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AN ELLIPTIC SURFACE COVERED BY MUMFORD'S FAKE PROJECTIVE PLANE

Dedicated to Professor Masayoshi Nagata on his sixtieth birthday

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Introduction. In [Mum], Mumford constructed an algebraic surface M of general type with $K_M^2 = 9$ and $p_g = q = 0$. This surface is called Mumford's fake projective plane because it has the same Betti numbers as the complex projective plane (see [BPV, Historical Note]). No other example of fake projective planes in this sense seems to be known up to now.

Since $c_1^2(M) = 3c_2(M) = 9$, the universal covering space of the complex surface M is isomorphic to the unit ball in $\boldsymbol{C}^2$ by Yau's result. However, Mumford's surface is constructed by means of the theory of the p-adic unit ball by Kurihara [Ku] and Mustafin [Mus]. By the construction of M, there exists an unramified Galois covering $V \to M$ of order eight. More precisely, a simple group G of order 168 acts on V, and M is the quotient of V by a 2-Sylow subgroup of G.

In this paper, we study the quotient surface $Y = V/G$. Since the action has fixed points, Y has some singular points. We prove that the minimal desingularization $\tilde{Y}$ of Y is an elliptic surface. We also determine the types of the singular fibers of the elliptic fibration.

Mumford's surface M is given as a $\boldsymbol{Z}_2$-scheme. Hence it has a modulo 2 reduction M_0. The normalization $\tilde{M}_0$ of M_0 is the blowing-up of $\boldsymbol{P}^2_{\boldsymbol{F}_2}$ at the seven $\boldsymbol{F}_2$-rational points. In Section 1, we describe explicitly how to recover M_0 from $\tilde{M}_0$.

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NOTATION. Let X be a scheme over an affine scheme Spec A. When a ring homomorphism $A \to B$ is given, we denote by X_B the fiber product $X \times_{\operatorname{Spec} A} \operatorname{Spec} B$ and by $X(B)$ the set of B-valued points of X. If X is of finite type and B is an algebraically closed field, then we sometimes treat $X(B)$ as a variety.

1. The closed fiber of Mumford's surface. We will recall some notation in Mumford's paper [Mum].

We always restrict ourselves to the case of the base ring $\boldsymbol{Z}_2$. Hence the maximal ideal is generated by 2, and the quotient field is the 2-adic number field $\boldsymbol{Q}_2$. We denote by η and 0 the generic point and the closed point of $\operatorname{Spec} \boldsymbol{Z}_2$, respectively.

A matrix $\alpha = (a_{i,j})_{i,j=0,1,2} \in GL(3, \boldsymbol{Q}_2)$ defines a linear automorphism of the vector space $\boldsymbol{Q}_2 X_0 + \boldsymbol{Q}_2 X_1 + \boldsymbol{Q}_2 X_2$ with indeterminates X_0, X_1, X_2 by

$$\alpha(c_0X_0 + c_1X_1 + c_2X_2) = (X_0, X_1, X_2)\alpha^t(c_0, c_1, c_2) = \sum_i (\sum_j a_{i,j}c_j)X_i .$$

Hence the induced automorphism $\alpha^\wedge$ of $\boldsymbol{P}^2_{\boldsymbol{Q}_2} = \operatorname{Proj} \boldsymbol{Q}_2[X_0, X_1, X_2]$ is given in terms of the homogeneous coordinates $(X_0: X_1: X_2)$ by

$$\alpha^\wedge(X_0: X_1: X_2) = (X_0: X_1: X_2)\alpha .$$

Thus the composite $\beta^\wedge \circ \alpha^\wedge$ is equal to $(\alpha\beta)^\wedge$.

The $\boldsymbol{Z}_2$-scheme $\mathscr{X}$ of Kurihara and Mustafin is defined as follows:

Let $\boldsymbol{P}^2_{\boldsymbol{Z}_2}$ be the projective plane with the homogeneous coordinates $(X_0: X_1: X_2)$. The closed fiber $\boldsymbol{P}^2_{\boldsymbol{F}_2}$ has seven $\boldsymbol{F}_2$-rational points and seven $\boldsymbol{F}_2$-rational lines. We first blow up $\boldsymbol{P}^2_{\boldsymbol{Z}_2}$ at these seven $\boldsymbol{F}_2$-rational points, and then blow up the resulting surface further along the proper transform of the union of the seven $\boldsymbol{F}_2$-rational lines. Let U be the union of the generic fiber $\boldsymbol{P}^2_{\boldsymbol{Q}_2}$ and a sufficiently small open neighborhood of the proper transform of $\boldsymbol{P}^2_{\boldsymbol{F}_2}$ in the blown-up scheme. For each α in $GL(3, \boldsymbol{Q}_2)$ we denote by U^α the $\boldsymbol{Z}_2$-scheme such that the generic fiber is equal to $\boldsymbol{P}^2_{\boldsymbol{Q}_2}$ and that there exists an isomorphism $U \xrightarrow{\sim} U^\alpha$ which induces $\alpha^\wedge$ on the generic fiber. Then the union $\cup_\alpha U^\alpha$ over all α in $GL(3, \boldsymbol{Q}_2)$ is patched together to a regular scheme $\mathscr{X}$ with the generic fiber $\boldsymbol{P}^2_{\boldsymbol{Q}_2}$.

By construction, the action of $GL(3, \boldsymbol{Q}_2)$ on $\boldsymbol{P}^2_{\boldsymbol{Q}_2}$ is extended to $\mathscr{X}$. Mumford found the following discrete subgroup Γ of $GL(3, \boldsymbol{Q}_2)$. Γ modulo scalar matrices acts on the closed fiber $\mathscr{X}_0$ freely and induces a quotient $\mathscr{X}/\Gamma$ as a formal scheme. $\mathscr{X}/\Gamma$ is algebraized to a projective regular scheme over $\boldsymbol{Z}_2$, and its generic fiber is the fake projective plane.

Γ is contained in the group Γ_1 generated by

$$\sigma = \begin{bmatrix} 1 & 0 & \lambda \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}, \quad \tau = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1+\lambda \\ 0 & 1 & \lambda \end{bmatrix},$$

$$\rho = \begin{bmatrix} 1 & 0 & \lambda \\ 0 & 1 & -\lambda^3/2 \\ 0 & 0 & \lambda^2/2 \end{bmatrix} \quad \text{and} \quad -I_3 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

where $\lambda = \zeta + \zeta^2 + \zeta^4 = (-1 + \sqrt{-7})/2$ for $\zeta = \exp(2\pi i/7)$. λ is embedded in $\boldsymbol{Z}_2$ so that $\lambda = (\text{unit})\cdot 2$, while its complex conjugate $\bar{\lambda}$ is a unit. There exists a homomorphism $\pi\colon \Gamma_1 \to GL(2, \boldsymbol{F}_7)$ and Γ is given as the inverse image $\pi^{-1}(S)$ of an arbitrary 2-Sylow subgroup S of $GL(2, \boldsymbol{F}_7)$.

By the matrices in [Mum, p. 243] which describe π, we see that the subgroup of Γ_1 generated by $\{\sigma, \tau, \rho\}$ is mapped onto $SL(2, \boldsymbol{F}_7)$ by π. Since $-I_3$ is a scalar matrix, the following change of notation does not affect the construction:

MODIFICATION OF THE NOTATION. Γ_1 is replaced by its subgroup of index 2 generated by $\{\sigma, \tau, \rho\}$. The homomorphism π is replaced by one from the new Γ_1 to $PSL(2, \boldsymbol{F}_7)$. More explicitly, $\pi\colon \Gamma_1 \to PSL(2, \boldsymbol{F}_7)$ is given by

$$\pi(\sigma) = \begin{bmatrix} 2 & 0 \\ 1 & 4 \end{bmatrix}, \quad \pi(\tau) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad \pi(\rho) = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$$

(see [Mum, p. 243]). The group Γ is also replaced by $\pi^{-1}(S)$ for a 2-Sylow subgroup S of $PSL(2, \boldsymbol{F}_7)$. In this case, the set of scalar matrices in Γ_1 is $\{(\lambda^2/2)^k I_3 = (\tau\rho)^{3k};\ k \in \boldsymbol{Z}\}$ (cf. [Mum, p. 241]).

From now on, we use this modified notation.

Let $\Gamma_0 = \operatorname{Ker} \pi$. Clearly, Γ_0 is a normal subgroup of Γ_1. The quotient $G = \Gamma_1/\Gamma_0$ is isomorphic to $PSL(2, \boldsymbol{F}_7)$ and hence is a simple group of order 168. Since Γ_0 modulo scalar matrices is also a torsionfree cocompact subgroup of $PGL(3, \boldsymbol{Q}_2)$, the quotient formal scheme $\mathscr{X}/\Gamma_0$ can also be algebraized to a projective regular $\boldsymbol{Z}_2$-scheme. We denote the algebraization by V. Then the action of Γ_1 on the scheme $\mathscr{X}$ induces an action of G on V. Since the scalar matrices in Γ_1 are contained in Γ_0, the induced action is effective. Mumford's fake projective plane is the generic fiber of the quotient $M = V/S$ by the 2-Sylow subgroup S of G.

Since V_η is an unramified cover of degree 8 of Mumford's fake projective plane, the following facts are easily checked.

(1) V_η is a surface of general type.

(2) $c_1^2(V_\eta) = 72$.

(3) $c_2(V_\eta) = 24$.

(4) $\chi(V_0) = \chi(V_\eta) = 8$.

(5) $q(V_\eta) = 0$ ([Mum, p. 238]).

(6) $p_g(V_\eta) = 7$.

In order to describe the closed fiber of M explicitly, we choose the 2-Sylow subgroup S of $G = \Gamma_1/\Gamma_0$ to be the subgroup generated by

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 1 \\ 6 & 0 \end{bmatrix},$$

where we identify G with $PSL(2, \boldsymbol{F}_7)$ by the isomorphism induced by π. S is isomorphic to the dihedral group of order 8. Indeed,

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^4 = I_2\,, \quad \begin{bmatrix} 0 & 1 \\ 6 & 0 \end{bmatrix}^2 = I_2 \quad \text{and} \quad \begin{bmatrix} 0 & 1 \\ 6 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 6 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1}$$

in $PSL(2, \boldsymbol{F}_7)$.

We denote by B the proper transform in $\mathscr{X}$ of the closed fiber $\boldsymbol{P}^2_{\boldsymbol{F}_2} \subset \boldsymbol{P}^2_{\boldsymbol{Z}_2} = \operatorname{Proj} \boldsymbol{Z}_2[X_0, X_1, X_2]$. B is an irreducible component of $\mathscr{X}_0$ and the projection $p: B \to \boldsymbol{P}^2_{\boldsymbol{F}_2}$ is the blowing-up $\boldsymbol{P}^2_{\boldsymbol{F}_2}$ at the seven $\boldsymbol{F}_2$-rational points. We denote by $C(a, b, c)$ the proper transform of the line $aX_0 + bX_1 + cX_2 = 0$ on $\boldsymbol{P}^2_{\boldsymbol{F}_2}$ to B and let $E(a, b, c) := p^{-1}((a, b, c))$ for each triple (a, b, c) of 0 or 1 with not all being zero.

The natural morphism from B to the closed fiber $M_0 = \mathscr{X}_0/\Gamma$ can be regarded as the normalization. Actually, we obtain M_0 by identifying each of suitable seven pairs of $C(a, b, c)$ and $E(a', b', c')$ in B. More precisely, we take $\{\rho\sigma^2\tau, \tau\rho\sigma\tau, \tau^2\rho\tau, \tau^3\rho\sigma\tau^6, \tau^4\rho\sigma^2\tau^5, \tau^5\rho\sigma^2, \tau^6\rho\sigma^2\tau^6\} \subset \Gamma$ as the set of representatives of $S \setminus \{1\}$. Then each element induces an isomorphism of curves on B as follows:

$$(\rho\sigma^2\tau)^\wedge: E(0, 0, 1) \xrightarrow{\sim} C(1, 1, 0)\,.$$
$$(\tau\rho\sigma\tau)^\wedge: E(1, 0, 0) \xrightarrow{\sim} C(1, 0, 0)\,.$$
$$(\tau^2\rho\tau)^\wedge: E(1, 1, 0) \xrightarrow{\sim} C(0, 1, 0)\,.$$
$$(\tau^3\rho\sigma\tau^6)^\wedge: E(1, 1, 1) \xrightarrow{\sim} C(0, 0, 1)\,.$$
$$(\tau^4\rho\sigma^2\tau^5)^\wedge: E(0, 1, 1) \xrightarrow{\sim} C(1, 0, 1)\,.$$
$$(\tau^5\rho\sigma^2)^\wedge: E(1, 0, 1) \xrightarrow{\sim} C(0, 1, 1)\,.$$
$$(\tau^6\rho\sigma^2\tau^6)^\wedge: E(0, 1, 0) \xrightarrow{\sim} C(1, 1, 1)\,.$$

In Figure 1, we explicitly describe how these seven pairs are identified. The three points to which the same symbol among $A, B, \cdots, G$ is attached are identified to a triple point of M_0. Here, by $\rho\sigma^2\tau$, the two rational curves $E(0, 0, 1)$ and $C(1, 1, 0)$ are identified in such a way that symbols A, A^*, B come to A^*, A, B, respectively. Consequently, the double curve obtained by this identification has a self-intersection point. Figure 2 indicates the configuration of the double curves on M_0.

We can check these results by calculating the corresponding action of Γ_1 on the Bruhat-Tits building which is isomorphic to the dual graph of the irreducible components of $\mathscr{X}_0$ [Mum, p. 235].

