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AN ELEMENTARY COUNTEREXAMPLE TO THE FINITENESS CONJECTURE*

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Abstract. We prove that there exist (infinitely many) values of the real parameters a and b for which the matrices

$$a\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad b\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

have the following property: all infinite periodic products of the two matrices converge to zero, but there exists a nonperiodic product that doesn't. Our proof is self-contained and fairly elementary; it uses only elementary facts from the theory of formal languages and from linear algebra. It is not constructive in that we do not exhibit any explicit values of a and b with the stated property; the problem of finding explicit matrices with this property remains open.

Key words. spectral radius, generalized spectral radius, joint spectral radius, finiteness conjecture, matrix semigroup

AMS subject classifications. 15A18, 15A60, 93D09

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1. Introduction.

The Lagarias–Wang finiteness conjecture was introduced in 1995 in connection with problems related to spectral radius computation of finite sets of matrices. Let $\rho(A)$ be the spectral radius[1] of the matrix A and let Σ be a finite set of matrices. The generalized spectral radius of Σ is defined by

$$\rho(\Sigma) = \limsup_{k\to+\infty} \max\{\rho(A_1 \cdots A_k)^{1/k} : A_i \in \Sigma,\ i = 1, \ldots, k\}.$$

This quantity was introduced in [7, 8]. The generalized spectral radius is known to coincide (see [1]) with the earlier defined joint spectral radius [13]; we refer to these quantities simply as "spectral radius." The notion of the spectral radius of a set of matrices appears in a wide range of contexts and has led to a number of recent contributions (see, e.g., [2, 3, 6, 8, 11, 15, 16, 17]); a list of over a hundred related contributions is given in [14]. We describe below one particular occurrence in a dynamical system context.

We consider systems of the form $x_{t+1} = A_t x_t$, where Σ is a finite set of matrices, and $A_t \in \Sigma$ for every $t \geq 0$. We do not impose any restrictions on the sequence of matrices A_t. These are exactly the discrete-time linear time-varying systems for

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[1]The spectral radius of a matrix is equal to the magnitude of its largest eigenvalue.

which the dynamics is taken from a finite set at every time instant. Starting from the initial state x_0, we obtain

$$x_{t+1} = A_t \cdots A_1 A_0 x_0.$$

The spectral radius of Σ is known to characterize how fast x_t can possibly grow with t; see [6, 7]. In particular, the trajectories all converge to the origin if and only if $\rho(\Sigma) < 1$.

We now describe the finiteness conjecture. It is known that

$$\rho(\Sigma) \geq \max\{\rho(A_1 \cdots A_k)^{1/k} : A_i \in \Sigma,\ i = 1, \dots, k\}$$

for all $k \geq 0$.

Finiteness conjecture. Let Σ be a finite set of matrices. Then there exists some $k \geq 1$ and a matrix $A = A_1 \dots A_k$ with $A_i \in \Sigma$ such that $\rho(A)^{1/k} = \rho(\Sigma)$.

This conjecture appears in [12]. The problem of determining if the conjecture is true appears under a different guise in [5], where it is attributed to E. S. Pyatnicky.

In terms of the dynamical system interpretation given above, this conjecture can be restated as saying that the convergence to zero of all periodic products of a given finite set of matrices implies the same for all possible products.

The conjecture has recently been proved to be false [4]. The existence of a counterexample is proved in [4] by using iterated function systems, topical maps, and Sturmian sequences. The proof relies in part on a particular fixed point theorem known as Mañé's lemma. In this contribution, we provide an alternative proof. We prove that there are uncountably many values of the real parameter α for which the pair of matrices

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \alpha \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

does not satisfy the finiteness conjecture. Our proof is not constructive in that we do not exhibit any particular value of α for which the corresponding pair of matrices violates the finiteness conjecture. The problem of finding an explicit counterexample and the problem of determining if there exist matrices with rational entries that violate the conjecture remain open questions. As compared to the proof in [4], our proof has the advantage of being self-contained and fairly elementary; it uses only elementary facts from linear algebra.

