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AN ELEMENTARY PROOF OF ABRAMOV'S RESULT ON THE ENTROPY OF A FLOW[1]

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§1. Preliminaries

Let T be an automorphism on a probability space (Ω, B, P). Given a finite partition α of Ω, into disjoint measurable sets $A_1, A_2, \cdots, A_n$, its entropy is

$$\text{(D.1)} \qquad H(\alpha) = -\sum_{i=1}^{n} P(A_i)\log P(A_i).$$

Put

$$\text{(D.2)} \qquad h(\alpha, T) = \lim_{n\to\infty} \frac{H(\alpha \vee T\alpha \vee \cdots \vee T^{n-1}\alpha)^{2)3)}}{n}$$

then, the entropy $h(T)$ of T is defined by

$$\text{(D.3)} \qquad h(T) = \sup_{\alpha} h(\alpha, T),$$

where the supremum is taken over all finite measurable partitions.

When $\{T_t\}$ is a measurable flow on a Lebesgue space (Ω, B, P), L.M. Abramov [2] proved the formula

$$\text{(F)} \qquad h(T_t) = |t|\, h(T_1) \text{ for every } t \in R.$$

He proved with use of representation of measurable flow and formulas of entropy of derived automorphism and skew product automorphism. Recently, G. Maruyama [4] pointed out that measurability for $\{T_t\}$ can be raplaced by the continuity of $\{T_t\}$.

In this paper we will given a simple proof of (F), when $\{T_t\}$ is subject to the continuity assumption

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1) This paper is chiefly due to Pinsker's work [1].

2) $\alpha \vee \beta$ means the partition formed by the sets $A_i \cap B_j A_i \in \alpha$, $B_j \in \beta$.

3) According to [3], the limit exists.

(D.4) $$\lim_{t\to 0} P(T_tA\triangle A) = 0 \text{ for every } A\in B.$$

In what follow, (Ω, B, P) denotes an arbitrary abstract probability space.

For later use, we will mention the well-known relations for the entropy [3]:

(1.1) $0 \leqslant H(\alpha/\beta)$,

(1.2) $H(\alpha\vee\beta/\gamma) = H(\alpha/\gamma) + H(\beta/\gamma\vee\alpha)$,

(1.3) if $\alpha \geqslant \beta$ (mod 0) then $H(\alpha/\gamma) \geqslant H(\beta/\gamma)$ and $H(\gamma/\alpha) \leqslant H(\gamma/\beta)$,

(1.4) if T is an automorphism on Ω then $H(\alpha/\beta) = H(T\alpha/T\beta)$.

§2. The theorem.

We need the following lemma.

LEMMA. *Let $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$, $\{\alpha_1', \alpha_2', \cdots, \alpha_n'\}$ be sequences of finite measurable partitions, and ε be an arbitrary positive number.*

If $H(\alpha_i'/\alpha_i) < \varepsilon$ for $i = 1, 2, \cdots, n$, then $H(\bigvee_{i=1}^n \alpha_i) - H(\bigvee_{i=1}^n \alpha_i') > -n\varepsilon$.

Proof. (1.1) (1.2) and (1.3) together imply

$$
\begin{aligned}
& H(\bigvee_{i=1}^n \alpha_i) - H(\bigvee_{i=1}^n \alpha_i') \\
&= H((\bigvee_{i=1}^n \alpha_i) \vee (\bigvee_{i=1}^n \alpha_i')) - H(\bigvee_{i=1}^n \alpha_i' / \bigvee_{i=1}^n \alpha_i) \\
&\quad - H((\bigvee_{i=1}^n \alpha_i) \vee (\bigvee_{i=1}^n \alpha_i') + H(\bigvee_{i=1}^n \alpha_i / \bigvee_{i=1}^n \alpha_i') \\
&\geqslant - H(\bigvee_{i=1}^n \alpha_i' / \bigvee_{i=1}^n \alpha_i) \\
&\geqslant - \sum_{k=1}^n H(\alpha_k' / (\bigvee_{i=1}^n \alpha_i) \vee (\bigvee_{i=1}^{k-1} \alpha_i')) \\
&\geqslant - \sum_{k=1}^n H(\alpha_k'/\alpha_k) > -n\varepsilon.
\end{aligned}
$$

PROPOSITION. *If $\{T_t\}$ is a continuous flow on a probability space (Ω, B, P) and if α is a finite measurable partition, then*

(2.1) $$\sup_{t\neq 0} \frac{1}{|t|} h(\alpha, T_t) = \lim_{t\to 0} \frac{1}{|t|} h(\alpha, T_t) .$$