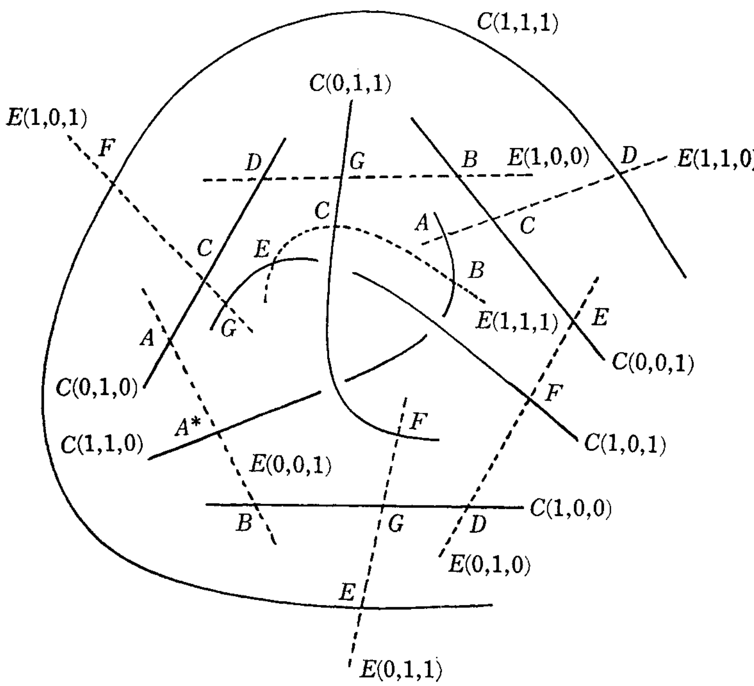


FIGURE 1

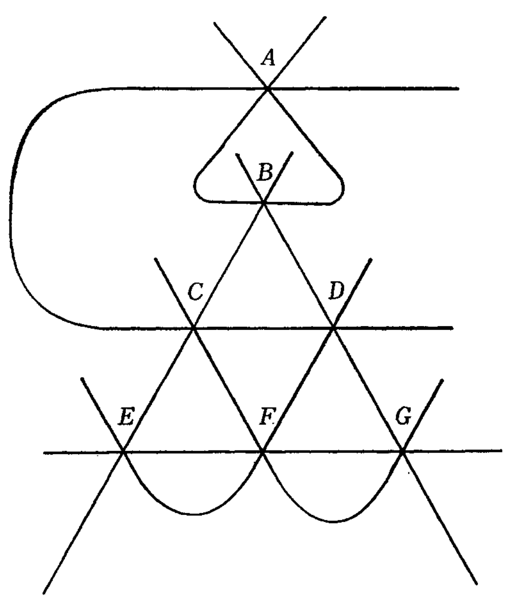


FIGURE 2

2. Singularities of the quotient surface. Since V is projective and G is finite, the quotient $Y=V/G$ is also a projective $\boldsymbol{Z}_2$-scheme. Although V is regular, Y has some singularities, since the action has fixed points. In this section, we study the singularities of Y.

Let $\bar{\boldsymbol{Q}}_2$ be the algebraic closure of the 2-adic number field $\boldsymbol{Q}_2$. The discrete valuation v of $\boldsymbol{Q}_2$ with $v(2)=1$ is uniquely extended to a valuation

$$v\colon \bar{\boldsymbol{Q}}_2 \to \boldsymbol{Q} \cup \{\infty\} .$$

The non-noetherian valuation ring $\bar{\boldsymbol{Z}}_2=\{a \in \bar{\boldsymbol{Q}}_2;\, v(a) \geqq 0\}$ is equal to the integral closure of $\boldsymbol{Z}_2$ in $\bar{\boldsymbol{Q}}_2$. For the maximal ideal $\mathfrak{m}=\{a \in \bar{\boldsymbol{Z}}_2;\, v(a)>0\}$, the residue field $\bar{\boldsymbol{Z}}_2/\mathfrak{m}$ is equal to the algebraic closure $\bar{\boldsymbol{F}}_2$ of the prime field $\boldsymbol{F}_2$.

In order to describe the geometric points of V_η and Y_η, it is convenient to use the $\bar{\boldsymbol{Z}}_2$-valued points of the $\boldsymbol{Z}_2$-scheme $\mathscr{X}$.

Let $\mathscr{D}:=\mathscr{X}(\bar{\boldsymbol{Z}}_2)$ be the set of $\bar{\boldsymbol{Z}}_2$-valued points of $\mathscr{X}$. Since $\bar{\boldsymbol{Q}}_2$ is the quotient field of $\bar{\boldsymbol{Z}}_2$, we have an injection

$$\mathscr{D} \to \mathscr{X}(\bar{\boldsymbol{Q}}_2)=\boldsymbol{P}^2(\bar{\boldsymbol{Q}}_2) ,$$

where $\boldsymbol{P}^2(\bar{\boldsymbol{Q}}_2)$ is the projective plane with the coordinates $(X_0\colon X_1\colon X_2)$. Hence we use this coordinate system to represent the points of $\mathscr{D}$ through this injection. As we see later, Mumford's fake projective plane is set-theoretically the quotient of $\mathscr{D}$ by $\Gamma \subset GL(3, \bar{\boldsymbol{Q}}_2)$.

Let $x\colon \mathrm{Spec}(\bar{\boldsymbol{Z}}_2) \to \mathscr{X}$ be a point of $\mathscr{D}$. Then by composing it with the inclusion $\mathrm{Spec}(\bar{\boldsymbol{F}}_2) \hookrightarrow \mathrm{Spec}(\bar{\boldsymbol{Z}}_2)$, we get an $\bar{\boldsymbol{F}}_2$-valued point of $\mathscr{X}_0 \subset \mathscr{X}$. We denote it by 2-red$(x)$. Let $y \in \mathscr{X}$ be the support point of 2-red(x). Then we get the associated local homomorphism $\mathscr{O}_{y,\mathscr{X}} \to \bar{\boldsymbol{Z}}_2$. By this observation, we see that $\mathscr{D}$ is equal to the sum

$$\bigcup_{y \in \mathscr{X}_0} \{x\colon \mathscr{O}_{y,\mathscr{X}} \to \bar{\boldsymbol{Z}}_2;\, x \text{ is a local } \boldsymbol{Z}_2\text{-homomorphism}\} .$$

We would like to know which points of $\boldsymbol{P}^2(\bar{\boldsymbol{Q}}_2)$ are in $\mathscr{D}$. Since $\mathscr{X}_0$ is a normal crossing divisor in $\mathscr{X}$, the points of $\mathscr{X}_0$ are classified into the following three types: (1) Smooth points of $\mathscr{X}_0$. (2) Points lying only on a double curve of $\mathscr{X}_0$. (3) Triple points.

Recall that the dual graph which describes the intersections of the components of $\mathscr{X}_0$ is known as the Bruhat-Tits building. Each irreducible component E of $\mathscr{X}_0$ corresponds to a free $\boldsymbol{Z}_2$-module $M \subset \boldsymbol{Q}_2X_0+\boldsymbol{Q}_2X_1+\boldsymbol{Q}_2X_2$ of rank three modulo the equivalence relation $M \sim 2^kM$. More explicitly, $\mathrm{Proj}\, S^*M \simeq \boldsymbol{P}^2_{\boldsymbol{Z}_2}$ for the symmetric algebra S^*M is dominated by $\mathscr{X}$, and E is the proper transform of the closed fiber. For the detail, see [Mum, p. 235].

(1) Let B be the irreducible component of $\mathscr{X}_0$ which corresponds to the module $M_0 = \boldsymbol{Z}_2X_0 + \boldsymbol{Z}_2X_1 + \boldsymbol{Z}_2X_2$. The smooth points of $\mathscr{X}_0$ which are contained in B are exactly those points of $\boldsymbol{P}^2_{\boldsymbol{F}_2} = \operatorname{Proj} \boldsymbol{F}_2[X_0, X_1, X_2] \subset \operatorname{Proj} \boldsymbol{Z}_2[X_0, X_1, X_2]$ which are not on the seven $\boldsymbol{F}_2$-rational lines on it. These lines are given by $(X_0 = 0)$, $(X_1 = 0)$, $(X_2 = 0)$, $(X_0 + X_1 = 0)$, $(X_0 + X_2 = 0)$, $(X_1 + X_2 = 0)$ and $(X_0 + X_1 + X_2 = 0)$. Hence, a point $x = (x_0 : x_1 : x_2) \in \boldsymbol{P}^2(\bar{\boldsymbol{Q}}_2)$ is in $\mathscr{D}$ with 2-red(x) in this smooth part if and only if

$$v(x_0) = v(x_1) = v(x_2) = v(x_0 + x_1) = v(x_0 + x_2) = v(x_1 + x_2) = v(x_0 + x_1 + x_2) .$$

(2) Let C be the double curve which corresponds to the pair $\boldsymbol{Z}_2X_0 + \boldsymbol{Z}_2X_1 + \boldsymbol{Z}_2X_2/2 \supset \boldsymbol{Z}_2X_0 + \boldsymbol{Z}_2X_1 + \boldsymbol{Z}_2X_2$. It can be shown easily that 2-red(x) of a point $x \in \boldsymbol{P}^2(\bar{\boldsymbol{Q}}_2)$ is on C and that it is not a triple point if and only if

$$v(x_2) - 1 < v(x_0) = v(x_1) = v(x_0 + x_1) < v(x_2) .$$

(3) The triple point P which corresponds to the triple $\boldsymbol{Z}_2X_0 + \boldsymbol{Z}_2X_1/2 + \boldsymbol{Z}_2X_2/2 \supset \boldsymbol{Z}_2X_0 + \boldsymbol{Z}_2X_1 + \boldsymbol{Z}_2X_2/2 \supset \boldsymbol{Z}_2X_0 + \boldsymbol{Z}_2X_1 + \boldsymbol{Z}_2X_2$ is the point $X_1/X_0 = X_2/X_1 = 2X_0/X_2 = 0$ of $\operatorname{Spec} \boldsymbol{Z}_2[X_1/X_0, X_2/X_1, 2X_0/X_2]$ (see [Mum, p. 234]). Then 2-red(x) is equal to P if and only if

$$v(x_2) - 1 < v(x_0) < v(x_1) < v(x_2) .$$

$PGL(3, \boldsymbol{Q}_2)$ acts transitively on the sets of the irreducible components, the double curves and triple points of $\mathscr{X}_0$, respectively. Hence we have the following description of $\mathscr{D}$.

PROPOSITION 2.1. *Let $x = (x_0 : x_1 : x_2)$ be a point of $\boldsymbol{P}^2(\bar{\boldsymbol{Q}}_2)$. Then x is in $\mathscr{D}$ if and only if there exists $\alpha \in GL(3, \boldsymbol{Q}_2)$ such that $(y_0, y_1, y_2) = (x_0, x_1, x_2)\alpha$ satisfies either*

(i) $v(y_0) = v(y_1) = v(y_2) = v(y_0 + y_1) = v(y_0 + y_2) = v(y_1 + y_2) = v(y_0 + y_1 + y_2)$,

(ii) $v(y_2) - 1 < v(y_0) = v(y_1) = v(y_0 + y_1) < v(y_2)$ *or*

(iii) $v(y_2) - 1 < v(y_0) < v(y_1) < v(y_2)$.

By the above criterion, it is easy to see that any $\boldsymbol{Q}_2$-rational point of $\boldsymbol{P}^2(\bar{\boldsymbol{Q}}_2)$ is not in $\mathscr{D}$. In fact, we have the following stronger result.

PROPOSITION 2.2. *Let K be an arbitrary quadratic extension of $\boldsymbol{Q}_2$. If x_0, x_1, x_2 are elements of K, then the point $(x_0 : x_1 : x_2) \in \boldsymbol{P}^2(\bar{\boldsymbol{Q}}_2)$ is not contained in $\mathscr{D}$.*

PROOF. Let α be an element of $GL(3, \boldsymbol{Q}_2)$ and let $(y_0, y_1, y_2) = (x_0, x_1, x_2)\alpha$. Clearly, y_0, y_1, y_2 are also in K. Let $\mathscr{O}_K$ be the integral closure of $\boldsymbol{Z}_2$ in K. Since $\boldsymbol{Z}_2$ is Henselian, $\mathscr{O}_K$ is also a discrete valuation ring. Let $u\mathcal{O}_K$ be the maximal ideal of $\mathcal{O}_K$. Since the ramification index e and the relative degree f satisfy the relation $ef = [K: \boldsymbol{Q}_2] = 2$, we have two possibilities: Namely,

(1) $e = 1$ and $f = 2$, i.e., $v(u) = 1$ and $\mathcal{O}_K/u\mathcal{O}_K = \boldsymbol{F}_4$, or

(2) $e = 2$ and $f = 1$, i.e., $v(u) = 1/2$ and $\mathcal{O}_K/u\mathcal{O}_K = \boldsymbol{F}_2$.

We now show that in both cases none of the three conditions in Proposition 2.1 is satisfied. We may assume $y_0, y_1, y_2 \in \mathcal{O}_K$ and one of them is 1 by dividing them by some y_i, if necessary. Let $\bar{y}_0, \bar{y}_1, \bar{y}_2$ be the images of y_0, y_1, y_2 in $\mathcal{O}_K/u\mathcal{O}_K$, respectively.