2. Proof outline.

Let us now briefly outline our proof. We define

$$A_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

and

$$A_0^\alpha = \frac{1}{\rho_\alpha} A_0, \quad A_1^\alpha = \frac{\alpha}{\rho_\alpha} A_1$$

with $\rho_\alpha = \rho(\{A_0, \alpha A_1\})$. Since $\rho(\lambda\Sigma) = |\lambda|\, \rho(\Sigma)$, the spectral radius of the set $\Sigma_\alpha = \{A_0^\alpha, A_1^\alpha\}$ is equal to one. Let $I = \{0, 1\}$ be a two-letter alphabet and let $I^+ = \{0, 1, 00, 01, 10, 11, 000, \dots\}$ be the set of finite nonempty words. We will also denote the empty word by $\emptyset$ and use the notation $I^* = I^+ \cup \{\emptyset\}$. To the word $w = w_1 \dots w_t \in I^+$ we associate the products $A_w = A_{w_1} \dots A_{w_t}$ and $A_w^\alpha = A_{w_1}^\alpha \dots A_{w_t}^\alpha$.

A word $w \in I^+$ will be said to be *optimal* for some α if $\rho(A_w^\alpha) = 1$. We use J_w to denote the set of α's for which $w \in I^+$ is optimal. If the finiteness conjecture is true, the union of the sets J_w for $w \in I^+$ covers the real line. We show that this union does not cover the interval $[0, 1]$.

In section 4, we show that if two words $u, v \in I^+$ are essentially equal, then $J_u = J_v$. Two words $u, v \in I^+$ are *essentially equal* if the periodic infinite words $U = uu\dots$ and $V = vv\dots$ can be decomposed as $U = xww\dots$ and $V = yww\dots$ for some $x, y, w \in I^+$. Words that are not essentially equal are *essentially different.* Obviously, if u and v are essentially different, then so are cyclic permutations of u and v. We show in the same section that the sets J_u and J_v are disjoint if u and v are essentially different. This part of the proof requires some properties of infinite words presented in section 3. The proof is then almost complete. To conclude, we observe in section 5 that the sets $J_w \cap [0, 1]$ are closed subintervals of $[0, 1]$. There are countably many words in I^+, and so $\cup_{w \in I^+}(J_w \cap [0, 1])$ is a countable union of disjoint closed subintervals of $[0, 1]$. Except for a trivial case that we can exclude here, there are always uncountably many points in $[0, 1]$ that do not belong to such a countable union. Each of these points provides a particular counterexample to the finiteness conjecture.

3. Palindromes in infinite words. The *length* of a word $w = w_1 \dots w_t \in I^*$ is equal to $t \geq 0$ and is denoted by $|w|$. The *mirror image* of w is the word $\tilde{w} = w_t \dots w_1 \in I^*$. A *palindrome* is a word in I^* that is identical to its mirror image. In particular, the empty word is a palindrome. For $u, v \in I^*$, we write $u > v$ if u is lexicographically larger than v, that is, $u_i = 1$, $v_i = 0$ for some $i \geq 1$ and $u_j = v_j$ for all $j < i$. This is only a partial order since, for example, 101000 and 1010 are not comparable. For an infinite word U, we denote by $F(U)$ the set of all finite factors of U.

LEMMA 3.1. *Let $u, v \in I^+$ be two words that are essentially different. We denote $U = uuu\dots$ and $V = vvv\dots$. Then there exists a pair of words $0p0$ and $1p1$ in the set $F(U) \cup F(V)$ such that $p \in I^*$ is a palindrome.*

Proof. Let m and n be the minimal periods of U and V, respectively. The values of m and n are invariant under cyclic permutations of u and v. Let us use induction on $m + n$. The result is obvious for $m + n = 2$ since in this case U and V must be equal to $111\dots$ and $000\dots$, and we may then take $p = \emptyset$. Consider now $u, v \in I^+$. If the words 00 and 11 both belong to the set $F(U) \cup F(V)$, then we can set $p = \emptyset$. So assume without loss of generality that 11 does not belong to $F(U) \cup F(V)$. We may also assume that both u and v begin with 0; otherwise, we can take appropriate cyclic permutations of u and v. Then u and v can be factorized in a unique way by factors $0' = 0$ and $1' = 01$.

In the new alphabet $\{0', 1'\}$, the resulting words u' and v' are still essentially different and the minimal periods m' and n' of $U' = u'u'\dots$ and $V' = v'v'\dots$ satisfy $m' + n' < m + n$. By induction, there exists a pair of words $0'q'0'$ and $1'q'1'$ in $F(U') \cup F(V')$ such that $q' = \tilde{q'}$. Let q be the word obtained from q' by replacing $0'$ by 0 and $1'$ by 01. From $1'q'1' \in F(U') \cup F(V')$ we get $01q01 \in F(U) \cup F(V)$. Since $0'q'0' \in F(U') \cup F(V')$ we have $0'q'0'0' \in F(U') \cup F(V')$ or $0'q'0'1' \in F(U') \cup F(V')$, and thus $0q00 \in F(U) \cup F(V)$. Define now $p = q0$ and observe that $0p0, 1p1 \in F(U) \cup F(V)$.

