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AN ELEMENTARY PROOF OF AN ESTIMATE FOR THE KAKEYA MAXIMAL OPERATOR ON FUNCTIONS OF PRODUCT TYPE

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Abstract. In this paper we shall give an elementary proof of a norm estimate given by Igari for the Kakeya maximal operator restricted to functions of product type. Our proof also gives an improvement of the result.

Introduction. For an integer $N>1$ let $\mathscr{B}_N$ be the class of all rectangles in $\boldsymbol{R}^d$, $d\geq 2$, of eccentricity N, that is, congruent to any dilate of the rectangle $(0,1)^{d-1}\times(0,N)$ and let B_N be the sub-class of $\mathscr{B}_N$ consisting of all rectangles which are congruent to the rectangle $(0,1)^{d-1}\times(0,N)$.

The Kakeya maximal operator M_N is defined on locally integrable functions f of $\boldsymbol{R}^d$ by

$$M_N f(x)=\sup_{x\in R\in\mathscr{B}_N}\frac{1}{|R|}\int_R |f(y)|dy\,,$$

where $|A|$ represents the Lebesgue measure of a set A. The smaller Kakeya maximal operator K_N is defined by

$$K_N f(x)=\sup_{x\in R\in B_N}\frac{1}{|R|}\int_R |f(y)|dy\,.$$

It is conjectured that M_N is bounded on $L^d(\boldsymbol{R}^d)$ with the norm which grows no faster than $O((\log N)^{\alpha_d})$ for some $\alpha_d>0$ as $N\to\infty$. This conjecture was solved in the affirmative in the case $d=2$ by Córdoba [2], with the exponent $\alpha_2=2$ but seems to remain unsolved for $d\geq 3$. For K_N Córdoba [2] proved the above estimate with the exponent $\alpha_2=1/2$ and used that estimate in the proof of the estimate for M_N.

In the higher dimensional case these estimates were proved so far only for some restricted class of functions. Restricting to functions of product type, Igari [3] proved the estimate for K_N with the exponent $\alpha_d=3/2$ ($d\geq 3$). Restricting to the functions of radial type, Carbery, Hernández, and Soria [1] proved the estimate for M_N with the exponent $\alpha_d=1$.

Igari based his proof on an interpolation theorem given in [4]. The purpose of this note is to present an elementary proof of Igari's result with a better value of α_d.

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THEOREM 1. *Let $d \geq 2$. There exists a constant C, independent of N, such that*

$$\|K_N f\|_d \leq C(\log N)^{1-1/d} \|f\|_d$$

holds for all f in $L^d(\boldsymbol{R}^d)$ of the form

$$f(x_1, x_2, \dots, x_d) = \prod_{l=1}^{d} f_l(x_l) .$$

Here $\|f\|_d$ denotes the L^d-norm of f.

I would like to express my gratitude to my teacher Professor S. T. Kuroda, who introduced me to this problem and helped me in this work.

1. Proof of the theorem. We may assume that $f_l \geq 0$. We divide $\boldsymbol{R}^d$ into open unit cubes Q_i (and their boundaries) which have center at lattice points $i \in \boldsymbol{Z}^d$ and whose sides are parallel to the axes. By the local integrability of f we can find for every cube Q_i a rectangle $R_i \in B_N$ such that

(i) $Q_i \cap R_i \neq \emptyset$,

(ii) $K_N f(x) \leq \dfrac{2}{|R_i|} \displaystyle\int_{R_i} f(y)dy , \qquad \forall x \in Q_i$

(see [2]). From (ii) we obtain

$$K_N f(x) \leq \sum_{i \in \boldsymbol{Z}^d} \frac{2}{|R_i|} \int_{R_i} f(y)dy \chi_{Q_i}(x) = \frac{2}{N} \sum_{i \in \boldsymbol{Z}^d} \int_{R_i} f(y)dy \chi_{Q_i}(x) . \tag{1}$$

So, it suffices for the proof of the theorem to estimate the right hand side of (1). To this end we define γ_i as

$$\gamma_i = \{ j \in \boldsymbol{Z}^d \mid Q_j \cap R_i \neq \emptyset \}$$

and denote the projection of Q_j onto the l-th axis by J_l, that is

$$J_l = (j_l - 1/2, j_l + 1/2) \qquad \text{for} \quad j = (j_1, j_2, \dots, j_d) \in \boldsymbol{Z}^d .$$