Case (1). $v(y_0) = v(y_1) = v(y_2) = 0$ implies $\bar{y}_0, \bar{y}_1, \bar{y}_2 \neq 0$. $v(y_0 + y_1) = v(y_0 + y_2) = v(y_1 + y_2) = 0$ implies that $\bar{y}_0, \bar{y}_1, \bar{y}_2$ are distinct elements of $\boldsymbol{F}_4$. Since the sum of the three distinct non-zero elements of $\boldsymbol{F}_4$ is zero, we have $v(y_0 + y_1 + y_2) > 0$. Hence (i) of Proposition 2.1 is impossible. Both (ii) and (iii) are obviously impossible, since $v(y_i)$'s are integers.

Case (2). (i) and (ii) are impossible, since $v(y_0) = v(y_1)$ and $\mathcal{O}_K/u\mathcal{O}_K \simeq \boldsymbol{F}_2$ imply $v(y_0 + y_1) > v(y_0)$. (iii) is also impossible, since $v(y_i)$'s are half integers. q.e.d.

Although $\bar{\boldsymbol{Z}}_2$ is neither complete nor noetherian, we have the following:

LEMMA 2.3. *Let $(A, \mathfrak{m}_A)$ be a local $\boldsymbol{Z}_2$-algebra essentially of finite type with $2 \in \mathfrak{m}_A$. Then, for the 2-adic completion $i: A \to A[\![2]\!]$, the induced map*

$$i^*: \{f: A[\![2]\!] \to \bar{\boldsymbol{Z}}_2;\ \textit{local } \boldsymbol{Z}_2\textit{-homomorphism}\}$$
$$\to \{f: A \to \bar{\boldsymbol{Z}}_2;\ \textit{local } \boldsymbol{Z}_2\textit{-homomorphism}\}$$

is bijective.

PROOF. Let $f, g: A[\![2]\!] \to \bar{\boldsymbol{Z}}_2$ be two local $\boldsymbol{Z}_2$-homomorphisms. Suppose that their restrictions to A are equal. Then they induce the same homomorphism $A/2^nA \to \bar{\boldsymbol{Z}}_2/2^n\bar{\boldsymbol{Z}}_2$ for every $n > 0$. By taking their projective limits, we have a homomorphism $A[\![2]\!] \to \bar{\boldsymbol{Z}}_2[\![2]\!]$. Since the natural homomorphism $\bar{\boldsymbol{Z}}_2 \to \bar{\boldsymbol{Z}}_2[\![2]\!]$ is injective, f and g are equal. Hence i^* is injective. We now show the surjectivity. Let $f: A \to \bar{\boldsymbol{Z}}_2$ be a local $\bar{\boldsymbol{Z}}_2$-homomorphism. Since A is essentially of finite type, the image $f(A)$ is contained in a finite extension of $\boldsymbol{Q}_2$ and hence it is a finite $\boldsymbol{Z}_2$-algebra. Hence it is complete in the 2-adic topology. Hence the homomorphism $f: A \to f(A) \hookrightarrow \bar{\boldsymbol{Z}}_2$ can be extended to $A[\![2]\!] \to f(A)$. q.e.d.

Recall that Γ_0 is a normal subgroup of Γ_1 such that $G = \Gamma_1/\Gamma_0$ is isomorphic to $PSL(2, \boldsymbol{F}_7)$. For an element α of Γ_1, we denote by α^- the induced automorphism of the $\boldsymbol{Z}_2$-scheme $V = \mathscr{X}/\Gamma_0$.

PROPOSITION 2.4. *There exists a natural map*

$$\varphi: \mathscr{D} \to V(\bar{\boldsymbol{Q}}_2)$$

such that the action of Γ_1 on $\mathscr{D}$ and $V(\bar{\boldsymbol{Q}}_2)$ are compatible with this map, i.e., for an arbitrary element $\alpha \in \Gamma_1$, the diagram

$$\begin{array}{ccc} \mathscr{D} & \xrightarrow{\alpha^\wedge} & \mathscr{D} \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ V(\bar{\boldsymbol{Q}}_2) & \xrightarrow{\alpha^-} & V(\bar{\boldsymbol{Q}}_2) \end{array}$$

commutes. Furthermore, the induced map $\bar{\varphi}: \mathscr{D}/\Gamma_0 \to V(\bar{\boldsymbol{Q}}_2)$ is bijective.

PROOF. Note that $V(\bar{\boldsymbol{Q}}_2) = V(\bar{\boldsymbol{Z}}_2)$, since V is proper over $\boldsymbol{Z}_2$. By Lemma 2.3, we have natural bijections

$$\mathscr{D} \simeq \bigcup_{y \in \mathscr{X}_0} \{x: \mathscr{O}^h_{y,\mathscr{X}} \to \bar{\boldsymbol{Z}}_2;\ \text{local } \boldsymbol{Z}_2\text{-homomorphism}\}$$

and

$$V(\bar{\boldsymbol{Q}}_2) \simeq \bigcup_{\bar{y} \in \mathscr{X}_0/\Gamma_0} \{\bar{x}: \mathscr{O}^h_{\bar{y},\mathscr{X}/\Gamma_0} \to \bar{\boldsymbol{Z}}_2;\ \text{local } \boldsymbol{Z}_2\text{-homomorphism}\}\ ,$$

where $\mathscr{O}^h_{y,\mathscr{X}}$ (resp. $\mathscr{O}^h_{\bar{y},\mathscr{X}/\Gamma_0}$) is the local ring at y (resp $\bar{y}$) of $\mathscr{X}$ (resp. $\mathscr{X}/\Gamma_0$) as a formal scheme, i.e., the 2-adic completion of the usual algebraic local ring. Let $x: \mathscr{O}_{y,\mathscr{X}} \to \bar{\boldsymbol{Z}}_2$ be an element of $\mathscr{D}$. Then, for the image $\bar{y}$ of y in the free quotient $\mathscr{X}_0/\Gamma_0$, we have a natural isomorphism

$$\mathscr{O}^h_{\bar{y},\mathscr{X}/\Gamma_0} \xrightarrow{\sim} \mathscr{O}^h_{y,\mathscr{X}}\ .$$

We define $\varphi(x)$ to be the composite

$$\mathscr{O}_{\bar{y},\mathscr{X}/\Gamma_0} \to \mathscr{O}^h_{\bar{y},\mathscr{X}/\Gamma_0} \xrightarrow{\sim} \mathscr{O}^h_{y,\mathscr{X}} \xrightarrow{x'} \bar{\boldsymbol{Z}}_2\ ,$$

where x' is the homomorphism which satisfies $i^*(x') = x$ for the embedding $i: \mathscr{O}_{y,\mathscr{X}} \to \mathscr{O}^h_{y,\mathscr{X}}$. Then it is obvious that φ satisfies the assertion of the proposition since $\mathscr{X}/\Gamma_0$ is the quotient of the formal scheme $\mathscr{X}$ with respect to a free action. q.e.d.

Now, we study the ramification of the quotient $V_\eta \to (V/H)_\eta$ with respect to a subgroup $H \subset G$. We need the following elementary ring-theoretic lemmas.

LEMMA 2.5. *Let B be a $\boldsymbol{Z}_2$-algebra of finite type. Assume that a finite group G acts on B as a $\boldsymbol{Z}_2$-algebra, and that a G-invariant maximal ideal $\mathfrak{p}$ contains 2. Then, for the local ring $A = B_\mathfrak{p}$, the ring A^G of G-*

invariant elements of A is essentially of finite type over $\boldsymbol{Z}_2$, and $A^G[\![2]\!]$ is equal to $A[\![2]\!]^G$.

PROOF. Since B is of finite type and G is a finite group, the subring B^G is also of finite type over $\boldsymbol{Z}_2$ and B is finite over B^G. Let $\mathfrak{p}^G = B^G \cap \mathfrak{p}$. Then since $\mathfrak{p}$ is G-invariant, $B \setminus \mathfrak{p}$ is a G-invariant multiplicative set with $(B \setminus \mathfrak{p})^G = B^G \setminus \mathfrak{p}^G$. Since G is finite, A is equal to $(B^G \setminus \mathfrak{p}^G)^{-1}B$ and $A^G = (B^G)_{\mathfrak{p}^G}$. Hence A^G is essentially of finite type and A is finite over A^G. There is an exact sequence

$$0 \to A^G \to A \xrightarrow{\delta} A^{\oplus |G|}/\Delta(A)$$

of finite A^G-modules, where $\Delta(A)$ is the diagonal and $\delta(a) = (ga)_{g \in G}$. Since $A^G[\![2]\!]$ is flat over A^G, and since $A \otimes_{A^G} A^G[\![2]\!]$ is equal to $A[\![2]\!]$, we get $A^G[\![2]\!] = A[\![2]\!]^G$ by tensoring this exact sequence with $A^G[\![2]\!]$. q.e.d.

LEMMA 2.6. *Let A be a local $\boldsymbol{Z}_2$-algebra essentially of finite type with $2 \in \mathfrak{m}_A$. Let $\mathfrak{p}$ be a prime ideal of A with $2 \notin \mathfrak{p}$ and $A/\mathfrak{p}$ is finite over $\boldsymbol{Z}_2$. Then, for $A' = A[\![2]\!]$, we have $A'_{\mathfrak{p}A'}[\![\mathfrak{p}]\!] = A_{\mathfrak{p}}[\![\mathfrak{p}]\!]$.*

PROOF. Since $A/\mathfrak{p}$ is finite over $\boldsymbol{Z}_2$, the finite $A/\mathfrak{p}$-module $\mathfrak{p}^n/\mathfrak{p}^{n+1}$ is also a finite $\boldsymbol{Z}_2$-module for every $n \geqq 0$. Hence $A/\mathfrak{p}^n$ is a finite $\boldsymbol{Z}_2$-algebra, and is complete in the 2-adic topology. Namely, we have $A/\mathfrak{p}^n = (A/\mathfrak{p}^n)[\![2]\!] = A'/\mathfrak{p}^n A'$. Since $(A/\mathfrak{p}^n)_{\mathfrak{p}/\mathfrak{p}^n} = A_{\mathfrak{p}}/\mathfrak{p}^n A_{\mathfrak{p}}$ and $(A'/\mathfrak{p}^n A')_{\mathfrak{p}A'/\mathfrak{p}^n A'} = A'_{\mathfrak{p}A'}/\mathfrak{p}^n A'_{\mathfrak{p}A'}$, we have $A_{\mathfrak{p}}/\mathfrak{p}^n A_{\mathfrak{p}} = A'_{\mathfrak{p}A'}/\mathfrak{p}^n A'_{\mathfrak{p}A'}$. The lemma is just the projective limit with respect to n of this equality. q.e.d.

Let H be a subgroup of $G = \Gamma_1/\Gamma_0$, and let Γ_H be the pull-back $\pi^{-1}(H) \subset \Gamma_1$. Let x be a point in $\mathscr{D}$ and let $\bar{x} = \varphi(x) \in V(\bar{\boldsymbol{Q}}_2)$. We denote by $\bar{\Gamma}_1$, $\bar{\Gamma}_0$ and $\bar{\Gamma}_H$ the images of Γ_1, Γ_0 and Γ_H in $PGL(3, \boldsymbol{Q}_2)$ as in Mumford [Mum, p. 240]. Since $\bar{\Gamma}_H/\bar{\Gamma}_0 \simeq H$ and since $\bar{\Gamma}_0$ acts freely on $\mathscr{D}$, the isotropy groups

$$T(x, \bar{\Gamma}_H) = \{\alpha^\wedge \in \bar{\Gamma}_H; \alpha^\wedge(x) = x\} \quad \text{and}$$
$$T(\bar{x}, H) = \{\alpha^- \in H; \alpha^-(\bar{x}) = \bar{x}\}$$

are isomorphic.

PROPOSITION 2.7. *The singularity of the quotient of $\boldsymbol{P}^2_{\boldsymbol{Q}_2}$ with respect to $T(x, \Gamma_H)$ at the image of x is formally isomorphic to that of the quotient of V with respect to $T(\bar{x}, H)$ at the image of $\bar{x}$.*

PROOF. Set $T = T(x, \bar{\Gamma}_H)$ and $\bar{T} = T(\bar{x}, H)$. Let y be the support point of 2-red(x) and let $\bar{y} \in V_0$ be the specialization of the support point of $\bar{x}$. Then by Lemma 2.5, we have

$$(\mathcal{O}_{y,\mathscr{X}})^T[\![2]\!] = (\mathcal{O}_{y,\mathscr{X}}^h)^T \simeq (\mathcal{O}_{\bar{y},V}^h)^{\bar{T}} = (\mathcal{O}_{\bar{y},V})^{\bar{T}}[\![2]\!] .$$

Let $\mathfrak{p}$ and $\mathfrak{p}'$ be the kernel of the composite homomorphisms

$$(\mathcal{O}_{y,\mathscr{X}})^T \to \mathcal{O}_{y,\mathscr{X}} \xrightarrow{x} \bar{\boldsymbol{Z}}_2 \quad \text{and} \quad (\mathcal{O}_{\bar{y},V})^{\bar{T}} \to \mathcal{O}_{\bar{y},V} \xrightarrow{\bar{x}} \bar{\boldsymbol{Z}}_2 ,$$

respectively. Then $((\mathcal{O}_{y,\mathscr{X}})^T)_{\mathfrak{p}}$ and $((\mathcal{O}_{\bar{y},V})^{\bar{T}})_{\mathfrak{p}'}$ are the local rings of the support points of x and $\bar{x}$, respectively. By Lemma 2.6 and the above equality, we have an isomorphism $((\mathcal{O}_{y,\mathscr{X}})^T)_{\mathfrak{p}}[\![\mathfrak{p}]\!] \xrightarrow{\sim} ((\mathcal{O}_{\bar{y},V})^{\bar{T}})_{\mathfrak{p}'}[\![\mathfrak{p}']\!]$. q.e.d.

