AN ELEMENTARY PROOF OF THE PICK-NEVANLINNA INTERPOLATION THEOREM

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1. INTRODUCTION

Let $\{z_1, \cdots, z_n\}$ be an n-tuple of distinct points in the open unit disk Δ. Let $\{w_1, \cdots, w_n\}$ be an n-tuple of complex numbers. The problem is to formulate a necessary and sufficient condition for the existence of a function f, analytic in Δ, bounded in modulus by 1, and such that $f(z_i) = w_i$ $(1 \leq i \leq n)$. Such an f is said to interpolate the sequences $\{z_i\}$ and $\{w_i\}$. The problem was originally solved by G. Pick [4] in 1916. His necessary and sufficient condition was that the n-by-n matrix

$$(1) \qquad M = \left[\frac{1 - w_i \bar{w}_j}{1 - z_i \bar{z}_j}\right] \quad (1 \leq i, j \leq n)$$

be positive semidefinite (nonnegative). R. Nevanlinna [2], [3] also solved the problem independently of Pick in 1919; however his conditions were rather implicit. He developed the following recursive relationship: If E is the set of analytic functions in Δ whose modulus is bounded by 1, then f is in E if and only if

$$f_1(z) = \frac{f(z) - f(z_1)}{1 - \overline{f(z_1)} f(z)} \Big/ \frac{z - z_1}{1 - \bar{z}_1 z}$$

is in E. In other words, f interpolates the n-tuples $\{z_i\}$ and $\{w_i\}$ $(1 \leq i \leq n)$ if and only if f_1 interpolates the (n - 1)-tuples

$$\{z_i\} \quad \text{and} \quad \left\{\frac{w_i - w_1}{1 - \bar{w}_1 w_i} \Big/ \frac{z_i - z_1}{1 - \bar{z}_1 z_i}\right\} \quad (2 \leq i \leq n).$$

Repeating the process, we then obtain $f_2, f_3, \cdots, f_n$. Nevanlinna's theorem asserts that a necessary and sufficient condition for E to contain a function f with the property that $f(z_i) = w_i$ $(1 \leq i \leq n)$ is that the corresponding functions $f_1, f_2, \cdots, f_n$ belong to E. In 1956, B. Sz.-Nagy and A. Korányi [6] gave a proof of Pick's condition, using Hilbert-space techniques. In 1967, D. Sarason [5] gave a proof by means of operator theory. A host of others have considered similar problems for infinite sequences.

We shall give an elementary constructive proof that (1) is sufficient, and we shall show that the interpolating functions can be taken from a much smaller class than the analytic functions bounded by 1. Using these considerations, we also give a proof that (1) is necessary. Finally, in Section 3, we draw some consequences of these results.

Received June 26, 1974.

Michigan Math. J. 21 (1974).

2. MAIN RESULT

An analytic function of the form

$$B(z) = c \prod_{i=1}^{n} \frac{z - a_i}{1 - \bar{a}_i z}, \qquad \text{where } |c| = 1 \text{ and } |a_i| < 1,$$

is called a *finite Blaschke product.* It is easy to see that the finite Blaschke products are precisely the functions analytic in Δ whose absolute value tends to 1 at every point of the unit circle. Unless they are constant, they map Δ onto Δ.

Construction. Let $B_0(z) = 1$ for all z in Δ. If $|w_n| = 1$, let $B_n(z) = w_n$ for all z in Δ. If $|w_n| < 1$, let $B_n(z) = C_n^{-1} \circ (zB_{n-1}) \circ A_n(z)$, where

$$A_n(z) = (z - z_n)/(1 - \bar{z}_n z), \qquad C_n(z) = (z - w_n)/(1 - \bar{w}_n z),$$

and B_{n-1} is a Blaschke product of degree at most $n - 1$, arranged so that

$$B_{n-1}(A_n(z_i)) = C_n(w_i)/A_n(z_i) \qquad (1 \le i \le n - 1).$$

THEOREM 1. *Suppose* $xMx^* \ge 0$ *for all* n-*vectors* x. *Then the construction above gives a finite Blaschke product* B_n *of degree at most* n *such that* $B_n(z_i) = w_i$ $(1 \le i \le n)$.

Proof. The idea of the proof is to show that the interpolation $B_n(z_i) = w_i$ $(1 \le i \le n)$ is equivalent to the interpolation $B_{n-1}(A_n(z_i)) = C_n(w_i)/A_n(z_i)$ $(1 \le i \le n - 1)$. The result will then follow by induction.