Finally, let us show that if q' is a palindrome in $\{0', 1'\}$, then $q0$ is a palindrome in $\{0, 1\}$. We use induction on $|q'|$. For $|q'| = 0, 1$ the statement is obviously true. Suppose that $|q'| \geq 2$. Then $q' = 0's'0'$ or $q' = 1's'1'$ for $s' \in \{0', 1'\}^*$, and s' is a palindrome in $\{0', 1'\}$. By induction hypothesis, $s0$ is then a palindrome in $\{0, 1\}$. We then have that either $q = 0s00$ or $q = 01s010$, but since $s0$ is a palindrome it follows that p is also a palindrome. ∎

COROLLARY 3.2. *Let $u, v \in I^+$ be two essentially different words and let $U = uuu\ldots$ and $V = vvv\ldots$. Then there exist words $a, b, x, y \in I^+$ satisfying $|x| = |y|$, $x > y$, $\tilde{x} > \tilde{y}$, $x > \tilde{y}$, $\tilde{x} > y$, and a palindrome $p \in I^*$ such that*

$$U = apxpxp\ldots \quad \text{and} \quad V = bpypyp\ldots$$

or one of the words U and V, say U, can be decomposed as

$$U = apxpxp\cdots = bpypyp\ldots.$$

Proof. By Lemma 3.1, there exists a pair of words $0p0$ and $1p1$ in the set $F(U) \cup F(V)$ such that p is a palindrome. Without loss of generality, assume that $1p1$ occurs in U. Then it occurs in U infinitely many times because U is periodic. Let us write

$$U = a'1p1d1p1d\ldots$$

and, analogously,

$$W = b'0p0f0p0f\ldots,$$

where W is either U or V. Without loss of generality we may assume $|d| = |f|$; otherwise, we can always take $d' = d1p1d\ldots1p1d$ instead of d and $f' = f0p0f\ldots0p0f$ instead of f in such a way that $|f'| = |d'|$. It remains to set $a = a'1$, $b = b'0$, $x = 1d1$, and $y = 0f0$. ∎

4. Optimal words are essentially equal. For a given word $w \in I^+$ we define $J_w = \{\alpha : \rho(A_w^\alpha) = 1\}$. Our goal in this section is to prove that J_u and J_v are equal when u and v are essentially equal, and have otherwise empty intersection.

LEMMA 4.1. *Let $u, v \in I^+$ be two words that are essentially equal. Then $J_u = J_v$.*

Proof. Assume $u, v \in I^+$ are essentially equal. Then $U = uu\ldots$ and $V = vv\ldots$ can be written as $U = ss\ldots$ and $V = tt\ldots$ with $|s| = |t|$ and t a cyclic permutation of s. The spectral radius satisfies $\rho(AB) = \rho(BA)$, and so the spectral radius of a product of matrices is invariant under cyclic permutations of the product factors. From this it follows that $\rho(A_s^\alpha) = \rho(A_t^\alpha)$, and hence u is optimal whenever v is. ∎

We need two preliminary lemmas for proving the next result.

LEMMA 4.2. *For any word $w \in I^+$ we have*

$$A_{\tilde{w}} - A_w = k(w)T,$$

where $k(w)$ is an integer and

$$T = A_0A_1 - A_1A_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Moreover, $k(w)$ is positive if and only if $w > \tilde{w}$.

Proof. Let us prove by induction that

$$A_w = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

implies

$$A_{\tilde{w}} = \begin{pmatrix} d & b \\ c & a \end{pmatrix}.$$

Indeed, this is true for $w = 0$ and $w = 1$. Notice also that

$$A_0 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+c & b+d \\ c & d \end{pmatrix}$$

and

$$\begin{pmatrix} d & b \\ c & a \end{pmatrix} A_0 = \begin{pmatrix} d & b+d \\ c & a+c \end{pmatrix},$$

and similarly for A_1. From this it follows that $A_{\tilde{w}} - A_w = k(w)T$. The sign relation follows from the fact that

$$A_0 \begin{pmatrix} a & b \\ c & d \end{pmatrix} A_1 - A_1 \begin{pmatrix} d & b \\ c & a \end{pmatrix} A_0 = \begin{pmatrix} a+b+c & 0 \\ 0 & -(a+b+c) \end{pmatrix}. \qquad \square$$