Now, we study the case $H = G$ and hence $\Gamma_H = \Gamma_1$. We denote by Y the $\boldsymbol{Z}_2$-scheme V/G. Since $T(x, \bar{\Gamma}_1) \simeq T(\bar{x}, G) \subset G$, each element of $T(x, \bar{\Gamma})$ is of finite order. Mumford [Mum, p. 241] has already shown that every element of $\bar{\Gamma}_1$ of finite order is conjugate to one of $\sigma^i\tau^j$ or $(\rho\tau)^i$ for some $0 \leqq i \leqq 2$ and $0 \leqq j \leqq 6$. Since $\{\sigma, \tau\}$ generates a non-commutative group of order 21, they are conjugate to one of

$$1, \sigma, \sigma^2, \tau, \tau^2, \cdots, \tau^6, (\tau\rho), (\tau\rho)^2 .$$

Since the fixed points of conjugate elements come to the same points in Y, it is sufficient to determine the fixed points of σ, τ and $\tau\rho$ in $\mathscr{X}_0$ or $\mathscr{D}$ in order to find out all the ramification points of $f: V \to Y$.

Before determining the ramification points of $f: V \to Y$, we have to reformulate some of Mumford's results in a different way.

REMARK 2.8. Mumford has shown the following in his paper.

(i) For the component B of $\mathscr{X}_0$ which corresponds to the module $M_0 = \boldsymbol{Z}_2X_0 + \boldsymbol{Z}_2X_1 + \boldsymbol{Z}_2X_2$, the stabilizer $\{\alpha^\wedge \in \bar{\Gamma}_1;\ \alpha^\wedge(B) = B\}$ is equal to $\bar{\Gamma}_2$ which is the group of order 21 generated by σ and τ (cf. [Mum, p. 241]).

(ii) $\bar{\Gamma}_2$ acts on the $\boldsymbol{F}_2$-rational points on B simply transitively (cf. [Mum, p. 242]).

(iii) In particular, if $\alpha^\wedge \in \bar{\Gamma}_1$ fixes B and one $\boldsymbol{F}_2$-rational point on it, then $\alpha^\wedge = 1$.

We first determine the fixed points of σ, τ and $\tau\rho$ in the closed fiber $\mathscr{X}_0$. We can do so by looking at the corresponding action on the Bruhat-Tits building as follows:

Let x_0 be a fixed point of σ on $\mathscr{X}_0$. Then there exists an irreducible component B' of $\mathscr{X}_0$ which is stable under σ and which contains x_0. Actually if x_0 is the triple point corresponding to the triple of distinct $\boldsymbol{Z}_2$-submodules $M_0' \supset M_1' \supset M_2'$ of $\boldsymbol{Q}_2X_0 + \boldsymbol{Q}_2X_1 + \boldsymbol{Q}_2X_2$ with $M_2' \supsetneqq 2M_0'$, then since $\det\sigma = 1$ we have $\sigma(M_i') = M_i'$ for every i. If x_0 is not triple and is on a double curve of $\mathscr{X}_0$, then σ fixes the two components of $\mathscr{X}_0$ which are adjacent along the double curve since σ is of order three. If x_0 is not on any double curve, then σ stabilizes the unique component which contains x_0.

Let γ be an element of $\bar{\Gamma}_1$ with $\gamma^\wedge(B) = B'$. Then $(\gamma\sigma\gamma^{-1})^\wedge$ stabilizes B. Since the subgroups of order three of $\bar{\Gamma}_2$ are mutually conjugate, $\gamma\sigma\gamma^{-1}$ is conjugate to σ or σ^2. Hence the fixed points of σ in B' and B give the same ramification points on Y_0. It is easy to see that σ has just two fixed points on B. One of them is on $C(1, 0, 0)$ and the other is on $E(1, 0, 0)$, and they are identified by $(\tau\rho\sigma\tau)^\wedge$ in M_0. The point on $C(1, 0, 0)$ is mapped to the point defined by $X_1^2 + X_1X_2 + X_2^2 = 0$ on the line $X_0 = 0$ in $\boldsymbol{P}^2_{F_2}$ by the natural isomorphism. We denote by w the corresponding ramification point of Y. Clearly, w is of degree two and splits into two points in $Y(\bar{\boldsymbol{F}}_2)$.

Since τ is of order seven, any fixed point of τ in $\mathscr{X}_0$ is on a stabilized component. Let M_0' be the module associated to a component of $\mathscr{X}_0$ stabilized by τ. We may assume $M_0 \supset M_0'$ and $2M_0 \not\supset M_0'$. Since the group generated by τ acts transitively on $(M_0/2M_0)\setminus\{0\}$, we have $M_0' = M_0$. Hence the fixed points of τ are in B. Later we explicitly determine the fixed points of τ together with those in $\mathscr{D}$.

Since $\det\tau\rho = \lambda^2/2$, $\tau\rho$ stabilizes no component of $\mathscr{X}_0$. Hence it stabilizes no double curve of $\mathscr{X}_0$ since it is of order three. It is easy to see that $P \in B$ is the unique triple point fixed by $\tau\rho$.

The fixed points of σ, τ and $\tau\rho$ in $\boldsymbol{P}^2(\bar{\boldsymbol{Q}}_2)$ are calculated easily as follows.

(1) $$\sigma = \begin{bmatrix} 1 & 0 & \lambda \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}, \quad \det(tI_3 - \sigma) = t^3 - 1\,,$$

eigenvalues	1	ω	ω^2
eigenvectors	$(3, \lambda, \lambda)$	$(0, 1, \omega)$	$(0, 1, \omega^2)$,

where $\omega = (-1 + \sqrt{-3})/2$.

(2) $$\tau = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1+\lambda \\ 0 & 1 & \lambda \end{bmatrix},$$

$$\det(tI_3 - \tau) = t^3 - \lambda t^2 - (\lambda + 1)t - 1 = (t - \zeta)(t - \zeta^2)(t - \zeta^4)\,.$$

eigenvalues	ζ	ζ^2	ζ^4
eigenvectors	$(1, \zeta, \zeta^2)$	$(1, \zeta^2, \zeta^4)$	$(1, \zeta^4, \zeta)$,

where $\zeta = \exp(2\pi i/7)$.

$$(3) \qquad \tau\rho = \begin{bmatrix} 0 & 0 & \lambda^2/2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \det(tI_3 - \tau\rho) = t^3 - \lambda^2/2 ,$$

eigenvalues	ε	$\omega\varepsilon$	$\omega^2\varepsilon$
eigenvectors	$(1, \varepsilon, \varepsilon^2)$	$(1, \omega\varepsilon, \omega^2\varepsilon^2)$	$(1, \omega^2\varepsilon, \omega\varepsilon^2)$,

where $\varepsilon = (\lambda^2/2)^{1/3}$.

In case (1), since every component of the eigenvectors are in $\boldsymbol{Q}_2(\sqrt{-7})$ or $\boldsymbol{Q}_2(\sqrt{-3})$, the fixed points of σ in $\boldsymbol{P}^2(\bar{\boldsymbol{Q}}_2)$ are outside $\mathscr{D}$ by Proposition 2.2.

In case (2), set $\tilde{q} = (1, \zeta, \zeta^2)$. Note that $\sigma^\wedge(\tilde{q}) = (1, \zeta^2, \zeta^4)$ and $(\sigma^\wedge)^2(\tilde{q}) = (1, \zeta^4, \zeta)$. Let ζ_0 be the image of ζ in $\boldsymbol{Z}_2(\zeta)/(2) \simeq \boldsymbol{F}_8$. Then since $\zeta_0 \in \boldsymbol{F}_8 \setminus \boldsymbol{F}_4$, we see that $1 + \zeta_0$, $1 + \zeta_0^2$, $\zeta_0 + \zeta_0^2$ and $1 + \zeta_0 + \zeta_0^2$ are not zero. This implies that $v(1) = v(\zeta) = v(\zeta^2) = v(1+\zeta) = v(1+\zeta^2) = v(\zeta+\zeta^2) = v(1+\zeta+\zeta^2) = 0$. Hence $\tilde{q}$ is a point of $\mathscr{D}$ by Proposition 2.1. We denote by q the image $f \circ \varphi(\tilde{q}) \in Y(\bar{\boldsymbol{Q}}_2)$.

Since 2-red$(\tilde{q})$ is a smooth point of $\mathscr{X}_0$ and is on the component B, the isotropy group $T(\tilde{q}, \bar{\Gamma}_1)$ is a subgroup of $\bar{\Gamma}_2$ by Remark 2.8. Since σ does not fix $\tilde{q} \in \mathscr{D}$, we have $T(\tilde{q}, \bar{\Gamma}_1) = \langle \tau \rangle$. As we see later in Remark 2.10, the linear map τ is given locally at $\tilde{q}$ by $(y_1, y_2) \mapsto (\zeta y_1, \zeta^3 y_2)$. Hence the singularity of the quotient at this point is the cyclic quotient singularity of type (7, 3). By Proposition 2.7, the singularity of $Y(\bar{\boldsymbol{Q}}_2)$ at q is also a cyclic quotient singularity of type (7, 3).

These τ-invariant points of $\boldsymbol{P}^2(\bar{\boldsymbol{Q}}_2)$ are $\boldsymbol{Q}_2(\zeta)$-valued and they are identified to q in $Y(\bar{\boldsymbol{Q}}_2)$ by σ. Since the action of σ on these three points is compatible with the automorphism of $\boldsymbol{Q}_2(\zeta)$ defined by $\zeta \mapsto \zeta^2$, we see that q is a $\boldsymbol{Q}_2$-valued point. Since Y is proper over $\boldsymbol{Z}_2$, there exists a $\boldsymbol{Z}_2$-valued point $\bar{q}$: Spec $\boldsymbol{Z}_2 \to Y$ such that $\bar{q}(\eta) = q$. We see easily that $\bar{q}(0) \in Y_0$ is also a cyclic quotient singularity of type (7, 3). We can see similarly that the fixed points of τ on B are only $(1, \zeta_0, \zeta_0^2)$, $(1, \zeta_0^2, \zeta_0^4)$ and $(1, \zeta_0^4, \zeta_0)$. Since B is the only component of $\mathscr{X}_0$ stabilized by τ, we see that $\bar{q} \subset Y$ is the unique ramification locus given by τ.

Finally in case (3), set $\tilde{p}_0 = (1, \varepsilon, \varepsilon^2)$, $\tilde{p}_1 = (1, \omega\varepsilon, \omega^2\varepsilon^2)$ and $\tilde{p}_2 = (1, \omega^2\varepsilon, \omega\varepsilon^2)$. Since $v(\lambda^2/2) = 1$, we have $v(\varepsilon) = 1/3$, while $v(\omega) = 0$. Hence these points are in $\mathscr{D}$ by Proposition 2.1. In this case, 2-red$(\tilde{p}_i)$'s are the same triple point $P \in \mathscr{X}_0$. At this point P, the three components of $\mathscr{X}_0$ which correspond to $\boldsymbol{Z}_2X_0 + \boldsymbol{Z}_2X_1/2 + \boldsymbol{Z}_2X_2/2$, $\boldsymbol{Z}_2X_0 + \boldsymbol{Z}_2X_1 + \boldsymbol{Z}_2X_2/2$ and $\boldsymbol{Z}_2X_0 + \boldsymbol{Z}_2X_1 + \boldsymbol{Z}_2X_2$ meet together. In particular, the component B contains P. Suppose $\alpha^\wedge \in \bar{\Gamma}_1$ fixes P. Then since $\tau\rho$ cyclically permutes the three components, $(\tau\rho)^{-i}\alpha^\wedge$ stabilize B for $i=0, 1$ or 2. By (iii) of Remark 2.8, we get $\alpha^\wedge=(\tau\rho)^i$. Since the isotropy group of $\tilde{p}_i$'s are contained in that of P, we have $T(\tilde{p}_i, \bar{\Gamma}_1)=\langle\tau\rho\rangle$.

No $\alpha\in\bar{\Gamma}_1$ maps $\tilde{p}_i$ to another $\tilde{p}_j$ since $\alpha^\wedge(\tilde{p}_i)=\tilde{p}_j$ implies $\alpha\in\langle\tau\rho\rangle$. Hence, the points $\tilde{p}_0, \tilde{p}_1, \tilde{p}_2$ are mapped to distinct points in $Y(\bar{\boldsymbol{Q}}_2)$. Let them be p_0, p_1 and p_2, respectively. As in case (2), we see that $Y(\bar{\boldsymbol{Q}}_2)$ has cyclic quotient singularities of type (3, 2) at these points.

The points $\tilde{p}_0, \tilde{p}_1, \tilde{p}_2$ are solutions of the system of equations $(X_1/X_0)=(X_2/X_1)=(\varepsilon^3X_0/X_2)$. Since the local ring of $\mathscr{X}$ at P is $\boldsymbol{Z}_2[X_1/X_0, X_2/X_1, \varepsilon^3X_0/X_2]_{\mathfrak{m}}$ for the maximal ideal $\mathfrak{m}=(X_1/X_0, X_2/X_1, \varepsilon^3X_0/X_2)$, the equations give a $\boldsymbol{Z}_2[\varepsilon]$-valued point $\bar{p}$ of Y such that $\bar{p}(0)=P$ and that the image of $\bar{p}(\eta)$ in Y is a $\boldsymbol{Q}_2(\varepsilon)$-valued point which splits into the three points p_0, p_1, p_2 in $Y(\bar{\boldsymbol{Q}}_2)$. Since P is the unique fixed point of $\tau\rho$ in $\mathscr{X}_0$, we see that $\bar{p}$ is the unique ramification locus of $f\colon V\to Y$ caused by $\tau\rho$.

