

An elementary proof of the hitting time theorem

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In this note, we give an elementary proof of the random walk hitting time theorem, which states that, for a left-continuous random walk on \mathbb{Z} starting at a nonnegative integer k , the conditional probability that the walk hits the origin for the first time at time n , given that it does hit zero at time n , is equal to k/n .

We start by introducing some notation. Let \mathbb{P}_k denote the law of a random walk starting in $k \geq 0$, let $\{Y_i\}_{i=1}^\infty$ be the i.i.d. steps of the random walk, let $S_n = k + Y_1 + \dots + Y_n$ be the position of the random walk starting in k after n steps, and let

$$T_0 = \inf\{n : S_n = 0\} \tag{1}$$

denote the walk's first hitting time of the origin. Clearly, $T_0 = 0$ \mathbb{P}_0 -a.s., that is, $T_0 = 0$ a.s. when the walker starts in the origin. Then, the hitting time theorem is the following result:

Theorem 1 (Hitting time theorem). *For a random walk starting in $k \geq 1$ with i.i.d. steps $\{Y_i\}_{i=1}^\infty$ satisfying that $Y_i \geq -1$ almost surely, the distribution of T_0 under \mathbb{P}_k is given by*

$$\mathbb{P}_k(T_0 = n) = \frac{k}{n} \mathbb{P}_k(S_n = 0). \tag{2}$$

Theorem 1 gives the remarkable conclusion that, conditionally on the event $\{S_n = 0\}$, and regardless of the precise distribution of the steps of the walk $\{Y_i\}_{i=1}^\infty$ as long as $Y_i \geq -1$ a.s., the probability of the walk to be at 0 *for the first time* is equal to $\frac{k}{n}$. Theorem 1 has particular importance in the context of branching processes, where, for $k = 1$, the law of T_0 is the same as the total progeny of a branching process with offspring distribution equal to the law of $Y_i + 1$ (see [4] or [9, Problem 12, page 234] for this connection, and [1, Section 10.4] for a modern application in random graph theory).

The first proofs of Theorem 1 and the related result for $k = 1$ can be found in [8]. The extension of $k \geq 2$ is in [6], or in [4] using a result in [3]. Most of these proofs make unnecessary use of generating functions, in particular, the Lagrange inversion formula, which the simple proof below does not employ. See also [5, Page 165-167] for a more recent version of the generating function proof. In [10], various proofs of the Hitting time theorem are given, including a combinatorial proof making use of a relation in [2]. A proof for random walks making only steps of size ± 1 using the reflection principle can for example be found in [5, Page 79].

The hitting time theorem is closely related to the ballot theorem, which has a long history dating back to Bertrand in 1887 (see [7] for an excellent overview of the history and literature). The version of the ballot theorem in [7] states that, for a random walk $\{S_n\}_{n=0}^\infty$ starting at 0, with exchangeable, non-negative steps, the probability that $S_m < m$ for all $m = 1, \dots, n$, conditionally on $S_n = k$, equals k/n . This proof borrows upon queueing theory methodology, and is related to, yet slightly different from, our proof below.

Proof. We prove (2) for all $k \geq 0$ by induction in n . When $n = 1$, both sides are equal to 0 when $k > 1$ and $k = 0$, and are equal to $\mathbb{P}(Y_1 = -1)$ when $k = 1$. This initializes the induction.

To advance the induction, we take $n \geq 2$, and note that both sides are equal to 0 when $k = 0$. Thus, we may assume that $k \geq 1$. We condition on the first step to obtain

$$\mathbb{P}_k(T_0 = n) = \sum_{s=-1}^{\infty} \mathbb{P}_k(T_0 = n | Y_1 = s) \mathbb{P}(Y_1 = s). \quad (3)$$

By the random walk Markov property,

$$\mathbb{P}_k(T_0 = n | Y_1 = s) = \mathbb{P}_{k+s}(T_0 = n - 1) = \frac{k+s}{n-1} \mathbb{P}_{k+s}(S_{n-1} = 0), \quad (4)$$

where in the last equality we have used the induction hypothesis, which is allowed since $k \geq 1$ and $s \geq -1$, so that $k+s \geq 0$. This leads to

$$\mathbb{P}_k(T_0 = n) = \sum_{s=-1}^{\infty} \frac{k+s}{n-1} \mathbb{P}_{k+s}(S_{n-1} = 0) \mathbb{P}(Y_1 = s). \quad (5)$$

We undo the law of total probability, noting that

$$\sum_{s=-1}^{\infty} (k+s) \mathbb{P}_{k+s}(S_{n-1} = 0 | Y_1 = s) \mathbb{P}(Y_1 = s) = \mathbb{P}_k(S_n = 0) \left(k + \mathbb{E}_k[Y_1 | S_n = 0] \right), \quad (6)$$

where $\mathbb{E}_k[Y_1 | S_n = 0]$ is the conditional expectation of Y_1 given that $S_n = 0$ occurs. We next note that the conditional expectation of $\mathbb{E}_k[Y_i | S_n = 0]$ is independent of i , so that

$$\mathbb{E}_k[Y_1 | S_n = 0] = \frac{1}{n} \sum_{i=1}^n \mathbb{E}_k[Y_i | S_n = 0] = \frac{1}{n} \mathbb{E}_k \left[\sum_{i=1}^n Y_i | S_n = 0 \right] = -\frac{k}{n}, \quad (7)$$

since $\sum_{i=1}^n Y_i = S_n - k = -k$ when $S_n = 0$. Therefore, we arrive at

$$\mathbb{P}_k(T_0 = n) = \frac{1}{n-1} \left[k - \frac{k}{n} \right] \mathbb{P}_k(S_n = 0) = \frac{k}{n} \mathbb{P}_k(S_n = 0). \quad (8)$$

This advances the induction, and completes the proof of Theorem 1. \square

We close this paper by deriving a slightly stronger result than Theorem 1. We first introduce some notation. We write $\vec{m} = (m_{-1}, m_0, m_1, \dots)$ where m_i are non-negative integers for all $i \geq -1$. Further, for \vec{m} and $n \geq 1$, let $A_{\vec{m}}(n)$ denote the event that $S_n = 0$, and the random walk $\{S_i\}_{i=0}^{\infty}$ has made m_{-1} steps of size -1 , m_0 steps of size 0 , m_1 steps of size 1 , etc. In other words, we let

$$A_{\vec{m}}(n) = \{S_n = 0\} \cap \{ \#\{i \in \{1, \dots, n\} : Y_i = s\} = m_s \forall s \geq -1 \}. \quad (9)$$

Naturally, $\sum_{i=-1}^{\infty} m_i = n$, and, since $S_n = 0$, we must have that $\sum_{i=-1}^{\infty} i m_i = -k$. Then we obtain the following extension of Theorem 1:

Theorem 2. *Under the assumptions in Theorem 1, for all $k \geq 1$, \vec{m} and $n \geq 1$,*

$$\mathbb{P}_k(\{T_0 = n\} \cap A_{\vec{m}}(n)) = \frac{k}{n} \mathbb{P}_k(A_{\vec{m}}(n)). \quad (10)$$

The proof of Theorem 2 is a minor modification of that of Theorem 1, again using that, conditionally on $A_{\vec{m}}(n)$, the steps (Y_1, \dots, Y_n) are exchangeable. We omit further details.

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