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AN ELEMENTARY PROOF OF TITCHMARSH'S CONVOLUTION THEOREM

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ABSTRACT. We give an elementary proof of the following theorem of Titchmarsh. Suppose f, g are integrable on the interval $(0, 2T)$ and that the convolution $f * g(t) = \int_0^t f(t-x)g(x)\,dx = 0$ on $(0, 2T)$. Then there are nonnegative numbers α, β with $\alpha + \beta \geq 2T$ for which $f(x) = 0$ for almost all x in $(0, \alpha)$ and $g(x) = 0$ for almost all x in $(0, \beta)$.

Suppose f, g are integrable on the interval $(0, 2T)$. If $f = 0$ a.e. on $(0, \alpha)$, $g = 0$ a.e. on $(0, \beta)$ with $\alpha + \beta = 2T$, then the convolution

$$f * g(t) = \int_0^t f(t-x)g(x)\,dx = 0 \quad \text{for } 0 \leq t \leq 2T.$$

The converse of this statement is the famous Titchmarsh's Convolution Theorem. *Let f, g belong to $L^1(0, 2T)$ and vanish on $(-\infty, 0)$. Suppose that $\int_0^t f(t-x)g(x)\,dx = 0$ on $(0, 2T)$. Then there are nonnegative numbers α, β with $\alpha + \beta \geq 2T$ for which $f(x) = 0$ for almost all α in $(0, \alpha)$ and $g(x) = 0$ for almost all α in $(0, \beta)$.*

There are many proofs of this theorem. The first three: Titchmarsh [**6**], Crum [**1**] and Dufresnoy [**2**] were based on the theory of analytic or harmonic functions. An elaborate real variable proof was later given by Mikusinski and Ryll-Nardzewski in [**4**, (a), (b), (c)]. An entirely different proof, now classical, by the same authors, appears in [**5**]. Unfortunately it is valid only for $T = \infty$. Recently, Helson [**3**] gave an elegant proof using the theory of Hardy's $H^p(R)$ spaces and invariant subspaces. The present proof, valid for any T, is elementary, using no machinery beyond Fubini's theorem and Parseval's formula for trigonometric series.

Put $F(x) = \int_0^x f(s)\,ds$. Then, by Fubini's theorem

$$0 = \int_0^t \int_0^u f(u-x)g(x)\,dx\,du = \int_0^t \int_x^t f(u-x)g(x)\,du\,dx = \int_0^t F(t-x)g(x)\,dx.$$

This is $F * g(t) = 0$, $t \in (0, 2T)$. Similarly, putting $G(x) = \int_0^x g(s)\,ds$ we get $F * G(t) = 0$, $t \in (0, 2T)$.

Now if Titchmarsh's theorem is true for the functions F, G, it is also true for f, g. Hence, *we may assume, in the proof of Titchmarsh's Theorem, that f, g are several times differentiable and satisfy e.g. relations like $f' * g = 0$ and $g(0) = 0$.*

LEMMA 1. *Suppose that h is continuous on $(0, 2T)$ and that*

$$\left| \int_0^{2t} e^{2n(t-x)} h(x)\,dx \right| \leq C_t n^{-1/2} \tag{1}$$

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for $0 \le t \le T$ *and all* $n \in \{1,2,3,\dots\}$ *where* C_t *is independent of* n. *Then* $h = 0$ *on* $(0,T)$.

PROOF. We have boundedly

$$\lim_{n\to\infty}\left[1 - e^{-e^{2n(t-x)}}\right] = \begin{cases} 1 & \text{for } 0 \le x < t, \\ 0 & \text{for } t < x \le 2t. \end{cases}$$

Hence, by dominated convergence

$$\int_0^t h(x)\,dx = \lim_n \int_0^{2t}\left[1 - e^{-e^{2n(t-x)}}\right] h(x)\,dx. \tag{2}$$

On the other hand, for fixed t and n, the sums

$$-\sum_{k=1}^{N} \frac{(-1)^k}{k!} e^{2nk(t-x)}$$

converge uniformly on $(0,2T)$ to $[1 - e^{-e^{2n(t-x)}}]$ as $N \to \infty$. Hence, by (1)

$$\left|\int_0^{2t}\left[1 - e^{-e^{2n(t-x)}}\right] h(x)\,dx\right| = \lim_N \left|\int_0^{2t} \sum_{k=1}^{N} \frac{(-1)^k}{k!} e^{2nk(t-x)} h(x)\,dx\right|$$

$$\le \limsup_N \sum_1^N \frac{1}{k!} C_t (nk)^{-1/2} \le C_t n^{-1/2} e.$$

By (2)

$$\int_0^t h(x)\,dx = 0, \qquad t \in (0,T).$$

Hence $h = 0$ on $(0,T)$.

LEMMA 2. *If* $f * g(t) = 0$ *for* $t \in (0,2T)$ *then*

$$f(x)g(x) = 0 \quad \textit{for all } x \in (0,T).$$

PROOF. For $t \le T$ we calculate in two ways the integral

$$I_t = \iint_{\Delta\cup\Delta'} e^{n(2t-y)} e^{-iky} f(y-x) g(x)\,dx\,dy,$$

where Δ is the triangle $0 \le x \le y$, $0 \le y \le 2t$ and Δ' is the triangle $2t \le y \le 4t$, $y - 2t \le x \le 2t$. Thus $\Delta \cup \Delta'$ is the parallelogram $x \le y \le 2t + x$, $0 \le x \le 2t$. We have

$$I_t = \int_0^{2t}\left\{\int_x^{2t+x} e^{n(2t-y)} e^{-iky} f(y-x)\,dy\right\} g(x)\,dx.$$