Let $x = (0, \cdots, 0, 1)$; then $xMx^* = (1 - |w_n|^2)/(1 - |z_n|^2)$. Hence $|w_n| \le 1$. If $|w_n| = 1$, then $M_{nn} = 0$. Because M is hermitian, $|M_{ij}|^2 \le M_{ii} M_{jj}$. Hence $0 = M_{in} = (1 - w_i \bar{w}_n)/(1 - z_i \bar{z}_n)$, so that $w_i = w_n$ $(1 \le i \le n)$. Thus B_n interpolates the sequences $\{z_i\}$ and $\{w_i\}$ $(1 \le i \le n)$.

If $|w_n| < 1$, and if we can find such a B_{n-1}, then B_n is a Blaschke product of degree at most n interpolating the sequences $\{z_i\}$ and $\{w_i\}$ $(1 \le i \le n)$. We need only show $M \ge 0$ implies B_{n-1} exists.

The proof is by induction. The function B_0 always exists. Suppose that for each positive semidefinite $(n - 1)$-by-$(n - 1)$ matrix of the form

$$[(1 - v_i \bar{v}_j)/(1 - u_i \bar{u}_j)],$$

the construction yields a finite Blaschke product B_{n-1} of degree at most $n - 1$ such that $B_{n-1}(u_i) = v_i$ $(1 \le i \le n - 1)$. Then it suffices to show that $M \ge 0$ implies the matrix

$$N = \left[\frac{1 - \dfrac{C_n(w_i)\overline{C_n(w_j)}}{A_n(z_i)\overline{A_n(z_j)}}}{1 - A_n(z_i)\overline{A_n(z_j)}} \right] \qquad (1 \le i, j \le n - 1)$$

is positive semidefinite. We see that

$$N_{ij} = \frac{1}{A_n(z_i)\overline{A_n(z_j)}}\left(\frac{1 - C_n(w_i)\overline{C_n(w_j)}}{1 - A_n(z_i)\overline{A_n(z_j)}} - 1\right)$$

$$= \frac{1}{A_n(z_i)\overline{A_n(z_j)}}\left(\frac{(1 - z_i\bar{z}_n)(1 - \bar{z}_j z_n)(1 - |w_n|^2)(1 - w_i\bar{w}_j)}{(1 - w_i\bar{w}_n)(1 - \bar{w}_j w_n)(1 - |z_n|^2)(1 - z_i\bar{z}_j)} - 1\right).$$

Now let D_1 be the n-by-n diagonal matrix with diagonal elements

$$(D_1)_{ii} = \left(\frac{1 - z_i\bar{z}_n}{1 - w_i\bar{w}_n}\right)\sqrt{\frac{1 - |w_n|^2}{1 - |z_n|^2}}\,.$$

Then

$$(D_1 MD_1^*)_{ij} = \frac{(1 - z_i\bar{z}_n)(1 - \bar{z}_j z_n)(1 - |w_n|^2)(1 - w_i\bar{w}_j)}{(1 - w_i\bar{w}_n)(1 - \bar{w}_j w_n)(1 - |z_n|^2)(1 - z_i\bar{z}_j)}\,.$$

Notice that the last column and the bottom row are all 1's. Now let

$$I_1 = \left[\begin{array}{c|c} I & \begin{matrix}-1\\ \vdots \\ -1\end{matrix} \\ \hline 0 \cdots 0 & 1 \end{array}\right],$$

where I is the (n - 1)-by-(n - 1) identity matrix. Then

$$(I_1 D_1 MD_1^* I_1^*)_{ij} = \begin{cases} \dfrac{(1 - z_i\bar{z}_n)(1 - \bar{z}_j z_n)(1 - |w_n|^2)(1 - w_i\bar{w}_j)}{(1 - w_i\bar{w}_n)(1 - \bar{w}_j w_n)(1 - |z_n|^2)(1 - z_i\bar{z}_j)} - 1 & \text{for } (i, j) \neq (n, n), \\ 1 & \text{for } i = j = n\,. \end{cases}$$

Now let D_2 be the n-by-n diagonal matrix with diagonal elements $1/A_n(z_i)$ for $1 \leq i \leq n - 1$ and 1 for $i = n$. Then

$$\left[\begin{array}{c|c} N & \begin{matrix}0\\ \vdots \\ 0\end{matrix} \\ \hline 0 \cdots 0 & 1 \end{array}\right] = D_2 I_1 D_1 MD_1^* I_1^* D_2^*\,.$$

Hence, if $x = (x_1, \cdots, x_{n-1})$, $y = (x_1, \cdots, x_{n-1}, 0)$, and $P = D_2 I_1 D_1$, then $xNx^* = (yP)M(yP)^* \geq 0$ by assumption. (Note that since $|w_n| < 1$, the matrix P is invertible. Therefore, in fact, $N \geq 0$ if and only if $M \geq 0$; moreover, rank N = rank M - 1.)