Thus we conclude:

THEOREM 2.9. *The morphism $f\colon V\to Y$ is ramified along $\bar{q}$, $\bar{p}$ and at the point $w\in Y_0$ of degree two. The restriction to the geometric fibers $f_{\bar{Q}_2}\colon V(\bar{\boldsymbol{Q}}_2)\to Y(\bar{\boldsymbol{Q}}_2)$ is ramified at the point p_0, p_1, p_2 and q. p_0, p_1 and p_2 (resp. q) are cyclic quotient singularities of type* (3, 2) (*resp. of type* (7, 3)).

REMARK 2.10. Let R be the étale finite ring extension $\boldsymbol{Z}_2[\zeta, \omega]$ of $\boldsymbol{Z}_2$. We can describe the minimal resolution of the singularities along $\bar{q}$ and $\bar{p}$ after the étale base extension $Y_R\to\operatorname{Spec}R$ of $Y\to\operatorname{Spec}\boldsymbol{Z}_2$ as follows:

By the coordinate change

$$(Y_0, Y_1, Y_2):=(X_0, X_1, X_2)\begin{bmatrix}1 & \zeta & \zeta^2\\ 1 & \zeta^2 & \zeta^4\\ 1 & \zeta^4 & \zeta\end{bmatrix}^{-1},$$

of $\boldsymbol{P}_R^2$, τ is diagonalized as

$$\begin{bmatrix}\zeta & 0 & 0\\ 0 & \zeta^2 & 0\\ 0 & 0 & \zeta^4\end{bmatrix}$$

and the eigenvectors are (1, 0, 0), (0, 1, 0) and (0, 0, 1). Hence the local ring of Y_R at $\bar{q}(0)$ is formally isomorphic to the localization of the ring of invariants $R[Y_1/Y_0, Y_2/Y_0]^\tau$ in the polynomial ring $R[Y_1/Y_0, Y_2/Y_0]$ with respect to the action of τ defined by $Y_1/Y_0\mapsto\zeta Y_1/Y_0$ and $Y_2/Y_0\mapsto\zeta^3Y_2/Y_0$. One can resolve it minimally by the standard method. For any geometric fiber, the exceptional set is a chain of nonsingular rational curves with the self-intersection numbers -3, -2, -2.

The local ring of Y_R at $\bar{p}(0)$ is formally isomorphic to the localization of the ring of invariants $R[X_1/X_0, X_2/X_1, \varepsilon^3X_0/X_2]^{\tau\rho} \subset R[X_1/X_0, X_2/X_1, \varepsilon^3X_0/X_2]$ with respect to the automorphism $\tau\rho$ given by $X_1/X_0 \mapsto X_2/X_1$, $X_2/X_1 \mapsto \varepsilon^3X_0/X_2$, $\varepsilon^3X_0/X_2 \mapsto X_1/X_0$. Note that $\varepsilon^3 = \lambda^2/2$ is a generator of the maximal ideal of the discrete valuation ring R. By the coordinate change

$$(T_0, T_1, T_2) = (X_1/X_0, X_2/X_1, \varepsilon^3X_0/X_2)\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{bmatrix},$$

we have

$$\tau\rho = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{bmatrix}.$$

Then the ring of invariants is $R[T_0, T_1^3, T_2^3, T_1T_2]$ with the relation $T_0^3 + T_1^3 + T_2^3 - 3T_0T_1T_2 = 27\varepsilon^3$. We see easily that this is a complete intersection of a regular ring. In particular, this is a Gorenstein ring. This singularity is resolved by the blowing up along the prime ideal (T_1^3, T_2^3, T_1T_2). For the geometric fiber $Y(\bar{\boldsymbol{Q}}_2)$, this is the blowing-up at $\{p_0, p_1, p_2\}$. Since these are cyclic quotient singularities of type (3, 2), this blowing-up gives the minimal resolution of these singular points and each exceptional set is the union of two nonsingular rational curves with the self-intersection numbers -2 intersecting each other at one point.

Thus we minimally resolved the singularities of Y_R along $\bar{q}$ and $\bar{p}$. Since this resolution is canonical, it descends to a scheme Y' over $\boldsymbol{Z}_2$. Clearly, $Y'(\bar{\boldsymbol{Q}}_2)$ is the minimal resolution of $Y(\bar{\boldsymbol{Q}}_2)$.

3. The plurigenera of the quotient surface.

In this section, we study pluri-canonical line bundles on V and its quotients.

The component B of $\mathscr{X}_0$ is a smooth rational surface, and the fourteen rational curves $C(a, b, c)$'s and $E(a, b, c)$'s form a divisor $A = \cup_{a,b,c}(C(a, b, c) \cup E(a, b, c))$ with only normal crossings in B. For the unramified covering $\mathscr{X}_0 \to V_0$, we denote by B_1, $C(a, b, c)_1$, $E(a, b, c)_1$, P_1 and A_1 the image of B, $C(a, b, c)$, $E(a, b, c)$, P and A in V_0, respectively. Note that the fixed point $P \in \mathscr{X}_0$ of $\tau\rho$, is the intersection point of $C(0, 0, 1)$ and $E(1, 0, 0)$. One can check that B_1 has no self-intersection. Hence B_1 is isomorphic to B.

From now on, we mainly treat V and its quotient with respect to a subgroup of G. Hence, for simplicity, we denote also by σ, τ, ρ their images in G. For an element $\alpha \in G$, we denote by α^- the associated automorphism of V as in Section 2.

Since $M_0 = V_0/S$ consists of only one irreducible component, we have

$$V_0 = \bigcup_{\alpha \in S} B_\alpha \quad \text{where} \quad B_\alpha = \alpha^-(B_1) .$$

Here B_α's cross each other normally and the normalization $\tilde{V}_0$ is equal to the disjoint union $\coprod_{\alpha \in S} B_\alpha$. Let $\varphi: \tilde{V}_0 \to V_0$ be the natural morphism.

Since the induced action of G on the set of double curves is transitive, and since the stabilizer of the double curve $D_1 = C(1, 0, 0)_1$ is $\{1, \sigma, \sigma^2\}$, we see that the union D of the double curves is

$$D = \bigcup_{\beta \in G/\langle\sigma\rangle} D_\beta ,$$

where $G/\langle\sigma\rangle$ is the set of left cosets $\{\langle\sigma\rangle g; g \in G\}$ and $D_\beta := \beta^-(D_1)$.

Similarly, the stabilizer of P_1 is $\{1, \tau\rho, (\tau\rho)^2\}$ and

$$\{P_\mu = \mu^-(P_1); \mu \in G/\langle\tau\rho\rangle\}$$

is the set of the triple points of V_0. Note that the set of $\boldsymbol{F}_2$-rational points of V_0 is exactly equal to this set.

For the union D of the double curves of V_0, let $\delta: \tilde{D} \to V_0$ be the natural morphism from the normalization $\tilde{D} = \coprod_{\beta \in G/\langle\sigma\rangle} D_\beta$ of D to V_0.

Since the double curves arise from the identification of (-1)-curves and (-2)-curves [Mum, p. 236], there exist morphisms $\varepsilon, \gamma: \tilde{D} \to \tilde{V}_0$ such that $\varepsilon(D_\beta)^2 = -1$ and $\gamma(D_\beta)^2 = -2$ for every component D_β of $\tilde{D}$ and $\varphi \circ \varepsilon = \varphi \circ \gamma = \delta$. The union $\varepsilon(\tilde{D}) \cup \gamma(\tilde{D})$ is equal to $\coprod_{\alpha \in S} A_\alpha$, where $A_\alpha = \alpha^-(A_1) \subset B_\alpha$.

For any line bundle L on V_0, the following diagram is exact:

$$H^0(V_0, L) \xrightarrow{\varphi^*} H^0(\tilde{V}_0, \varphi^* L) \underset{\gamma^*}{\overset{\varepsilon^*}{\rightrightarrows}} H^0(\tilde{D}, \delta^* L) .$$

For an equidimensional Gorenstein scheme Z, we denote by ω_Z its canonical invertible sheaf. As is well known for varieties with normal crossing singularities, we have

$$\varphi^* \omega_{V_0} = \omega_{\tilde{V}_0}(\varepsilon(\tilde{D}) \cup \gamma(\tilde{D})) = \bigoplus_{\alpha \in S} \omega_{B_\alpha}(A_\alpha) .$$

Hence we get the exact diagram

$$(1) \qquad H^0(V_0, \omega_{V_0}^{\otimes m}) \to \bigoplus_{\alpha \in S} H^0(B_\alpha, \omega_{B_\alpha}^{\otimes m}(mA_\alpha)) \underset{\gamma^*}{\overset{\varepsilon^*}{\rightrightarrows}} H^0(\tilde{D}, \delta^* \omega_{V_0}^{\otimes m})$$

for every integer m.

On the other hand, $\bigoplus_{\alpha \in S} \varphi_* \mathcal{O}_{B_\alpha}(-A_\alpha)$ is equal to the ideal $I_D \subset \mathcal{O}_{V_0}$

defining D. Hence by the projection formula, we have

$$\omega_{V_0}^{\otimes m} \otimes I_D = \bigoplus_{\alpha \in S} \varphi_* \omega_{B_\alpha}^{\otimes m}((m-1)A_\alpha) .$$

Hence we get an exact sequence of $\mathcal{O}_{V_0}$-modules

$$(2) \qquad 0 \to \bigoplus_{\alpha \in S} \varphi_* \omega_{B_\alpha}^{\otimes m}((m-1)A_\alpha) \to \omega_{V_0}^{\otimes m} \to \omega_{V_0}^{\otimes m} \otimes \mathcal{O}_D \to 0 .$$

Now we analyze the sections of $\omega_B^{\otimes m}(mA)$ and $\omega_B^{\otimes m}((m-1)A)$ more precisely.

For the projective plane $\boldsymbol{P}^2_{\boldsymbol{F}_2}$ with the homogeneous coordinate system $(X_0: X_1: X_2)$, we set $y = X_0/X_2$ and $z = X_1/X_2$. Then the rational 2-form $\omega_0 = (dy \wedge dz)/yz$ vanishes nowhere and has a pole of order one along the divisor $(X_0X_1X_2 = 0)$. Let $p^*\omega_0$ be the pull-back of ω_0 with respect to the natural morphism $p: B \to \boldsymbol{P}^2_{\boldsymbol{F}_2}$. Then, the divisor $(p^*\omega_0)$ is equal to

$$\begin{aligned} E(1,1,1) - C(1,0,0) - C(0,1,0) - C(0,0,1) \\ - E(1,0,0) - E(0,1,0) - E(0,0,1) . \end{aligned}$$

Hence $p^*\omega_0$ is a section of $\omega_B(A)$ with the zero divisor

$$\begin{aligned} F_0 = C(1,1,0) + C(1,0,1) + C(0,1,1) + C(1,1,1) \\ + E(1,1,0) + E(1,0,1) + E(0,1,1) + 2E(1,1,1) . \end{aligned}$$

Let F be a divisor on B which is linearly equivalent to F_0. Then the images $p(F_0)$ and $p(F)$ in $\boldsymbol{P}^2_{\boldsymbol{F}_2}$ are also linearly equivalent. Since $p(F_0) = (u_0 = 0)$ for $u_0 = (X_0 + X_1)(X_0 + X_2)(X_1 + X_2)(X_0 + X_1 + X_2)$, we see that $p(F)$ is equal to $(f = 0)$ for a homogeneous quartic polynomial $f \in \boldsymbol{F}_2[X_0, X_1, X_2]$.

Since $p^*(u_0 = 0) - F_0 = \sum_{a,b,c} E(a, b, c)$ should be equal to $p^*(f = 0) - F$, the divisor $(f = 0) \subset \boldsymbol{P}^2_{\boldsymbol{F}_2}$ contains all the seven $\boldsymbol{F}_2$-rational points of $\boldsymbol{P}^2_{\boldsymbol{F}_2}$. Conversely, if f is a quartic homogeneous polynomial with $f(a, b, c) = 0$ for all triple (a, b, c) of 0 or 1, then $p^*(f = 0) - \sum_{a,b,c} E(a, b, c)$ is effective and linearly equivalent to F_0. Hence $(f/u_0)p^*\omega_0$ is a section of $\omega_B(A)$.

Thus the space of section of $\omega_B(A)$ is described as

$$(3) \qquad H^0(\omega_B(A)) = \left\{ \frac{f}{u_0}\left(\frac{dy}{y} \wedge \frac{dz}{z}\right) \right\} ,$$

where f runs over the homogeneous polynomials in $\boldsymbol{F}_2[X_0, X_1, X_2]$ of degree 4 such that $f(a, b, c) = 0$ if $a, b, c = 0$ or 1.

Similarly for general $m \in \boldsymbol{Z}$, we get the following:

$$(4) \qquad H^0(\omega_B^{\otimes m}(mA)) = \left\{ \frac{f}{u_0^m}\left(\frac{dy}{y} \wedge \frac{dz}{z}\right)^{\otimes m} \right\} ,$$

where f runs over the homogeneous polynomials in $\boldsymbol{F}_2[X_0, X_1, X_2]$ of degree $4m$ which has zero of multiplicity at least m at each of the seven $\boldsymbol{F}_2$-rational points of $\boldsymbol{P}^2_{\boldsymbol{F}_2}$.

