

for any even  $n$  the maximal length of a centrally symmetric inscribed  $n$ -gon in  $\mathcal{E}$  having a given point  $P$  of  $\mathcal{E}$  as a vertex is independent of  $P$ . Theorem 2 can be proved and generalized along the same lines. The generalization reads as follows: if  $\mathcal{E}_1, \dots, \mathcal{E}_k$  are confocal ellipses, then the maximal length of a centrally symmetric  $2k$ -gon  $(P_1, \dots, P_k, -P_1, \dots, -P_k)$  with  $P_i$  in  $\mathcal{E}_i$ , as  $P_2, \dots, P_k$  vary with  $P_1$  fixed, is independent of  $P_1$ .

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# An Elementary Proof of the Wallis Product Formula for pi

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**1. THE WALLIS PRODUCT FORMULA.** In 1655, John Wallis wrote down the celebrated formula

$$\frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdots = \frac{\pi}{2}. \tag{1}$$

Most textbook proofs of (1) rely on evaluation of some definite integral like

$$\int_0^{\pi/2} (\sin x)^n dx$$

by repeated partial integration. The topic is usually reserved for more advanced calculus courses. The purpose of this note is to show that (1) can be derived using only the mathematics taught in elementary school, that is, basic algebra, the Pythagorean theorem, and the formula  $\pi \cdot r^2$  for the area of a circle of radius  $r$ .

Viggo Brun gives an account of Wallis's method in [1] (in Norwegian). Yaglom and Yaglom [2] give a beautiful proof of (1) which avoids integration but uses some quite sophisticated trigonometric identities.

**2. A NUMBER SEQUENCE.** We define a sequence of numbers by  $s_1 = 1$ , and for  $n \geq 2$ ,

$$s_n = \frac{3}{2} \cdot \frac{5}{4} \cdots \frac{2n-1}{2n-2}.$$

The partial products of (1) with an odd number of factors can be written as

$$o_n = \frac{2^2 \cdot 4^2 \cdots (2n-2)^2 \cdot (2n)}{1 \cdot 3^2 \cdots (2n-1)^2} = \frac{2n}{s_n^2}, \quad (2)$$

while those with an even number of factors are of the form

$$e_n = \frac{2^2 \cdot 4^2 \cdots (2n-2)^2}{1 \cdot 3^2 \cdots (2n-3)^2 \cdot (2n-1)} = \frac{2n-1}{s_n^2}. \quad (3)$$

Here  $e_1 = 1$  should be interpreted as an empty product. Clearly  $e_n < e_{n+1}$  and  $o_n > o_{n+1}$ , and by comparing (2) and (3) we see that  $e_n < o_n$ . Therefore we must have

$$e_1 < e_2 < e_3 < \cdots < o_3 < o_2 < o_1.$$

Thus if  $1 \leq i \leq n$ ,

$$\frac{2i}{s_i^2} = o_i \geq o_n$$

and

$$\frac{2i-1}{s_i^2} = e_i \leq e_n,$$

from which it follows that

$$\frac{2i-1}{e_n} \leq s_i^2 \leq \frac{2i}{o_n}. \quad (4)$$

It will be convenient to define  $s_0 = 0$ . Notice that with this definition, (4) holds also for  $i = 0$ . We denote the difference  $s_{n+1} - s_n$  by  $a_n$ . Observe that  $a_0 = 1$ , and for  $n \geq 1$ ,

$$a_n = s_{n+1} - s_n = s_n \left( \frac{2n+1}{2n} - 1 \right) = \frac{s_n}{2n} = \frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n}.$$

We first derive the identity

$$a_i a_j = \frac{j+1}{i+j+1} a_i a_{j+1} + \frac{i+1}{i+j+1} a_{i+1} a_j. \quad (5)$$

*Proof.* After the substitutions

$$a_{i+1} = \frac{2i+1}{2(i+1)} a_i$$

and

$$a_{j+1} = \frac{2j+1}{2(j+1)} a_j,$$

the right hand side of (5) becomes

$$a_i a_j \left( \frac{2j+1}{2(j+1)} \cdot \frac{j+1}{i+j+1} + \frac{2i+1}{2(i+1)} \cdot \frac{i+1}{i+j+1} \right) = a_i a_j. \quad \blacksquare$$

If we start from  $a_0^2$  and repeatedly apply (5), we obtain the identities

$$\begin{aligned} 1 &= a_0^2 = a_0a_1 + a_1a_0 = a_0a_2 + a_1^2 + a_2a_0 = \cdots \\ &= a_0a_n + a_1a_{n-1} + \cdots + a_na_0. \end{aligned} \tag{6}$$