We say that a matrix A *dominates* B if $A \geq B$ componentwise and $\mathrm{tr}A > \mathrm{tr}B$ (tr denotes the trace). The eigenvalues of the 2×2 matrix A are given by $(\mathrm{tr}A \pm \sqrt{(\mathrm{tr}A)^2 - 4\det A})/2$. For all words w, the matrix A_w satisfies $\det(A_w) = 1$ and $\mathrm{tr}(A_w) \geq 2$. We therefore have $\rho(A_u) > \rho(A_v)$ whenever A_u dominates A_v.

LEMMA 4.3. *For any word of the form $w = psq$, where $s > \tilde{s}$ and $q < \tilde{p}$, the matrix $A_{w'}$ with $w' = p\tilde{s}q$ dominates A_w.*

Proof. We have $A_{w'} - A_w = k(s)A_pTA_q$, $k(s) > 0$. The relations $A_iTA_i = T$, $i = 0, 1$, and

$$A_1TA_0 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

finish the proof. $\square$

Let $w = psq$. If $s > \tilde{s}$ and $q < \tilde{p}$, we say that $s \to \tilde{s}$ is a *dominating flip*. We are now ready to prove the main result of this section.

LEMMA 4.4. *Let $u, v \in I^+$ be two words that are essentially different. Then $J_u \cap J_v = \emptyset$.*

Proof. Let $u, v \in I^+$ be two words that are essentially different. We assume without loss of generality that neither $U = uu\ldots$ nor $V = vv\ldots$ is equal to $00\ldots$ or $11\ldots$ because $11\ldots$ is not optimal for any $\alpha \in [0, 1]$ and $00\ldots$ is only optimal for $\alpha = 0$, but no other word is optimal for $\alpha = 0$. In order to prove the result we show that if $\rho(A_u^\alpha) = \rho(A_v^\alpha)$ for some value of α, then there exists a word w satisfying $\rho(A_w^\alpha) > \rho(A_u^\alpha)$.

By Corollary 3.2, there exist words $a, b, x, y \in I^+$ satisfying $|x| = |y|$, $x > y$, $\tilde{x} > \tilde{y}, x > \tilde{y}, \tilde{x} > y$, and a palindrome $p \in I^*$ such that

$$U = apxpxp\ldots \quad \text{and} \quad V = bpypyp\ldots$$

or

$$U = apxpxp\ldots = bpypyp\ldots.$$

Since neither U nor V is equal to $00\ldots$ or $11\ldots$, the matrices A_{xp} and A_{yp} are strictly positive.

Let us consider the word $xpxpxpypypyp$. Setting $s = xpy$, we make a dominating flip in this word and get the word $xpxp\tilde{y}p\tilde{x}pypyp$. Then we set $s = xp\tilde{y}p\tilde{x}py$ and make another dominating flip. As a result, the matrix $A_{xpxpxpypypyp}$ is dominated by the matrix $A_{xp\tilde{y}pxpyp\tilde{x}pyp}$. Analogously, any matrix $A_s A_v A_r$, $v \in I^*$, is dominated by the matrix $A_{s'} A_v A_{r'}$ where $s = xpxpxp$, $r = ypypyp$, $s' = xp\tilde{y}pxp$, and $r' = yp\tilde{x}pyp$. Let us denote the linear operators $A \to A_s A A_r$ and $A \to A_{s'} A A_{r'}$ acting in $\mathbb{R}^4$ as well as their 4×4 matrices by L and L', respectively. It is known that $L = A_r^T \otimes A_s$ and $L' = (A_r')^T \otimes A_s'$, where $\otimes$ is used to denote the Kronecker (tensor) product (see [10, Lemma 4.3.1]). Both L and L' are strictly positive. The minimal closed convex cone in $\mathbb{R}^4$ containing all matrices A_v, $v \in I^*$, is the cone of all nonnegative 2×2 matrices. Indeed, any nonnegative matrix X with $\det(X) = 0$ can be approximated by matrices of the form βA_w, $\beta > 0$, $w \in I^*$. In particular, this is true for the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence $L' \geq L$ elementwise and $L \neq L'$. From the Perron–Frobenius theory (see, for instance, Problem 8.15 in [9]) we get $\rho(L') > \rho(L)$. The spectral radius of a Kronecker product is the product of the spectral radii (see [10, Theorem 4.2.12]), and so

