradict the fact that $D([0, 1], \lambda) = 1$. This completes the proof.

Our final lemma is a weak analogue for $a_e^{-1}(\lambda)$ of the triviality that $a^{-1}(\{\lambda_1\})$ and $a^{-1}(\{\lambda_2\})$ do not intersect when $\lambda_1 \neq \lambda_2$. A variant of this fact was used in [15]; we give the proof here for completeness.

LEMMA 4. *Let y be fixed in* $[0, 1]$ *and let X_y be the set of all points λ such that $a_e^{-1}(\lambda)$ is finite and contains y. Then X_y has ν-measure zero.*

PROOF. Assume that $0 < y < 1$. The cases $y = 0$ and $y = 1$ are entirely similar. Pick rational sequences $\{r_n\}$ and $\{t_n\}$ in $[0, 1]$ with $r_n < y < t_n$, $r_n \uparrow y$ and $t_n \downarrow y$. For any z in $R_e(a)$, $D((r_n, t_n), z)$ is non-increasing in n by Lemma 2 and so we can define a function g on $R_e(a)$ (and thus ν-*a.e.*) by

$$g(z) = \lim_{n \to \infty} D((r_n, t_n), z).$$

By the monotone convergence theorem and (1) we have

$$\int g \, d\nu = \lim_{n\to\infty} \int \frac{d\mu_{(r_n, t_n)}}{d\nu} \, d\nu$$
$$= \lim_{n\to\infty} m((r_n, t_n))$$
$$= 0.$$

It follows that $g = 0$ ν-a.e. We will be done if we can show that $g > 0$ ν-*a.e.* on X_y.

Let us fix λ in X_y; by keeping λ outside of an appropriate set of ν-measure zero we can and do assume that the conclusions of Lemmas 2 and 3 are in effect for this λ. Let N be so large that $[r_n, t_n]$ contains only the one point y of $a_e^{-1}(\lambda)$ whenever $n \geqq N$. Let V be an open set in $\mathscr{A}$, disjoint from $[r_n, t_n]$ whenever $n \geqq N$, and containing $a_e^{-1}(\lambda)\backslash\{y\}$. Now when $n \geqq N$, $F_n = [0, 1]\backslash((r_n, t_n) \cup V)$ is a compact set in $\mathscr{A}$ which contains no points of $a_e^{-1}(\lambda)$, and thus $D(F_n, \lambda) = 0$ by Lemma 3. Thus, by finite additivity we find, whenever $n \geqq N$,

$$\begin{aligned} 1 &= D([0, 1], \lambda) \\ &= D([0, 1]\backslash F_n, \lambda) \\ &= D((r_n, t_n) \cup V, \lambda) \\ &= D((r_n, t_n), \lambda) + D(V, \lambda). \end{aligned}$$

Thus $D((r_n, t_n), \lambda)$ is independent of n when $n \geqq N$, and it is certainly positive since (r_n, t_n) contains the point y in $a_e^{-1}(\lambda)$. Thus $g(\lambda) > 0$ and the proof is complete.

PROOF OF THE THEOREM. Let $\mathbf{Q}$ denote the rational numbers in $[0, 1]$, and for each t in $\mathbf{Q}$ let $\mathscr{X}_t$ denote the characteristic function of $[0, t]$. We will apply the general principle of §2 to the countable spanning set $\mathscr{S} = \{\mathscr{X}_t \colon t \in \mathbf{Q} \text{ and } t > 0\}$. We write $\nu_{s,t}$ for the measure $\nu_{\mathscr{X}_s, \mathscr{X}_t}$. For any Borel set G and positive s, t in $\mathbf{Q}$ we have

$$\begin{aligned} \nu_{s,t}(G) &= \langle E(G)\mathscr{X}_s, \mathscr{X}_t\rangle \\ &= \int_{a^{-1}(G)} \mathscr{X}_s \mathscr{X}_t \, dm \\ &= m([0, \min(s, t)] \cap a^{-1}(G)) \\ &= \mu_{[0, \min(s,t)]}(G). \end{aligned}$$

Thus by (1) we may take, for any positive s, t in $\mathbf{Q}$,

$$\frac{d\nu_{s,t}}{d\nu}(\lambda) = D([0, \min(s, t)], \lambda), \lambda \in R_e(a).$$