Let $\omega = (f/u_0^m)(dy/y \wedge dz/z)^{\otimes m}$ be an element of $H^0(\omega_B^{\otimes m}(mA))$. Then ω is in $H^0(\omega_B^{\otimes m}((m-1)A))$ if and only if f has the factor $u = X_0X_1X_2(X_0 + X_1)(X_0 + X_2)(X_1 + X_2)(X_0 + X_1 + X_2)$ and f has zero of multiplicity at least $m + 1$ at every $\boldsymbol{F}_2$-rational point of $\boldsymbol{P}^2_{\boldsymbol{F}_2}$. Since u has zeros of multiplicity three at these points, we see that

$$(5) \qquad H^0(\omega_B^{\otimes m}((m-1)A)) = \left\{\frac{ug}{u_0^m}\left(\frac{dy}{y} \wedge \frac{dz}{z}\right)^{\otimes m}\right\},$$

where g runs over the homogeneous polynomials in $\boldsymbol{F}_2[X_0, X_1, X_2]$ of degree $4m - 7$ which has zeros of multiplicity at least $m - 2$ at the seven $\boldsymbol{F}_2$-rational points of $\boldsymbol{P}^2_{\boldsymbol{F}_2}$.

Let m be an integer greater than one. Since $c_1^2(V_\eta) = 72$ and $\chi(\mathcal{O}_{V_\eta}) = 8$, we have $P_m(V_\eta) = \chi^0(\omega_{V_\eta}^{\otimes m}) = 36m(m-1) + 8$ by the plurigenus formula for surfaces of general type. Hence $H^0(V, \omega_V^{\otimes m})$ is a free $\boldsymbol{Z}_2$-module of rank $36m(m-1) + 8$. By Grothendieck's base change theorem, we have a natural injection

$$i_m\colon H^0(V, \omega_V^{\otimes m}) \bigotimes_{\boldsymbol{Z}_2} \boldsymbol{F}_2 \hookrightarrow H^0(V_0, \omega_{V_0}^{\otimes m}).$$

More generally, let H be a subgroup of G acting freely on V and let $V' = V/H$. Then we have an injection

$$i'_m\colon H^0(V', \omega_{V'}^{\otimes m}) \bigotimes_{\boldsymbol{Z}_2} \boldsymbol{F}_2 \hookrightarrow H^0(V'_0, \omega_{V'_0}^{\otimes m}).$$

Note that the left hand side is of dimension $(36m(m-1) + 8)/|H|$, since V'_η is also of general type.

PROPOSITION 3.1. *The above homomorphisms i_m and i'_m are isomorphisms for $m = 2$ and 3.*

PROOF. We give the proof only for i_m, since the proof for general i'_m is similar. Suppose $m = 2$. By (5), we have

$$H^0(\omega_B^{\otimes 2}(A)) = \left\{\frac{(aX_0 + bX_1 + cX_2)u}{u_0^2}\left(\frac{dy}{y} \wedge \frac{dz}{z}\right)^{\otimes 2}; a, b, c \in \boldsymbol{F}_2\right\}.$$

This is obviously three-dimensional. Hence $\bigoplus_{\alpha \in S} H^0(\omega_{B_\alpha}^{\otimes 2}(A_\alpha))$ is of dimension $8 \times 3 = 24$. On the other hand, V_0 has fifty-six $\boldsymbol{F}_2$-rational points $\{P_\mu\}_{\mu \in G/\langle\tau\rho\rangle}$. Hence there exists a natural homomorphism

$$(6) \qquad j_2\colon H^0(V_0, \omega_{V_0}^{\otimes 2}) \to \bigoplus_{\mu \in G/\langle\tau\rho\rangle} \omega_{V_2}^{\otimes 2}(P_\mu).$$

Here the right hand side is an $\boldsymbol{F}_2$-vector space of dimension 56. Hence it suffices to show that the kernel $\operatorname{Ker} j_2$ is contained in $\bigoplus_{\alpha\in S} H^0(\omega_{B_\alpha}^{\otimes 2}(A_\alpha))$, because then the dimension of $H^0(V_0, \omega_{V_0}^{\otimes 2})$ is at most $24 + 56 = 80$ which is the rank of $H^0(V, \omega_V^{\otimes 2})$.

Let ω be an element of $\operatorname{Ker} j_2$. We have to show that $\omega|_{D_\beta} = 0$ on each double curve D_β. Set $M_\beta = \delta^*\omega_{V_0}|_{D_\beta}$. Since $\delta^*\omega_{V_0}|_{D_\beta} = \gamma^*\omega_{B_\alpha}(A_\alpha)|_{D_\beta}$ for some $\alpha \in S$, and since $\gamma(D_\beta)$ is a nonsingular rational curve with $\gamma(D_\beta)^2 = -2$, we have

$$\deg M_\beta = \deg \omega_{B_\alpha}|_{\gamma(D_\beta)} + \gamma(D_\beta)\cdot A_\alpha = 0 + 1 = 1 .$$

Since $D_\beta \simeq \boldsymbol{P}^1(\boldsymbol{F}_2)$ has three $\boldsymbol{F}_2$-rational points and ω is zero there, $\omega|_{D_\beta} \in H^0(M_\beta^{\otimes 2})$ should be zero.

We now consider the case $m = 3$. By (5), $H^0(\omega_B^{\otimes 3}(2A))$ is isomorphic to the module of homogeneous quintic polynomials which have zeros at all the seven $\boldsymbol{F}_2$-rational points of $\boldsymbol{P}^2_{\boldsymbol{F}_2}$. It is easy to see that this is of dimesion $21-7=14$. Hence $\bigoplus_{\alpha\in S} H^0(\omega_{B_\alpha}^{\otimes 3}(2A_\alpha))$ is of dimension $8\times 14 = 112$. Let L be the kernel of the homomorphism

$$(7) \qquad j_3\colon H^0(V_0, \omega_{V_0}^{\otimes 3}) \to \bigoplus_{\mu\in G/\langle\tau\rho\rangle} \omega_{V_0}^{\otimes 3}(P_\mu) \simeq \boldsymbol{F}_2^{\oplus 56} .$$

Clearly, L is of codimension at most 56 in $H^0(V_0, \omega_{V_0}^{\otimes 3})$. Let D_β be a double curve of V_0, and let $0, 1, \infty$ be its $\boldsymbol{F}_2$-rational points. We consider the restriction map $L \to H^0(M_\beta^{\otimes 3})$. Since $\deg M_\beta^{\otimes 3} = 3$ and since each element $\omega \in L$ has zeros at $\{0, 1, \infty\}$, the image of this map is in $H^0(M_\beta^{\otimes 3}(-0-1-\infty)) \simeq \boldsymbol{F}_2$. Hence the kernel of the natural homomorphism

$$(8) \qquad L \to \bigoplus_{\beta\in G/\langle\sigma\rangle} H^0(M_\beta^{\otimes 3}(-0-1-\infty)) = \boldsymbol{F}_2^{\oplus 56}$$

is of codimension at most 56. Since the kernel is contained in $\bigoplus_{\alpha\in S} H^0(\omega_{B_\alpha}^{\otimes 3}(2A_\alpha))$, we see that the dimension of $H^0(V_0, \omega_{V_0}^{\otimes 3})$ is at most $112 + 56 + 56 = 224$ which is the rank of $H^0(V, \omega_V^{\otimes 3})$. Hence i_3 is an isomorphism. q.e.d.

REMARK 3.2. This proof implies that the homomorphisms (6), (7) and (8) are surjective. This is also true for the homomophism i'_m.

PROPOSITION 3.3. *Let H be a subgroup of G, and let ω be an element of $H^0(V_0, \omega_{V_0}^{\otimes m})$ for $m = 2$ or 3. If ω is H-invariant, then there exists an element $\tilde\omega \in H^0(V, \omega_V^{\otimes m})$ which is H-invariant and $\tilde\omega|_{V_0} = \omega$.*

PROOF. Let S_0 be a 2-Sylow subgroup of H. Then since S_0 is contained in a 2-Sylow subgroup of G, S_0 acts on V freely by a result of Mumford. Let V' be the quotient V/S_0. Since ω is S_0-invariant, it descends to an element of $H^0(V'_0, \omega_{V'_0}^{\otimes m})$. By Proposition 3.1, there exists

$\tilde{\omega}' \in H^0(V', \omega_{V'})$ with $\tilde{\omega}'|_{V'_0} = \omega$. We regard $\tilde{\omega}'$ as an S_0-invariant element of $H^0(V, \omega_V^{\otimes 2})$. Let $H = S_0\alpha_1 + \cdots + S_0\alpha_n$ be the left coset decomposition of H with respect to S_0. Let $\tilde{\omega} = \sum_{i=1}^n \alpha_i^*(\tilde{\omega}')$. Then $\tilde{\omega}$ is H-invariant, and $\tilde{\omega}|_{V_0} = n\omega = \omega$, since $n = [H: S_0]$ is an odd number. q.e.d.

THEOREM 3.4. *Let H be a subgroup of G and let m be 2 or 3. Then the homomorphism*

$$H^0(V, \omega_V^{\otimes m})^H \underset{\mathbf{Z}_2}{\otimes} \mathbf{F}_2 \to H^0(V_0, \omega_{V_0}^{\otimes m})^H$$

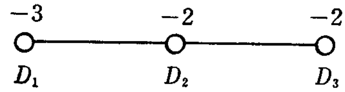
induced by i_m is an isomorphism.

PROOF. Since the quotient $H^0(V, \omega_V^{\otimes m})/H^0(V, \omega_V^{\otimes m})^H$ is contained in the $\mathbf{Q}_2$-module $H^0(V_\eta, \omega_{V_\eta}^{\otimes m})/H^0(V_\eta, \omega_{V_\eta}^{\otimes m})^H$, it is a free $\mathbf{Z}_2$-module. Hence $H^0(V, \omega_V^{\otimes m})^H$ is a direct summand of $H^0(V, \omega_V^{\otimes m})$. In particular, the homomorphism is injective. Since $m = 2$ or 3, it is surjective by Proposition 3.3. q.e.d.

The following shows that the bigenus P_2 of the desingularization of the quotient surface V_η/H is calculated only in terms of the closed fiber V_0.

PROPOSITION 3.5. *Let H be a subgroup of G, and let $\tilde{Z}$ be the minimal resolution of $Z = V_\eta/H$. Then $P_2(\tilde{Z}) = \dim H^0(V_0, \omega_{V_0}^{\otimes 2})^H$.*

PROOF. By Theorem 3.4, we have $\dim H^0(V_\eta, \omega_{V_\eta}^{\otimes 2})^H = \dim H^0(V_0, \omega_{V_0}^{\otimes 2})^H$. By Theorem 2.9, Z may only have at most cyclic quotient singularities of types (3, 2) or (7, 3), and the morphism $V_\eta \to Z$ is ramified only at these singular points. Hence an element $s \in H^0(V_\eta, \omega_{V_\eta}^{\otimes 2})^H$ can be regarded as a section of $\omega_{Z'}^{\otimes 2}$, where $Z' = Z \setminus \{\text{singular points}\}$. Note that $\tilde{Z}$ contains Z' as an open subset. It suffices to show that the rational section s of $\omega_{\tilde{Z}}^{\otimes 2}$ has no pole along the exceptional divisors. This is the case over the cyclic quotient singularities of type (3, 2), since they are rational double points. Let $y \in Z$ be a cyclic quotient singularity of type (7, 3) and let D_1, D_2, D_3 be the exceptional curves for the resolution of y with $D_1^2 = -3$, $D_2^2 = D_3^2 = -2$, $D_1 \cdot D_2 = D_2 \cdot D_3 = 1$ and $D_1 \cdot D_3 = 0$.



We can write the divisor (s) on $\tilde{Z}$ as $aD_1 + bD_2 + cD_3 + F$, where the support of F contains none of D_i's. Let d_i be the intersection number $D_i \cdot F$ for $i = 1, 2, 3$. Since (s) is linearly equivalent to $2K_{\tilde{Z}}$, we have

$$2 = (s)\cdot D_1 = -3a + b + d_1 \ ,$$
$$0 = (s)\cdot D_2 = a - 2b + c + d_2 \ ,$$
$$0 = (s)\cdot D_3 = b - 2c + d_3 \ .$$

By these equalities, we calculate easily that

$$7a = 3(d_1 - 2) + 2d_2 + d_3 \ ,$$
$$7b = 2(d_1 - 2) + 6d_2 + 3d_3 \ ,$$
$$7c = (d_1 - 2) + 3d_2 + 5d_3 \ .$$

Since a, b, c are integers and d_1, d_2, d_3 are nonnegative, we have $a, b, c \geqq 0$. Hence s has no pole on $\tilde{Z}$. q.e.d.

Recall that $\Gamma_2 = \langle \sigma, \tau \rangle$ stabilizes the component B of $\mathscr{X}_0$. We denote by G_{21} the injective image of Γ_2 in G. G_{21} is a group of order 21. Since $G_{21} \cap S = \{1\}$, G is equal to the disjoint union $\cup_{\alpha \in S} G_{21}\alpha$. If an element β is in $G_{21}\alpha$, then β induces an isomorphism $(\beta|_{B_1})\colon B_1 \to B_\alpha$.

The action of G on V_0 induces an action on the diagram (1). An element $(\omega_\alpha)_{\alpha \in S} \in \bigoplus_{\alpha \in S} H^0(B_\alpha, \omega_{B_\alpha}^{\otimes m}(mA_\alpha))$ is G-invariant if and only if $(\beta|_{B_1})^*\omega_\alpha = \omega_1$ for every $\beta \in G$, where α is the element of S with $\beta \in G_{21}\alpha$. This is also equivalent to the condition that ω_1 is G_{21}-invariant and $\omega_1 = (\alpha|_{B_1})^*\omega_\alpha$ for every $\alpha \in S$.