Then

$$\int_{\boldsymbol{R}^d} (K_N f)(x)^d dx \leq \int_{\boldsymbol{R}^d} \left(\frac{2}{N} \sum_{i \in \boldsymbol{Z}^d} \int_{R_i} f(y)dy \chi_{Q_i}(x) \right)^d dx = \left(\frac{2}{N} \right)^d \sum_i \left(\int_{R_i} f(y)dy \right)^d$$

$$\leq \left(\frac{2}{N} \right)^d \sum_i \left(\sum_{j \in \gamma_i} \int_{Q_j} f(y)dy \right)^d = \left(\frac{2}{N} \right)^d \sum_i \left(\sum_{j \in \gamma_i} \prod_{l=1}^{d} \int_{J_l} f_l(y_l)dy_l \right)^d .$$

By multiple Hölder's and ordinary Hölder's inequalities we obtain

$$\int_{\mathbf{R}^d} (K_N f)(x)^d dx \le \left(\frac{2}{N}\right)^d \sum_{i \in \mathbf{Z}^d} \prod_{l=1}^{d} \sum_{j \in \gamma_i} \left(\int_{J_l} f_l(y_l) dy_l\right)^d$$

$$\le \left(\frac{2}{N}\right)^d \sum_i \prod_{l=1}^{d} \sum_{j \in \gamma_i} \int_{J_l} f_l(y_l)^d dy_l = \left(\frac{2}{N}\right)^d \sum_i \prod_{l=1}^{d} \int_{\mathbf{R}} \left(\sum_{j \in \gamma_i} \chi_{J_l}(y_l)\right) f_l(y_l)^d dy_l$$

$$= \left(\frac{2}{N}\right)^d \int_{\mathbf{R}^d} \left(\sum_i \prod_{l=1}^{d} \sum_{j \in \gamma_i} \chi_{J_l}(y_l)\right) f(y)^d dy .$$

It is now clear that the theorem follows from the following lemma.

LEMMA 2. *Let $g(y)$ be defined as*

$$g(y) = \sum_{i \in \mathbf{Z}^d} \prod_{l=1}^{d} \sum_{j \in \gamma_i} \chi_{J_l}(y_l) . \tag{2}$$

Then

$$\|g\|_\infty \le C N^d (\log N)^{d-1} ,$$

where C does not depend on N and the choice of R_i.

By the definition of g we have $g(y)=0$ if $y \in \mathbf{R}^d \setminus \bigcup_{i \in \mathbf{Z}^d} Q_i$ and $g(y)=g(i)$ if $y \in Q_i$. Therefore, it suffices for the proof of the lemma to show that

$$g(0) \le C N^d (\log N)^{d-1} \tag{3}$$

for sufficiently large N, where C is a constant independent of the choice of R_i. More precisely we assume that $N > \sqrt{d-1}$.

Let Ω_l, $1 \le l \le d$, be the band-like domain defined by

$$\Omega_l = \mathbf{R}^{l-1} \times (-1/2, 1/2) \times \mathbf{R}^{d-l} .$$

Then, the second sum in (2) with $y=0$ is equal to $\operatorname{card}(\{j \in \mathbf{Z}^d \mid Q_j \cap (\Omega_l \cap R_i) \ne \emptyset\})$. Next, we define A as

$$A = \{(j_1, j_2, \dots, j_d) \in \mathbf{Z}^d \cap [0, 2N]^d \mid j_k \le j_d,\ 1 \le k \le d-1\} .$$

Then by symmetry and the definition of g

$$g(0) \le C \sum_{i \in A} \prod_{l=1}^{d} \operatorname{card}(\{j \in \mathbf{Z}^d \mid Q_j \cap (\Omega_l \cap R_i) \ne \emptyset\}) . \tag{4}$$

PROPOSITION 3. *Let $N > \sqrt{d-1}$. For any $i \in A$ and $R_i \in B_N$ such that $Q_i \cap R_i \ne \emptyset$ the estimate*

$$\prod_{l=1}^{d} \operatorname{card}(\{j \in \mathbf{Z}^d \mid Q_j \cap (\Omega_l \cap R_i) \ne \emptyset\}) \le C N^{d-1} \prod_{l=1}^{d-1} (i_l + 1)^{-1} \tag{5}$$

holds. Here, C depends only on d.

Inserting (5) into the right hand side of (4) and estimating the sum, we obtain (3) and complete the proof of the lemma.

