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AN ELEMENTARY PROOF OF A LIMA'S THEOREM FOR SURFACES

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Abstract

An elementary proof of the following theorem is given:

THEOREM. *Let M be a compact connected surface without boundary. Consider a C^∞ action of $\mathbb{R}^n$ on M. Then, if the Euler-Poincaré characteristic of M is not zero there exits a fixed point.*

The proof given here adapts for dimension two the ideas used by P. Molino and the author in [2] and [3]. Moreover we show that the theorem remains true if $\mathbb{R}^n$ is replaced by a connected nilpotent Lie group G.

In the slightly more general case, dealt with by E.L. Lima, of a surface with boundary, it is sufficient gluing together two copies of this surface in order to obtain a surface without boundary.

1. Actions of $\mathbb{R}^n$

Let V be the Lie algebra of $\mathbb{R}^n$. The action of $\mathbb{R}^n$ induces a Lie algebra homomorphism $v \in V \to X_v \in \mathcal{X}(M)$ called infinitesimal action. We recall that the infinitesimal isotropy of a point p is the set $I(p) = \{v \in V / X_v(p) = 0\}$. As V is abelian $I(p)$ depends only on the orbit.

Denote by Σ_k the set of points p of M whose orbit is k-dimensional, i.e. $\operatorname{codim} I(p) = k$.

Suppose Σ_0 empty. We will gradually arrive to a contradiction.

1) Set $C_2 = \{v \in V / X_v(p) = 0 \text{ for some } p \in \Sigma_2\}$.

As there are at most countably many 2-orbits because they are open sets, C_2 is at most countable union of $(n-2)$-planes of V.

2) The map on the grassmannian of $(n-1)$-planes $h : p \in \Sigma_1 \to I(p) \in g_{n-1}(V)$ is differentiable, i.e. it can be locally extended to a differentiable map.

Indeed, consider $p \in \Sigma_1$ and $u \in V$ such that $X_u(p) \neq 0$. We can find a coordinate system (A, x), $p \in A$, such that $X_u = \frac{\partial}{\partial x_1}$ and that the image of A on $\mathbb{R}^2$ is a rectangle.

Let $\{v_1, \dots v_{n-1}\}$ a basis of $I(p)$. Set $X_{v_j} = f_j \frac{\partial}{\partial x_1} + g_j \frac{\partial}{\partial x_2}$. We define the map

$$\begin{aligned} \widetilde{h} : A &\longrightarrow g_{n-1}(V) \\ x &\longrightarrow \mathbb{R}\{v_1 - f_1 u, \dots v_{n-1} - f_{n-1} u\} \end{aligned}$$

whose differentiability is clear.

Note that $w \in \widetilde{h}(x)$ if and only if $X_w(x)$ is proportional to $\frac{\partial}{\partial x_2}$. If $x \in A \cap \Sigma_1$ this means that $X_w(x) = 0$ because it is also proportional to $\frac{\partial}{\partial x_1}$. Then $\widetilde{h}$ is a local extension of h.

3) Let $Fr(\Sigma_1)$ be the boundary on M of Σ_1. Then $C_1 = \{v \in V / X_v(p) = 0$ for some $p \in Fr(\Sigma_1)\} = \bigcup_{p \in Fr(\Sigma_1)} I(p)$ is of the first category (i.e. it is contained in the union of a countable family of closed nowhere dense subsets of M).

Since $Fr(\Sigma_1)$ can be covered by a finite family of coordinate systems (A, x) as in 2), it will be sufficient to prove that $\bigcup_{p \in A \cap Fr(\Sigma_1)} I(p)$ is of the first category. Let T be a slice of A obtained by doing x_1 constant. As the isotropy is constant on the orbits:

$$\bigcup_{p \in A \cap Fr(\Sigma_1)} I(p) = \bigcup_{p \in T \cap Fr(\Sigma_1)} I(p)$$

Consider the vector bundle $\pi : E \to T$, subbundle of $T \times V$, given by the condition $\pi^{-1}(x) = \{x\} \times \widetilde{h}(x)$. Set $\varphi : (x, v) \in E \to v \in V$.

The set $\pi^{-1}(T \cap Fr(\Sigma_1))$ is of the first category in E because $T \cap Fr(\Sigma_1)$ is of the first category in T. As φ is differentiable and E and V are manifolds of the same dimension, it follows that

$$\varphi(\pi^{-1}(T \cap Fr(\Sigma_1))) = \bigcup_{p \in T \cap Fr(\Sigma_1)} I(p)$$

is of the first category in V.

4) Take now $v \in (V - C_1 \cup C_2)$. The set $Z(X_v)$ of the zeros of X_v is contained in $\overset{0}{\Sigma}_1$. On the other hand the 1–foliations given by:

(a) X_v on $M - Z(X_v)$

(b) the action of $\mathbb{R}^n$ on $\overset{0}{\Sigma}_1$.

agree on $(M - Z(X_v)) \cap \overset{0}{\Sigma}_1$. Then M admits an 1–foliation and $\mathcal{X}(M) = 0$, contradiction.

2. Case of a connected nilpotent Lie group G

It will be sufficient to adapt the proof of the abelian case. Let V be the Lie algebra of G. Since V is nilpotent every subalgebra of codimension one is an ideal. Therefore the isotropy is constant over each 1-orbit and C_1 will still be of the first category.

Let B be a 2-orbit. Given $p \in B$ there always exists an ideal I of codimension one which contains $I(p)$. As B is an orbit and I an ideal then $I(q) \subset I$ for all $q \in B$. Consequently C_2 is contained in a finite or countable union of $(n-1)$-planes of V. In particular $C_1 \cup C_2 \neq V$. The rest is similar.

Example 1. See $P(2, \mathbb{R})$ as the plane $\mathbb{R}^2$ plus the infinite points. The vector fields on $\mathbb{R}^2$: $\frac{\partial}{\partial x_1}$, $\frac{\partial}{\partial x_2}$ and $x_1 \frac{\partial}{\partial x_2}$ can be extended, in a natural way, to $P(2, \mathbb{R})$ because they are affine. These vector fields generate an action of a 3-dimensional nilpotent group on $P(2, \mathbb{R})$, whose orbits are $\mathbb{R}^2$; the set of all points of infinity except the vertical one (i.e. the point associated to the vertical direction); and the infinite vertical point, which is the only fixed point.

Example 2. Tare now $\frac{\partial}{\partial x_1}$, $\frac{\partial}{\partial x_2}$ and $-x_2 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2}$. One obtains an action of a 3-dimensional solvable group with no fixed point. Their orbits are $\mathbb{R}^2$ and the set of the infinite points.

See [**1**] for a 2-dimensional example with no fixed point.

References

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