Suppose (ω_α) is G-invariant. By the diagram (1), (ω_α) is in $H^0(V_0, \omega_{V_0}^{\otimes m})^G$ if and only if $\varepsilon^*((\omega_\alpha)) = \gamma^*((\omega_\alpha))$. Since the action of G on the set of double curves of V_0 is transitive, this equality holds if they coincide on a component of $\tilde{D}$. Recall that, for $\alpha = \tau\rho\sigma\tau$, $C(1, 0, 0) \subset B$ and $\alpha^\wedge(E(1, 0, 0)) \subset \alpha^\wedge(B)$ form a double curve of $\mathscr{X}_0$. The isomorphism κ of the identification $E(1, 0, 0) \to C(1, 0, 0)$ is given by $(X_1\colon X_2) \mapsto (X_2\colon X_1)$.

We set

$$L_m = \{\omega \in H^0(B, \omega_B^{\otimes m}(mA))^{\Gamma_2};\ \kappa^*(\omega|_{C(1,0,0)}) = \omega|_{E(1,0,0)}\} \ .$$

By the expression (4) for $H^0(B, \omega_B^{\otimes m}(mA))$, we see easily that L_m is naturally isomorphic to $L'_m \subset \boldsymbol{F}_2[X_0, X_1, X_2]$ consisting of Γ_2-invariant homogeneous polynomials f of degree $4m$ such that $f(1, X_1, X_2)$ has no terms of degree smaller than m and $f(0, X_2, X_1)/X_1^m X_2^m (X_1 + X_2)^m = [f(1, X_1, X_2)]_m$, where $[g]_m$ denotes the homogeneous part of degree m of a polynomial g. Note that f has zero of multiplicity at least m at $(1, 0, 0)$ if and only if $f(1, X_1, X_2)$ has no terms of degree smaller than m. By the above observation, we have the following:

PROPOSITION 3.6. *$H^0(V_0, \omega_{V_0}^{\otimes m})^G$ is isomorphic to L_m by the correspondence $(\omega_\alpha)_{\alpha \in S} \mapsto \omega'_1$ where ω'_1 is the pull-back of ω_1 by the natural isomorphism $B \xrightarrow{\sim} B_1$. Hence it is also isomorphic to L'_m.*

For any $\alpha \in GL(3, \boldsymbol{F}_2)$, we have $\alpha^*(f/u_0^m(dy/y\wedge dz/z)^{\otimes m}) = (\alpha^* f)/u_0^m(dy/y \wedge dz/z)^{\otimes m}$ for $f/u_0^m(dy/y\wedge dz/z)^{\otimes m} \in H^0(B, \omega_B^{\otimes m}(mA))$, where f is a homogeneous polynomial of degree $4m$. Hence, in order to determine the Γ_2-invariant elements of $H^0(\omega_B^{\otimes m}(mA))$, we have to know those of $\boldsymbol{F}_2[X_0, X_1, X_2]$.

Recall that $\lambda = (-1+\sqrt{-7})/2$ is embedded in $\boldsymbol{Z}_2$ so that $\lambda \equiv 0 \pmod 2$. Hence, for $\zeta = \exp(2\pi i/7)$, $\boldsymbol{Q}_2(\zeta)$ is a cubic extension of $\boldsymbol{Q}_2$ with the relation $\zeta^3 - \lambda\zeta^2 - (1+\lambda)\zeta - 1 = 0$. We denote by ζ_0 the modulo 2 reduction of ζ, i.e., ζ_0 is a root of the equation $X^3 + X + 1 = 0$ in $\boldsymbol{F}_2[X]$.

The following method to find Γ_2-invariant polynomials in $\boldsymbol{F}_2[X_0, X_1, X_2]$ is due to Nakamura.

We set

$$\begin{aligned} Y_0 &= X_0 + \zeta_0^2 X_1 + \zeta_0 X_2 , \\ Y_1 &= X_0 + \zeta_0^4 X_1 + \zeta_0^2 X_2 , \\ Y_2 &= X_0 + \zeta_0 X_1 + \zeta_0^4 X_2 . \end{aligned}$$

Note that this is the modulo 2 reduction of the coordinate change in Remark 2.10, since

$$\begin{bmatrix} 1 & 1 & 1 \\ \zeta_0^2 & \zeta_0^4 & \zeta_0 \\ \zeta_0 & \zeta_0^2 & \zeta_0^4 \end{bmatrix} = \begin{bmatrix} 1 & \zeta_0 & \zeta_0^2 \\ 1 & \zeta_0^2 & \zeta_0^4 \\ 1 & \zeta_0^4 & \zeta_0 \end{bmatrix}^{-1} .$$

Then we have

$$\begin{aligned} &\tau(Y_0) = \zeta_0 Y_0 , \quad \tau(Y_1) = \zeta_0^2 Y_1 , \quad \tau(Y_2) = \zeta_0^4 Y_2 , \\ &\sigma(Y_0) = Y_2 , \quad \sigma(Y_1) = Y_0 \quad \text{and} \quad \sigma(Y_2) = Y_1 . \end{aligned}$$

Thus, if a polynomial f in $\bar{\boldsymbol{F}}_2[Y_0, Y_1, Y_2]$ is τ-invariant, then it is a sum of τ-invariant monomials in Y_0, Y_1 and Y_2.

A monomial $Y_0^i Y_1^j Y_2^k$ is τ-invariant if and only if $i + 2j + 4k \equiv 0 \pmod 7$. If it is τ-invariant, then

$$F_{i,j,k} = Y_0^i Y_1^j Y_2^k + Y_0^k Y_1^i Y_2^j + Y_0^j Y_1^k Y_2^i$$

is Γ_2-invariant. Conversely, every Γ_2-invariant polynomial in $\bar{\boldsymbol{F}}_2[Y_0, Y_1, Y_2]$ is a linear combination of $F_{i,j,k}$'s.

PROPOSITION 3.7. *For any i, j, k with $i + 2j + 4k \equiv 0 \pmod 7$, $F_{i,j,k}$ is in $\boldsymbol{F}_2[X_0, X_1, X_2]$. Conversely, every Γ_2-invariant polynomial in $\boldsymbol{F}_2[X_0, X_1, X_2]$ is a sum of $F_{i,j,k}$'s.*

PROOF. Clearly, $F_{i,j,k} \in \boldsymbol{F}_2(\zeta_0)[X_0, X_1, X_2]$. Let u be the automorphism of $\boldsymbol{F}_2(\zeta_0)[X_0, X_1, X_2]$ defined by $u(X_i) = X_i$ for $i = 0, 1, 2$ and $u(\zeta_0) = \zeta_0^2$. Then, a polynomial f in $\boldsymbol{F}_2(\zeta_0)[X_0, X_1, X_2]$ is in $\boldsymbol{F}_2[X_0, X_1, X_2]$ if and only if $u(f) = f$. Since $u(Y_0) = Y_1$, $u(Y_1) = Y_2$, $u(Y_2) = Y_0$, we have $u(F_{i,j,k}) = F_{i,j,k}$.

Suppose $F \in \boldsymbol{F}_2[X_0, X_1, X_2]$ is Γ_2-invariant. Since $\boldsymbol{F}_2[X_0, X_1, X_2] \subset \boldsymbol{F}_2(\zeta_0)[Y_0, Y_1, Y_2]$, F is written uniquely as a linear combination of $F_{i,j,k}$'s with coefficients in $\boldsymbol{F}_2(\zeta_0)\setminus\{0\}$. Since $u(F_{i,j,k}) = F_{i,j,k}$, the coefficients are in $\boldsymbol{F}_2\setminus\{0\} = \{1\}$. q.e.d.

We denote by Inv_n the $\boldsymbol{F}_2$-vector space of Γ_2-invariant homogeneous polynomials of degree n in $\boldsymbol{F}_2[X_0, X_1, X_2]$. By the above proposition, we can easily find bases for Inv_n for small n's as follows:

$$\mathrm{Inv}_0 = (1) .$$
$$\mathrm{Inv}_1 = \mathrm{Inv}_2 = \{0\} .$$
$$\mathrm{Inv}_3 = (\phi_3) , \quad \phi_3 = Y_0 Y_1 Y_2 .$$
$$\mathrm{Inv}_4 = (\phi_4) , \quad \phi_4 = Y_0 Y_1^3 + Y_1 Y_2^3 + Y_2 Y_0^3 .$$
$$\mathrm{Inv}_5 = (\phi_5) , \quad \phi_5 = Y_0^3 Y_1^2 + Y_1^3 Y_2^2 + Y_2^3 Y_0^2 .$$
$$\mathrm{Inv}_6 = (\phi_3^2, \phi_6), \ \phi_6 = Y_0^5 Y_1 + Y_1^5 Y_2 + Y_2^5 Y_0 .$$
$$\mathrm{Inv}_7 = (\phi_3\phi_4, \phi_7), \ \phi_7 = Y_0^7 + Y_1^7 + Y_2^7 .$$
$$\mathrm{Inv}_8 = (\phi_4^2, \phi_3\phi_5) .$$

We can also show that Inv_{12} is generated by $\{F_{10,2,0}, F_{3,9,0}, F_{5,6,1}, F_{7,3,2}, F_{4,4,4}\}$. Hence

$$\mathrm{Inv}_{12} = (\phi_3^4, \phi_3^2\phi_6, \phi_3\phi_4\phi_5, \phi_5\phi_7, \phi_6^2) ,$$

TABLE 1

f	$f(0, X_2, X_1)$	$f(1, X_1, X_2) \bmod (X_1, X_2)^4$
ϕ_3	$X_1^3+X_1X_2^2+X_2^3$	$1+X_1^2+X_1X_2+X_2^2+X_1^3+X_1^2X_2+X_2^3$
ϕ_4	$X_1^4+X_1^2X_2^2+X_2^4$	$1+X_1^2+X_1X_2+X_2^2+X_1^2X_2+X_1X_2^2$
ϕ_5	$X_1^5+X_1X_2^4+X_2^5$	$1+X_1^2X_2+X_1X_2^2$
ϕ_6	$X_1^6+X_1^4X_2^2+X_2^6$	$1+X_1^2+X_1X_2+X_2^2$
ϕ_7	$X_1^7+X_1^4X_2^3+X_1^2X_2^5+X_1X_2^6+X_2^7$	$1+X_1^3+X_1^2X_2+X_2^3$

TABLE 2

f	$f(0, X_2, X_1)$	$f(1, X_1, X_2) \bmod (X_1, X_2)^4$
ϕ_6^2	$X_1^{12}+X_1^8X_2^4+X_2^{12}$	1
$\phi_5\phi_7$	$X_1^{12}+X_1^9X_2^3+X_1^8X_2^4+X_1^6X_2^6+X_1^4X_2^8+X_1^3X_2^9+X_2^{12}$	$1+X_1^3+X_1X_2^2+X_2^3$
$\phi_3\phi_4\phi_5$	$X_1^{12}+X_1^9X_2^3+X_1^8X_2^4+X_1^6X_2^6+X_1^4X_2^8+X_1^3X_2^9+X_2^{12}$	$1+X_1^3+X_1^2X_2+X_2^3$
ϕ_3^4	$X_1^{12}+X_1^4X_2^8+X_2^{12}$	1
$\phi_6\phi_3^2$	$X_1^{12}+X_1^{10}X_2^2+X_1^8X_2^4+X_1^6X_2^6+X_1^4X_2^8+X_1^2X_2^{10}+X_2^{12}$	$1+X_1^2+X_1X_2+X_2^2$

since $F_{10,2,0} = \phi_6^2$, $F_{3,9,0} = \phi_5\phi_7 + \phi_6^2 + \phi_3^2\phi_6$, $F_{5,6,1} = \phi_3\phi_4\phi_5 + \phi_3^4 + \phi_3^2\phi_6$, $F_{7,3,2} = \phi_3^2\phi_6$ and $F_{4,4,4} = \phi_3^4$.

In order to determine L'_m for $m = 2, 3$, we provide the Tables 1 and 2 of $f(0, X_2, X_1)$ and $f(1, X_1, X_2)$ for $f = \phi_i$ and each element of the basis for Inv_{12}. In the tables, we omit the part of degree greater than 3 of $f(1, X_1, X_2)$.

PROPOSITION 3.8. *We have* $L'_2=(\phi_4^2+\phi_3\phi_5)$ *and* $L'_3=(\phi_5\phi_7+\phi_3^4, \phi_3\phi_4\phi_5+\phi_6^2)$. *In particular,* $\dim H^0(V_0, \omega_{V_0}^{\otimes 2})^G = 1$ *and* $\dim H^0(V_0, \omega_{V_0}^{\otimes 3})^G = 2$.

PROOF. The second assertion follows from the first by Proposition 3.6. In $\mathrm{Inv}_8\setminus\{0\}$, only $\phi_4^2 + \phi_3\phi_5$ has zero of multiplicity 2 at $(1, 0, 0)$. For $f = \phi_4^2 + \phi_3\phi_5$, we calculate easily by Table 1 that $[f(1, X_1, X_2)]_2 = f(0, X_2, X_1)/X_1^2X_2^2(X_1 + X_2)^2 = X_1^2 + X_1X_2 + X_2^2$. Hence L'_2 is generated by $\phi_4^2 + \phi_3\phi_5$.

From Table 2, the F_2-vector space $\{f \in \mathrm{Inv}_{12};\ f$ has zero of multiplicity 3 at $(1, 0, 0)\}$ is of dimension 3 and is generated by $\{\phi_6^2 + \phi_3^4, \phi_5\phi_7 + \phi_3^4, \phi_3\phi_4\phi_5 + \phi_3^4\}$. Hence it is easy to see that $L'_3 = (\phi_5\phi_7 + \phi_3^4, \phi_3\phi_4\phi_5 + \phi_6^2)$. Actually, we have

$$f(0, X_2, X_1)/X_1^3X_2^3(X_1 + X_2)^3 = [f(1, X_1, X_2)]_3 = X_1^3 + X_1X_2^2 + X_2^3$$

for $f = \phi_5\phi_7 + \phi_3^4$, and

$$f(0, X_2, X_1)/X_1^3X_2^3(X_1 + X_2)^3 = [f(1, X_1, X_2)]_3 = X_1^3 + X_1^2X_2 + X_2^3$$

for $f = \phi_3\phi_4\phi_5 + \phi_6^2$. q.e.d.