Putting $y = x + u$ the inner integral becomes

$$\int_0^{2t} e^{n(t-u)} e^{-iku} f(u)\,du\, e^{n(t-x)} e^{-ikx}.$$

Hence

$$I_t = \int_0^{2t} e^{n(t-u)} e^{-iku} f(u)\,du \cdot \int_0^{2t} e^{n(t-x)} e^{-ikx} g(x)\,dx. \tag{3}$$

On the other hand, the integral over Δ is zero by $\int_0^y f(y-x)g(x)\,dx = 0$. Hence

$$I_t = \iint_{\Delta'} = \int_{2t}^{4t} e^{n(2t-y)} e^{-iky} h_t(y)\,dy \tag{4}$$

where

$$h_t(y) = \int_{y-2t}^{2t} f(y-x)g(x)\,dx.$$

We have

$$h_t(2t) = \int_0^{2t} f(2t-x)g(x)\,dx = 0, \qquad h_t(4t) = 0.$$

An integration by parts in (4) then gives

$$I_t = (n+ik)^{-1} \int_{2t}^{4t} e^{n(2t-y)} e^{-iky} h_t'(y)\,dy. \tag{4$'$}$$

We have

$$h_t'(y) = -f(2t)g(y-2t) + \int_{y-2t}^{2t} f'(y-x)g(x)\,dx,$$
$$h_t'(2t) = 0, \quad h_t'(4t) = -f(2t)g(2t).$$

Hence, by (4′)

$$\begin{aligned} I_t &= (n+ik)^{-2} e^{-2tn} e^{-i4tk} f(2t)g(2t) \\ &\quad + (n+ik)^{-2} \int_{2t}^{4t} e^{n(2t-y)} e^{-iky} h_t''(y)\,dy. \end{aligned} \tag{4$''$}$$

In the last integral the coefficient of $h_t''(y)$ has a modulus bounded by 1 on the interval of integration and we may suppose that $|f(x)g(x)| \le M$ for x in $(0, 2T)$, $|h_t''(y)| \le M$ for $t \in (0,T)$, $y \in (2t, 4t)$ where M is a constant.

Hence, by (4″), $|I_t| \le |n+ik|^{-2}(M + 2tM) \le |n+ik|^{-2}C$, where C is a constant. By (3)

$$\left|\int_0^{2t} e^{n(t-x)} e^{-ikx} f(x)\,dx\right| \cdot \left|\int_0^{2t} e^{n(t-x)} e^{-ikx} g(x)\,dx\right| \le |n+ik|^{-2}C. \tag{5}$$

We now use the polarized version of Parseval's formula, viz.

$$\left|\frac{1}{2\pi}\int_0^{2t} f\bar{g}\right| = \left|\sum_p \hat{f}(p)\overline{\hat{g}(p)}\right| \le \sum_p \left|\hat{f}(p)\hat{g}(p)\right|.$$

Putting $k = k_p = 2\pi(2t)^{-1}p$, we get from (5)

$$\begin{aligned} \left|\frac{1}{2\pi}\int_0^{2t} e^{2n(t-x)} f(x)g(x)\,dx\right| &\le \sum_{p=-\infty}^{\infty} |n+ik_p|^{-2}C \\ &\le n^{-1/2}\sum_p |n+ik_p|^{-3/2}C \le n^{-1/2}C_t, \end{aligned}$$

where C_t is independent of n. By Lemma 1, $fg = 0$ on $(0,T)$.

PROOF OF TITCHMARSH'S THEOREM. Observe first that if $\gamma > 0$ then the translate $f_\gamma(x) = f(x-\gamma)$ satisfies the equation $f_\gamma * g(t) = 0$ for $t \in (0, 2T)$.

In the other direction let $(0,\alpha)$ and $(0,\beta)$ respectively be the largest intervals on which f and g vanish.

We have

$$\int_0^t f_{-\alpha}(t-x)g(x)\,dx = \int_0^t f(t+\alpha-x)g(x)\,dx = \int_0^{t+\alpha} f(t+\alpha-x)g(x)\,dx = 0$$

for $t+\alpha \in (0,2T)$, hence for $t \in (0, 2T-\alpha)$. We infer that

$$f_{-\alpha} * g_{-\beta}(t) = 0 \quad \text{for } t \in (0, 2T-\alpha-\beta).$$

By the observation above we have

$$f_{-\alpha+\gamma} * g_{-\beta}(t) = 0 \quad \text{for } t \in (0, 2T-\alpha-\beta) \text{ and all } \gamma > 0,$$

and by Lemma 2

$$f_{-\alpha+\gamma}(x)g_{-\beta}(x) = 0 \quad \text{for } x \in \left(0, T - \frac{\alpha+\beta}{2}\right) \text{ and all } \gamma > 0. \tag{6}$$

Assume that $2T-\alpha-\beta > 0$; then $g_{-\beta}(x_0) \neq 0$ for some $x_0 \in (0, T-(\alpha+\beta)/2)$. If $0 < \gamma < x_0$ then, by (6), applied to x_0, we get $f_{-\alpha+\gamma}(x_0) = 0$, i.e. $f_{-\alpha}(x_0-\gamma) = 0$. This last is $f_{-\alpha}(u) = 0$ for $u \in (0, x_0)$ which is impossible. Thus $2T - (\alpha+\beta) \leq 0$ and the theorem is proved.

ADDED IN PROOF. J. G. Mikusinski, in his book *The Bochner integral*, Academic Press, New York, 1978, shows that the proof in [**5**] can be made valid for any $T > 0$.

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