2. Proof of Proposition 3. We shall first prove some simple geometric propositions. Hereafter, we denote by $[a]$ the largest integer not greater than a, use C to denote various constants depending only on d, and write for simplicity

$$D=d^{1/2}, \qquad D'=(d-1)^{1/2}.$$

PROPOSITION 4. *For any unit cube* $Q\subset \boldsymbol{R}^d$ *we have*

$$\operatorname{card}(\{j\in \boldsymbol{Z}^d \mid Q_j\cap(Q\cap\Omega_l)\neq\varnothing\})\leq([D]+2)^{d-1}. \tag{6}$$

PROOF. The proof is clear from the estimate

$$\operatorname{card}(\{j\in \boldsymbol{Z} \mid Q\cap\{x\in \boldsymbol{R}^d \mid x_k=j+1/2\}\neq\varnothing\})\leq[D]+1.$$

■

PROPOSITION 5. *Let* $i\in \boldsymbol{Z}^d$. *Let* $R\in B_N$ *be such that* $R\cap Q_i\neq\varnothing$. *Let* e *be a unit vector parallel to longer sides of* R. *Assume that* $R\cap\Omega_l\neq\varnothing$. *Then*

$$(|i_l|-(D'+1))/N\leq|e_l|. \tag{7}$$

PROOF. Let $i_l\geq 0$. It follows from $R\cap Q_i\neq\varnothing$ and $R\cap\Omega_l\neq\varnothing$ that the length of the projection of R on the x_l-axis is greater than i_l-1. On the other hand that length is less than $N|e_l|+D'$. ■

PROPOSITION 6. *Let* e *be a unit vector in* $\boldsymbol{R}^d$ *with* $e_l\neq 0$. *Let* L *be a line parallel to* e. *Then*

$$\operatorname{card}(\{j\in \boldsymbol{Z}^d \mid Q_j\cap(\Omega_l\cap L)\neq\varnothing\})\leq([D]+2)^{d-1}([|e_l|^{-1}]+1).$$

PROOF. The length of $\Omega_l\cap L$ is $|e_l|^{-1}$. Therefore, $\Omega_l\cap L$ is covered by $[|e_l|^{-1}]+1$ segments of length 1. To each segment we attach a unit cube containing it and apply Proposition 4. ■

PROPOSITION 7. *Let* e *be a unit vector in* $\boldsymbol{R}^d$ *with* $e_l\neq 0$. *Let* R *be a column congruent to* $(0, 1)^{d-1}\times \boldsymbol{R}$ *and parallel to* e. *Then*

$$\begin{aligned}\operatorname{card}(\{j\in \boldsymbol{Z}^d \mid Q_j\cap(\Omega_l\cap R)\neq\varnothing\})&\leq([D]+2)^{d-1}([(D'+1)|e_l|^{-1}]+1)\\&\leq C|e_l|^{-1}.\end{aligned}$$

PROOF. When an $\boldsymbol{R}^{d-1}$-dimensional unit cube intersects Ω_l, its point farthest from Ω_l has the distance at most D' from Ω_l. Considering this fact, we proceed as in the proof of Proposition 6. ■

Using these propositions we shall now prove Proposition 3. If the left hand side of (5) is not equal to 0, then $R_i\cap\Omega_l\neq\varnothing$ for all l, $1\leq l\leq d$. We define A_1 and A_2 as

$$A_1=\{j\in \boldsymbol{Z}^d\cap([D']+2,2N]^d)\,|\,j_k\le j_d,\ 1\le k\le d-1\}\,,$$
$$A_2=A-A_1\,.$$

Case 1. $i\in A_1$. From Proposition 5 with R_i in place of R we obtain $0<(i_l-([D']+2))/N\le|e_l|$ for all $1\le l\le d$. This implies that Proposition 7 is applicable so that it suffices to compute the maximum of $\prod_{l=1}^{d}(1/x_l)$ in the region

$$\sum_{l=1}^{d}x_l^2=1\,,\qquad (i_l-([D']+2))/N\le x_l\,.$$

Noting $[D']+3\le i_l$ in A_1, we see that the above domain is contained in the region $F\subset\boldsymbol{R}^d$ given by

$$\sum_{l=1}^{d}x_l^2=1\,,\qquad (i_l+1)\{([D']+4)3N\}^{-1}\le x_l\,. \tag{8}$$

For later convenience we compute the maximum in this extended region. The proof of (5) for the case 1 now follows from the following:

PROPOSITION 8. *Let $i\in A$ and let F be the region given by* (8). *Then,*

$$\max_{x\in F}\prod_{l=1}^{d}x_l^{-1}\le CN^{d-1}\prod_{l=1}^{d-1}(i_l+1)^{-1}\,.$$

The proof of the proposition will be given later.