$$\rho(L') = \rho(A_{s'})\rho(A_{r'}) > \rho(A_s)\rho(A_r) = \rho(L).$$

Since the flips performed do not change the average proportion of matrices A_0 and A_1 in the product, we can also write

$$\rho(L^\alpha) = \rho(A_s^\alpha)\rho(A_r^\alpha) \quad \text{and} \quad \rho(L'^\alpha) = \rho(A_{s'}^\alpha)\rho(A_{r'}^\alpha)$$

for each $\alpha > 0$, where L^α and L'^α are defined analogously to L and L'. Suppose that $\rho(A_u^\alpha) = \rho(A_v^\alpha) = 1$. Then $\rho(A_s^\alpha) = \rho(A_r^\alpha) = 1$ and, hence, either $\rho(A_{s'}^\alpha) > 1$ or $\rho(A_{r'}^\alpha) > 1$, which is a contradiction. ∎

5. Finiteness conjecture. We are now ready to prove the main result.

THEOREM 5.1. *There are uncountably many values of the real parameter α for which the pair of matrices*

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \alpha \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

does not satisfy the finiteness conjecture.

Proof. It is clearly equivalent to prove the statement for the matrices A_0^α and A_1^α. For $\alpha = 0$, all optimal words w are essentially equal to 0. For any other word w, the set $J_w \cap [0, 1]$ can be written as

$$J_w \cap [0,1] = \{\alpha \in (0,1] : \rho(A_w^\alpha) = 1\}$$

or, equivalently,

$$J_w \cap [0,1] = \left\{ \alpha \in (0,1] : (\rho(A_w)\alpha^{|w|_1})^{\frac{1}{|w|}} = \sup_{v \in I^+} (\rho(A_v)\alpha^{|v|_1})^{\frac{1}{|v|}} \right\}. \tag{5.1}$$

In this expression $|w|_1$ denotes the number of 1's in the word w. Associated to $w \in I^+$ we define the affine function

$$h_w(\beta) = \frac{1}{|w|}(\ln \rho(A_w) + |w|_1 \beta)$$

and let

$$h(\beta) = \sup_{w \in I^+} h_w(\beta).$$

Passing to the logarithmic scale in the expression (5.1), we get

$$J_w \cap [0,1] = \{e^\beta : \beta \in \mathbb{R}, h_w(\beta) = h(\beta)\} \cap [0,1]. \tag{5.2}$$

The functions h_w are affine, h is convex and continuous, and $h(\beta) \geq h_w(\beta)$ for all $w \in I^+$ and $\beta \in \mathbb{R}$. From this it follows that the set $\{\beta \in \mathbb{R} : h_w(\beta) = h(\beta)\}$ is an interval of the real line. This interval is the zero set of a continuous function, and it is therefore closed. From (5.2) we conclude that $J_w \cap [0,1]$ is a closed subinterval of $[0,1]$.

Let us finally show that $[0,1]$ cannot be covered by countably many disjoint closed intervals H_i, $i \geq 1$ (possibly, single points), unless this is a single interval, which is, obviously, not the case here.

We define a function $g(\alpha) : [0,1] \to [0,1]$ as follows. We set $g(0) = 0$, $g(1) = 1$ and then set $g(\alpha) = 1/2$ for all $\alpha \in H_1$. For each subsequent index i, we define $g(\alpha) = g_i = (a_+ + a_-)/2$ for all $\alpha \in H_i$, where a_- is the current highest value of $g(\cdot)$ at the left of H_i and a_+ is the current lowest value of $g(\cdot)$ at the right of H_i.

As a result, the function $g(\cdot)$ is well-defined on $[0,1] \cap (\cup_{i=1,2,\ldots} H_i)$ and nondecreasing. It can be then extended to the whole segment $[0,1]$ by continuity because between any two segments H_i and H_j there exists a segment H_k with $k > i, j$. Since $g(0) = 0$ and $g(1) = 1$, the range of $g(\alpha)$ coincides with $[0,1]$ for $\alpha \in [0,1]$. Therefore, there exist uncountably many values of $\alpha \in [0,1]$ such that $g(\alpha) \neq g_i$, $i = 1, 2, \ldots$. ∎

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