Proof. Since

$$\lim_{t\to 0} H(T_t\alpha/\alpha) = \lim_{t\to 0}(H(\alpha\vee T_t\alpha) - H(\alpha)) = 0$$

by (D.1) and (D.4), for any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$H(T_t\alpha/\alpha) < \varepsilon \quad \text{for} \quad |t| < \delta. \tag{2.2}$$

On the other hand, if T and t' are arbitrary positive numbers, there exists an integer n such that $T - t' \leqslant (n-1)t' < T$. For any t satisfying $0 < t < \min(\delta, t')$, there exists an integer m such that $(m-1)t \leqslant T < mt$. Consequently, there exists a subsequence $\{m_k\}_{k=1}^{n-1}$ of $1, 2, \cdots, m$ such that $|kt' - m_k t| < \delta$ for $k = 1, 2, \cdots, n-1$.

From (2.2) one obtains

$$H(T_{t'}{}^k\alpha/T_t^{m_k}\alpha) = H(T_{kt'-m_k t}\alpha/\alpha) < \varepsilon. \tag{2.3}$$

From Lemma and (2.3) follows

$$H(\bigvee_{k=0}^{m-1} T_t^k\alpha) \geqslant H(\bigvee_{k=0}^{n-1} T_{m_k t}\alpha) > H(\bigvee_{k=0}^{n-1} T_{t'}{}^k\alpha) - n\varepsilon. \tag{2.4}$$

Then, since $\lim_{T\to\infty} T/m = t$ and $\lim_{T\to\infty} T/n = t'$, on making $T\to\infty$ in (2.4) divided by T and in view of (D.2) we have the relation

$$\frac{1}{t} h(\alpha, T_t) > \frac{1}{t'} h(\alpha, T_{t'}) - \frac{\varepsilon}{t'}.$$

Since ε is arbitrary and since

$$h(\alpha, T_t) = h(\alpha, T_{-t}) \text{ for all } t\in R,$$

$$\varliminf_{t\to 0} \frac{1}{|t|} h(\alpha, T_t) \geqslant \frac{1}{|t'|} h(\alpha, T_{t'}) \tag{2.5}$$

holds. We obtain from (2.5),

$$\varliminf_{t\to 0} \frac{1}{|t|} h(\alpha, T_t) \geqslant \sup_{t'\neq 0} \frac{1}{|t'|} h(\alpha, T_{t'}).$$

Thus, there exists $\lim_{t\to 0} \frac{1}{|t|} h(\alpha, T_t)$

and

$$\lim_{t\to 0} \frac{1}{|t|} h(\alpha, T_t) = \sup_{t\neq 0} \frac{1}{|t|} h(\alpha, T_t).$$

THEOREM. *If $\{T_t\}$ is a continuous flow, then*

$$h(T_t) = |t| h(T_1) \text{ for any real } t.$$

Proof. Define now $h(\{T_t\}) = \sup_{\substack{t \neq 0 \\ \alpha}} \frac{1}{|t|} h(\alpha, T_t)$, where the supremum is taken over all finite measurable partitions and all non-zero numbers.

On the one hand, from (D.3)

$$h(\{T_t\}) = \sup_{t \neq 0} \frac{1}{|t|} h(T_t) \geqslant \overline{\lim_{t \to 0}} \frac{1}{|t|} h(T_t); \tag{2.6}$$

On the other hand, from (2.1)

$$\varliminf_{t \to 0} \frac{1}{|t|} h(T_t) \geqslant \varliminf_{t \to 0} \frac{1}{|t|} h(\alpha, T_t) \tag{2.7}$$

$$= \lim_{t \to 0} \frac{1}{|t|} h(\alpha, T_t) \text{ for all } \alpha.$$

According to (2.1),

$$h(\{T_t\}) = \sup_{\alpha} \left(\lim_{t \to 0} \frac{1}{|t|} h(\alpha, T_t) \right). \tag{2.8}$$

Combining (2.6), (2.7) and (2.8), we obtain

$$\lim_{t \to 0} \frac{1}{|t|} h(T_t) = h(\{T_t\}). \tag{2.9}$$

Finally by the formula $h(T^k) = |k| h(T)$ in [3], which holds for any automorphism T and integer k, one gets for any real t $(\neq 0)$

$$\frac{1}{|t|} h(T_t) = \frac{1}{\left|\frac{t}{2^n}\right|} h(T_{\frac{t}{2^n}}) \quad n = 1, 2, \cdots, ;$$

so that from (2.9),

$$\frac{1}{|t|} h(T_t) = \lim_{n \to \infty} \frac{1}{\left|\frac{t}{2^n}\right|} h(T_{\frac{t}{2^n}}) = h(\{T_t\})$$

for any real t $(\neq 0)$.

This proves the theorem.

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