Let us fix λ in $R_e(a)$, pick rationals $0 < t_1 < t_2 < \cdots < t_k \leqq 1$, and put $b_j = D([0, t_j], \lambda)$. Then our Gram matrix is

$$\left[\frac{d\nu_{t_i, t_j}}{d\nu}(\lambda)\right]_{i,j=1}^{k} = \begin{bmatrix} b_1 & b_1 & b_1 & \cdots & \cdots & b_1 \\ b_1 & b_2 & b_2 & \cdots & \cdots & b_2 \\ b_1 & b_2 & b_3 & \cdots & \cdots & b_3 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & b_{k-1} & b_{k-1} \\ b_1 & b_2 & b_3 & \cdots & b_{k-1} & b_k \end{bmatrix}.$$

The determinant of such an "L-shaped" matrix has been calculated **[8]**. Indeed, if one subtracts the $(k-1)^{\text{th}}$ column from the k^{th}, the $(k-2)^{\text{th}}$ from the $(k-1)^{\text{th}}$, etc., one arrives at a lower triangular matrix with diagonal entries $b_1, b_2 - b_1, \cdots, b_k - b_{k-1}$. Thus

$$\text{(2)} \qquad \det\left[\frac{d\nu_{t_i, t_j}}{d\nu}(\lambda)\right]_{i,j=1}^{k} = b_1(b_2 - b_1)(b_3 - b_2)\cdots(b_k - b_{k-1}),$$

and our general principle takes this form: $n(\lambda)$ *can be taken to be the supremum of those natural numbers* k *for which there exist rationals* $0 < t_1 <$

$t_2 < \cdots < t_k \leqq 1$ *such that the determinant* (2) *is positive.* Note that there are such natural numbers because

$$\frac{d\nu_{1,1}}{d\nu}(\lambda) = D([0, 1], \lambda) = 1.$$

Without loss of generality we can make some assumptions on λ. By avoiding an appropriate ν-null set we can and do assume that the conclusions of Lemmas 2 and 3 are in effect for λ. In addition, Lemma 4 implies that $\bigcup \{X_y : y \in \mathbf{Q}\}$ has ν-measure zero. We will stipulate that this union does not contain λ, which is to say: if $a_e^{-1}(\lambda)$ is finite, it contains no rational numbers. From Lemma 2 we have

$$\begin{aligned} b_i - b_{i-1} &= D([0, t_i], \lambda) - D([0, t_{i-1}], \lambda) \\ &= D((t_{i-1}, t_i), \lambda), i = 2, \cdots, k; \end{aligned} \tag{3}$$

this will allow us to interpret (2).

We know that $a_e^{-1}(\lambda)$ is non-empty, so first suppose that $a_e^{-1}(\lambda)$ contains at least k points, $s_1 < s_2 < \cdots < s_k$. Whether $a_e^{-1}(\lambda)$ is finite or not, we can assume that $s_k < 1$. Clearly we can choose rationals $0 < t_1 < t_2 < \cdots < t_k \leqq 1$ with $s_1 \in [0, t_1)$, and $s_i \in (t_{i-1}, t_i)$, $i = 2, \cdots, k$. Then by the definition of $a_e^{-1}(\lambda)$, $D([0, t_1), \lambda)$ and $D((t_{i-1}, t_i), \lambda)$, $i = 2, \cdots, k$, are all positive and (3) implies that our determinant (2) is positive. Our general principle then implies that $n(\lambda) \geqq k$, giving us half of the conclusion: $n(\lambda) \geqq \# a_e^{-1}(\lambda)$.

For the other direction, assume that $a_e^{-1}(\lambda) = \{s_1, s_2, \cdots, s_p\}$ with $s_{j-1} < s_j$ for $j = 2, \cdots, p$. Let $k > p$, consider any rationals $0 < t_1 < t_2 < \cdots < t_k \leqq 1$, and put $t_0 = 0$. To show that $n(\lambda) \leqq p$ (and thus complete the proof) it will suffice to argue that the determinant (2) is zero. Since each s_j is irrational, none of the t_i's coincides with any s_j; furthermore $0 < s_0$ and $s_1 < 1$. Since $k > p$, one of the closed intervals $[0, t_1]$, $[t_1, t_2]$, $\cdots$, $[t_{k-1}, t_k]$, call it $[t_{i-1}, t_i]$, must lie inside one of the intervals $[0, s_1)$, (s_1, s_2), $\cdots$, $(s_p, 1]$. It follows from Lemma 3 that $D([t_{i-1}, t_i], \lambda) = 0$. On invoking (3) and Lemma 2 we see that $b_1 = 0$ (if $i = 1$), or $b_i - b_{i-1} = 0$ (if $i > 1$). In either case the Gram determinent (2) vanishes and the proof is complete.

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