*Proof.* By applying (5) to every term, the sum  $a_0a_{n-1} + \cdots + a_{n-1}a_0$  becomes

$$\left(a_0a_n + \frac{1}{n}a_1a_{n-1}\right) + \left(\frac{n-1}{n}a_1a_{n-1} + \frac{2}{n}a_2a_{n-2}\right) + \cdots + \left(\frac{1}{n}a_{n-1}a_1 + a_na_0\right). \tag{7}$$

After collecting terms, this simplifies to  $a_0a_n + \cdots + a_na_0$ . ■

**3. A GEOMETRIC CONSTRUCTION.** We divide the positive quadrant of the  $xy$ -plane into rectangles by drawing the straight lines  $x = s_n$  and  $y = s_n$  for all  $n$ . Let  $R_{i,j}$  be the rectangle with lower left corner  $(s_i, s_j)$  and upper right corner  $(s_{i+1}, s_{j+1})$ . The area of  $R_{i,j}$  is  $a_i a_j$ . Therefore the identity (6) states that the total area of the rectangles  $R_{i,j}$  for which  $i + j = n$  is 1. We let  $P_n$  be the polygonal region consisting of all rectangles  $R_{i,j}$  for which  $i + j < n$ . Hence the area of  $P_n$  is  $n$  (see Figure 1).

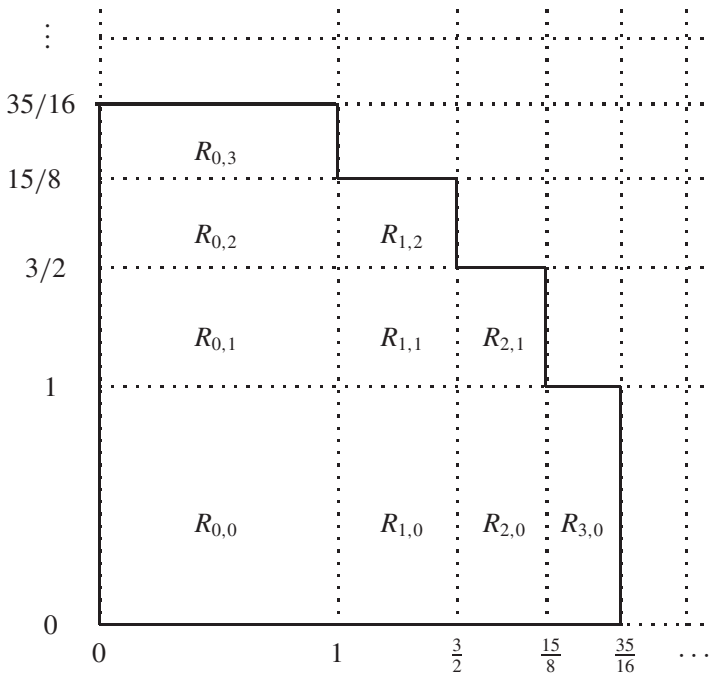


Figure 1. The region  $P_4$  of area 4.

The outer corners of  $P_n$  are the points  $(s_i, s_j)$  for which  $i + j = n + 1$  and  $1 \leq i, j \leq n$ . By the Pythagorean theorem, the distance of such a point to the origin is

$$\sqrt{s_i^2 + s_j^2}.$$

By (4), this is bounded from above by

$$\sqrt{\frac{2(i+j)}{o_n}} = \sqrt{\frac{2(n+1)}{o_n}}.$$

Similarly, the inner corners of  $P_n$  are the points  $(s_i, s_j)$  for which  $i+j = n$  and  $0 \leq i, j \leq n$ . The distance of such a point to the origin is bounded from below by

$$\sqrt{\frac{2(i+j-1)}{e_n}} = \sqrt{\frac{2(n-1)}{e_n}}.$$

Therefore  $P_n$  contains a quarter circle of radius  $\sqrt{2(n-1)/e_n}$ , and is contained in a quarter circle of radius  $\sqrt{2(n+1)/o_n}$ . Since the area of a quarter circle of radius  $r$  is equal to  $\pi r^2/4$  while the area of  $P_n$  is  $n$ , this leads to the bounds

$$\frac{(n-1)\pi}{2e_n} < n < \frac{(n+1)\pi}{2o_n},$$

from which it follows that

$$\frac{(n-1)\pi}{2n} < e_n < o_n < \frac{(n+1)\pi}{2n}.$$

It is now clear that as  $n \rightarrow \infty$ ,  $e_n$  and  $o_n$  both approach  $\pi/2$ .

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# Automorphisms of Finite Abelian Groups

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**1. INTRODUCTION.** In introductory abstract algebra classes, one typically encounters the classification of finite Abelian groups [1]:

**Theorem 1.1.** *Let  $G$  be a finite Abelian group. Then  $G$  is isomorphic to a product of groups of the form*

$$H_p = \mathbb{Z}/p^{e_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p^{e_n}\mathbb{Z},$$

*in which  $p$  is a prime number and  $1 \leq e_1 \leq \cdots \leq e_n$  are positive integers.*