Case 2. $i\in A_2$. We may assume that $i_l\le[D']+2$ for $1\le l\le p$ and $i_l>[D']+2$ for $p<l\le d$, where p is an integer such that $1\le p\le d$. For $l>p$ we use the estimate of Proposition 7. For dealing with the case $1\le l\le p$ we use the following relation (use Proposition 4).

$$\begin{aligned}\operatorname{card}(\{j\in\boldsymbol{Z}^d\,|\,Q_j\cap(\Omega_l\cap R_i)\ne\varnothing\})&\le([D]+2)^{d-1}([(N^2+d-1)^{1/2}]+1)\\&\le([D]+2)^{d-1}3N\,.\end{aligned}\tag{9}$$

Therefore we obtain

$$\operatorname{card}(\{j\in\boldsymbol{Z}^d\,|\,Q_j\cap(\Omega_l\cap R_i)\ne\varnothing\})\le C\min(3N,|e_l|^{-1})\,. \tag{10}$$

We may assume that $|e_l|<1/(3N)$ for $1\le l<q$ and $|e_l|\ge 1/(3N)$ for $q\le l\le p$, where $1\le q\le p+1$. Then by (10)

$$\begin{aligned}&\prod_{l=1}^{d}\operatorname{card}(\{j\in\boldsymbol{Z}^d\,|\,Q_j\cap(\Omega_l\cap R_i)\ne\varnothing\})\\&\qquad\le C(3N)^{q-1}\prod_{l=q}^{d}|e_l|^{-1}\le CN^{q-1}\prod_{l=1}^{q-1}(i_l+1)^{-1}\prod_{l=q}^{d}|e_l|^{-1}\,.\end{aligned}$$

When $1\le q\le d$ we want to apply Proposition 8 adapted to the $(d-q+1)$-dimensional case to $\prod_{l=q}^{d}|e_l|^{-1}$. (This use of inductive argument is due to Professor S. T. Kuroda.) To this end we must check the assumption (8). For l with $q \le l \le d$ we see by the above discussion and $(i_l+1)\{([D']+4)3N\}^{-1} \le 1/(3N) \le |e_l|$, $q \le l \le p$, that

$$(i_l+1)\{([D']+4)3N\}^{-1} \le |e_l| .$$

Furthermore, $\sqrt{8}/3 \le \sum_{l=q}^{d}|e_l|^2 \le 1$. Therefore we can apply Proposition 8 and obtain (5).

When $q=d$ we only need to note $|e_d|^{-1} \le 3/\sqrt{8}$. ■

Finally, we shall prove Proposition 8. Putting $y_l = x_l^2$, the assertion is converted to computing the minimum value of

$$\psi(y) = \prod_{l=1}^{d} y_l$$

in the region

$$\sum_{l=1}^{d} y_l = 1 , \qquad a_l \le y_l , \qquad 1 \le l \le d . \tag{11}$$

We have

$$\psi(y) = \prod_1^d y_l = \prod_1^{d-1} y_l\left(1 - \sum_1^{d-1} y_l\right) = \left(\prod_2^{d-1} y_l\right) y_1\left(\left(1 - \sum_2^{d-1} y_l\right) - y_1\right).$$

We shall fix all y_l other than y_1. Then, $\psi(y)$ is a quadratic polynomial of y_1. From (11) it follows that y_1 is restricted to

$$a_1 \le y_1 \le 1 - a_d - \sum_2^{d-1} y_l .$$

By a simple consideration using $a_1 \le a_d$ we see that $\psi(y)$ attains the minimum value at $y_1 = a_1$.

Next we put $\tilde{\psi}(y_2, \dots, y_d) = \psi(a_1, y_2, \dots, y_d)$ and repeat a similar process with $\tilde{\psi}$ instead of ψ. In this way we conclude that $\psi(y)$ attains the minimum value

$$\prod_1^{d-1} a_l\left(1 - \sum_1^{d-1} a_l\right)$$

at $y = (a_1, a_2, \dots, a_{d-1}, 1 - \sum_1^{d-1} a_l)$.

Therefore, the maximum value in Proposition 8 is

$$\left(\prod_1^{d-1} a_l\left(1 - \sum_1^{d-1} a_l\right)\right)^{-1/2} \tag{12}$$

$$= \{([D']+4)3N\}^d \prod_1^{d-1} (i_l+1)^{-1}\left((3([D']+4))^2N^2 - \sum_1^{d-1} (i_l+1)^2\right)^{-1/2} .$$

From $i \in A$ the quantity inside the square root on the right side of (12) is estimated as

$$(3([D']+4))^2 N^2 - \sum_1^{d-1} (i_l+1)^2 \geq (3([D']+4))^2 N^2 - 2(d-1) - 8(d-1)N^2 \geq CN^2 .$$

Hence, the right hand side of (12) is not greater than $CN^{d-1}\prod_1^{d-1}(i_l+1)^{-1}$. This completes the proof of Proposition 8. ∎

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