Pick proved the converse fairly easily, using Cauchy's integral formula, after transforming the problem into the consideration of functions mapping the unit disk into the right half-plane. His proof can also be found in L. Ahlfors [1, pp. 3-4]. We offer another proof.

THEOREM 2. *If* f *is analytic in* Δ, $|f(z)| \leq 1$, *and* $f(z_i) = w_i$ $(1 \leq i \leq n)$, *then* $xMx^* \geq 0$ *for all* n-*vectors* x.

Proof. The idea, again, is that interpolating the n-tuples $\{z_i\}$ and $\{w_i\}$ $(1 \leq i \leq n)$ is equivalent to interpolating the $(n - 1)$-tuples $\{A_n(z_i)\}$ and $\{C_n(w_i)/A_n(z_i)\}$ $(1 \leq i \leq n - 1)$.

If $|w_n| = 1$, the maximum-modulus principle implies $f(z) = w_n$ for all z in Δ; hence

$$M_{ij} = \frac{1 - |w_n|^2}{1 - z_i \bar{z}_j} = 0.$$

Therefore $xMx^* \geq 0$ trivially.

If $|w_n| < 1$, let

$$A_n(z) = (z - z_n)/(1 - \bar{z}_n z) \quad \text{and} \quad C_n(z) = (z - w_n)/(1 - \bar{w}_n z),$$

and let $g(z) = (C_n \circ f \circ A_n^{-1}(z))/z$. Schwarz's lemma implies $|g(z)| \leq 1$. Note that $g(A_n(z_i)) = C_n(w_i)/A_n(z_i)$ $(1 \leq i \leq n - 1)$.

The proof now follows by induction. Assume that for each analytic function g on Δ, bounded by 1 and with $g(u_i) = v_i$ $(1 \leq i \leq n - 1)$, the $(n - 1)$-by-$(n - 1)$ matrix

$$\left[\frac{1 - v_i \bar{v}_j}{1 - u_i \bar{u}_j}\right]$$

is positive semidefinite. Our function g satisfies these requirements with $u_i = A_n(z_i)$ and $v_i = C_n(w_i)/A_n(z_i)$. Thus the matrix N, as in the proof of Theorem 1, is positive semidefinite. As we noted in the proof of Theorem 1, this implies that M is positive semidefinite.

3. FURTHER RESULTS

COROLLARY 1. *Suppose* M *is positive semidefinite. Then* $\det(M) = 0$ *if and only if the interpolating function is unique. In this case the interpolating function is a Blaschke product whose degree is the rank of* M.

This follows from the facts that rank N = rank M - 1, the interpolating function is unique if $|w_n| = 1$, and as in the proof of Theorem 2, if f is a Blaschke product then deg g = deg f - 1.

COROLLARY 2. *Let* F *be the set of analytic functions in* Δ *whose modulus is bounded by* 1, *such that* $f(z_i) = w_i$ $(1 \leq i \leq n)$. *Fix* z_0 *in* Δ. *Then the solutions to the problems*

$$(1)\ \max \Re f(z_0) \quad \textit{and} \quad (2)\ \max |f(z_0)|,$$

where the maximum is taken with respect to all functions f *in the class* F, *are Blaschke products whose degree is the rank of* M. *In the first problem, the solution is unique.*

Proof. We can find the most general member of F from the construction, by letting B_0 be an arbitrary analytic function on Δ bounded in modulus by 1. The possible values at z_0 of these functions fills a closed disk D (a point is considered a closed disk). This was known to both Pick and Nevanlinna; see also Ahlfors [1, pp. 4-5]. It can be seen from our considerations by induction. Assume the possible values of $B_{n-1}(A_n(z_0))$, as B_0 varies, fill a closed disk. Now

$$B_n(z_0) = C_n^{-1}(A_n(z_0)B_{n-1}(A_n(z_0))),$$

$A_n(z_0)$ is constant, and C_n^{-1} maps circles into circles; therefore the possible values of $B_n(z_0)$ fill a closed disk. Let

$$M'(w_0) = \left[\frac{1 - w_i\bar{w}_j}{1 - z_i\bar{z}_j}\right] \quad (0 \leq i, j \leq n),$$

and let M be as before. If F is not empty, M is positive semidefinite. Thus $M'(w_0)$ is positive semidefinite if and only if $\det(M'(w_0)) \geq 0$. Since $\det(M'(w_0))$ is a continuous function of w_0, the solutions to both problems must occur when $\det(M'(w_0)) = 0$, that is, when w_0 is on the boundary of D. Now apply Corollary 1. The solution to the second problem is unique if and only if D is not centered at the origin.

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