We now prove the following:

THEOREM 3.9. *For the minimal resolution* Y'_η *of* $Y_\eta = V_\eta/G$, *we have* $P_2(Y'_\eta) = P_3(Y'_\eta) = 1$. *We can choose as generators of* $H^0(Y'_\eta, \omega_{Y'_\eta}^{\otimes 2})$ *and* $H^0(Y'_\eta, \omega_{Y'_\eta}^{\otimes 3})$, *the elements which corresponds to the* Γ_2*-invariant polynomilals* $\phi_4^2 + \phi_3\phi_5$ *and* $\phi_3^4 + \phi_3\phi_4\phi_5 + \phi_5\phi_7 + \phi_6^2$ *by the modulo 2 reduction, respectively.*

PROOF. We have $P_2(Y'_\eta) = 1$ by Propositions 3.5, 3.6 and 3.8. By Proposition 3.8, $H^0(Y'_\eta, \omega_{Y'_\eta}^{\otimes 2})$ is generated by the lifting of the element of $H^0(V_0, \omega_{V_0}^{\otimes 2})^G$ which corresponds to $\phi_4^2 + \phi_3\phi_5$.

By Theorem 2.9 and Remark 2.10, Y has cyclic quotient singularity of type (7, 3) along $\bar{q}$, and it is minimally resolved simultaneously in Y'.

Let s be a section of $\omega_{Y''_\eta}^{\otimes 3}$, where Y'' is the smooth part $Y\setminus\{\bar{p}, \bar{q}, w\}$ of Y. For the resolution of Y_η at q, we define the exceptional divisors D_1, D_2, D_3 in Y'_η and integers a, b, c and $d_1, d_2, d_3 \geqq 0$ similarly as in the proof of Proposition 3.5. Then we have

$$7a = 3(d_1 - 3) + 2d_2 + d_3 ,$$

$$7b = 2(d_1 - 3) + 6d_2 + 3d_3 ,$$
$$7c = (d_1 - 3) + 3d_2 + 5d_3 .$$

Hence $b, c \geqq 0$ and $a \geqq -1$. In other words, s is regular at the divisors D_2, D_3 and may have a pole of order at most one along D_1.

Let L_i be the intersection of the closure of D_i with Y'_0 for $i = 1, 2, 3$. Then $L = L_1 \cup L_2 \cup L_3$ is the exceptional curve of $\bar{q}(0) \in Y_0$. Let $U \subset Y'_0$ be a smooth neighborhood of L and let ω_0 be a rational section of $\omega_U^{\otimes 3}$ which is regular outside L. Then, as in the case of the generic fiber, ω_0 may have a pole of order at most one along L_1 When ω_0 is represented by an element of $H^0(B, \omega_B^{\otimes 3}(3A))^{\Gamma_2}$, its regularity at L_1 is examined as follows:

Let $f(Y_0, Y_1, Y_2)$ be the corresponding Γ_2-invariant homogeneous polynomial of degree 12 in Y_i's. We take the local coordinate $(y_1, y_2) = (Y_1/Y_0, Y_2/Y_0)$ of the point $(1: \zeta_0: \zeta_0^2) \in \boldsymbol{P}^2_{\bar{F}_2} = \operatorname{Proj} \bar{F}_2[X_0, X_1, X_2]$. Then the action of τ is given by $(y_1, y_2) \mapsto (\zeta_0 y_1, \zeta_0^3 y_2)$ (cf. Remark 2.10). In the resolution, L is covered by four affine open sets with coordinates $(y_1^7, y_1^{-3}y_2)$, $(y_1^3 y_2^{-1}, y_1^{-2}y_2^3)$, $(y_1^2 y_2^{-3}, y_1^{-1}y_2^5)$ and $(y_1 y_2^{-5}, y_2^7)$, where the second and the third coordinates are of the neighborhoods of $L_1 \cap L_2$ and $L_2 \cap L_3$, respectively. The divisor L_1 is described as the line $(s = 0)$ with respect to the coordinate $(s, t) = (y_1^7, y_1^{-3}y_2)$. ω_0 is equal to $v \cdot f(1, y_1, y_2)(dy_1 \wedge dy_2)^{\otimes 3}$ for a non-vanishing regular function v on U. In view of the equality $dy_1 \wedge dy_2 = (1/7)s^{-3/7} ds \wedge dt$, we see that ω_0 has a pole at L_1 if and only if $s^{-9/7}g(s, t)$ has a pole along $(s = 0)$, where $g(s, t) = f(1, y_1, y_2)$.

Among τ-invariant monomials of degree 12 in Y_i's only $s^{-9/7}g(s, t)$ for $Y_0^{10}Y_1^2$ has a pole along $(s = 0)$. Hence ω_0's which correspond to $\phi_5\phi_7 + \phi_3^4 = F_{10,2,0} + F_{3,9,0} + F_{7,3,2} + F_{4,4,4}$ and $\phi_3\phi_4\phi_5 + \phi_6^2 = F_{10,2,0} + F_{5,6,1} + F_{7,3,2} + F_{4,4,4}$ have a pole along L_1, while ω_0 for $\phi_3^4 + \phi_3\phi_4\phi_5 + \phi_5\phi_7 + \phi_6^2 = F_{3,9,0} + F_{5,6,1}$ does not.

Let ω_η be an element of $H^0(Y''_\eta, \omega_{Y''_\eta}^{\otimes 3})$ which has nontrivial reduction ω_0 to Y''_0. Then ω_η has a pole at D_1, if so does ω_0 at L_1. Hence, by Theorem 3.4, there exists $\omega \in H^0(Y''_\eta, \omega_{Y''_\eta}^{\otimes 3})$ with a pole along D_1. Since D_1 is a nonsingular rational curve and $D_1^2 = -3$, we have $\omega_{Y'}^{\otimes 3}(D_1)|_{D_1} \simeq \mathcal{O}_{D_1}$. Hence $H^0(Y'_\eta, \omega_{Y'_\eta}^{\otimes 3})$ is of codimension one in $H^0(Y'_\eta, \omega_{Y'_\eta}^{\otimes 3}(D_1))$, which is isomorphic to $H^0(Y''_\eta, \omega_{Y''_\eta}^{\otimes 3})$, since the other singularities p_0, p_1, p_2 are rational double points. Since $H^0(Y''_\eta, \omega_{Y''_\eta}^{\otimes 3}) \simeq H^0(V_\eta, \omega_{V_\eta}^{\otimes 3})^G$ is of dimension two by Theorem 3.4 and Proposition 3.8, we have $\dim H^0(Y'_\eta, \omega_{Y'_\eta}^{\otimes 3}) = 1$. q.e.d.

REMARK 3.10. The Γ_2-invariant polynomials $f_2 = \phi_4^2 + \phi_3\phi_5$ and $f_3 = \phi_3^4 + \phi_3\phi_4\phi_5 + \phi_5\phi_7 + \phi_6^2$ are equal to $F_{2,6,0} + F_{4,3,1}$ and $F_{3,9,0} + F_{5,6,1}$, respectively. By expressing these polynomials in terms of the coordinates at $L_1 \cap L_2$ and $L_2 \cap L_3$ in the proof of the above theorem, we see that generators of $H^0(Y'_\eta, \omega_{Y'_\eta}^{\otimes m})$ for $m = 2, 3$ and their modulo 2 reductions have no zero along D_i's and L_i's, respectively.

4. The minimal resolution of $Y(\bar{\boldsymbol{Q}}_2)$. In this section, we denote by X the normal surface $Y(\bar{\boldsymbol{Q}}_2)$. By Theorem 2.9, X has cyclic quotient singularities p_0, p_1, p_2 and q. Let $\pi: \tilde{X} \to X$ be the minimal resolution of these singularities. Hence $\tilde{X} = Y'(\bar{\boldsymbol{Q}}_2)$ for Y' in Remark 2.10. We denote by $D_1, \cdots, D_9$ the irreducible divisors of $\tilde{X}$ such that $\pi^{-1}(q) = D_1 + D_2 + D_3$, $\pi^{-1}(p_0) = D_4 + D_5$, $\pi^{-1}(p_1) = D_6 + D_7$ and $\pi^{-1}(p_2) = D_8 + D_9$. We assume $D_1^2 = -3$ and $D_1 \cap D_3 = \emptyset$ as in Section 3. Hence we have $D_i^2 = -2$ for $2 \leqq i \leqq 9$. Let K_X be a canonical divisor of X. Since X has only cyclic quotient singularities, K_X is a $\boldsymbol{Q}$-Cartier divisor. In fact, $21K_X$ is a Cartier divisor.

PROPOSITION 4.1. *The Chern numbers of the nonsingular surface* $\tilde{X}$ *are* $c_1^2(\tilde{X}) = 0$ *and* $c_2(\tilde{X}) = 12$.

PROOF. Let $K_{\tilde{X}}$ be the canonical divisor of $\tilde{X}$ which is equal to K_X on $X \setminus \{p_0, p_1, p_2, q\}$. Then $\pi^* K_X - K_{\tilde{X}}$ is a $\boldsymbol{Q}$-divisor supported in $D_1 \cup \cdots \cup D_9$, i.e., $\pi^* K_X - K_{\tilde{X}} = a_1 D_1 + \cdots + a_9 D_9$ for some $a_1, \cdots, a_9 \in \boldsymbol{Q}$. Since D_i's are nonsingular rational curves, we have $(\pi^* K_X - K_{\tilde{X}}) \cdot D_i = -K_{\tilde{X}} \cdot D_i = 2 + D_i^2$ for every i.

Then we see easily that

$$\pi^* K_X - K_{\tilde{X}} = (3/7)D_1 + (2/7)D_2 + (1/7)D_3 .$$

In particular, we have

$$(1) \qquad K_X{}^2 - K_{\tilde{X}}{}^2 = (\pi^* K_X - K_{\tilde{X}}) \cdot K_{\tilde{X}} = 3/7 .$$

On the other hand, by Theorem 2.9, there exists a finite morphism $f: V(\bar{\boldsymbol{Q}}_2) \to X$ of degree 168 ramified only at $\{p_0, p_1, p_2, q\}$. Since $c_1^2(V(\bar{\boldsymbol{Q}}_2)) = 72$, we have

$$(2) \qquad K_X{}^2 = 72/168 = 3/7 .$$

Hence $c_1^2(\tilde{X}) = K_{\tilde{X}}{}^2 = 0$ by (1) and (2).

For $c_2(\tilde{X})$, we may let $\bar{\boldsymbol{Q}}_2 = \boldsymbol{C}$ and calculate it as the topological Euler number $e(\tilde{X})$. By Theorem 2.9, $f^{-1}(p_i)$ for $i = 0, 1, 2$ and $f^{-1}(q)$ consist of $168/3 = 56$ and $168/7 = 24$ points, respectively. Since $c_2(V(\bar{\boldsymbol{Q}}_2)) = 24$, we have

$$\begin{aligned} c_2(\tilde{X}) &= (c_2(V(\bar{\boldsymbol{Q}}_2)) - {}^\sharp f^{-1}(\{p_0, p_1, p_3, q\}))/168 + e(\pi^{-1}(\{p_0, p_1, p_2, q\})) \\ &= (24 - (3 \cdot 56 + 24))/168 + (3 \cdot 3 + 4) = 12 . \end{aligned}$$

q.e.d.

REMARK 4.2. The above proposition implies $\chi(\mathcal{O}_{\tilde{X}}) = 1$ by Noether's formula. In fact, we have $p_g(\tilde{X}) = q(\tilde{X}) = 0$, since X has a finite covering $M(\bar{\boldsymbol{Q}}_2) \to X$ from Mumford's fake projective plane $M(\bar{\boldsymbol{Q}}_2)$ ramified only at finite points.

PROPOSITION 4.3. *$\tilde{X}$ is a minimal elliptic surface, i.e., the Kodaira dimension of $\tilde{X}$ is equal to one.*

PROOF. Suppose $\tilde{X}$ were of general type, and let X' be its minimal model. By the plurigenus formula, we have $P_m(\tilde{X}) = (m(m-1)/2)K_{X'}^2 + \chi(\mathcal{O}_{\tilde{X}})$ for $m \geqq 2$. In particular $P_2(\tilde{X}) \geqq 2$. This contradicts Theorem 3.9.

If $\tilde{X}$ were of Kodaira dimension zero, then $\tilde{X}$ is either a $K3$ surface or an Enriques surface, since $q(\tilde{X}) = 0$. These are impossible since $p_g(\tilde{X}) = 0$ and $P_3(\tilde{X}) = 1$ by Theorem 3.9.

Hence $\tilde{X}$ is an elliptic surface and it is minimal by $K_{\tilde{X}}^2 = 0$. q.e.d.

Recall that the $\boldsymbol{Z}_2$-scheme Y' is regular outside the point w in the closed fiber. For each integer m, we denote by $\omega_{Y'}^{\otimes m}$ the maximal torsion-free extension of $\omega_{Y'}^{\otimes m} \setminus \{w\}$ to Y'. We fix sections F_2 and F_3 of $\omega_{Y'}^{\otimes 2}$ and $\omega_{Y'}^{\otimes 3}$ with non-trivial modulo 2 reductions, respectively, which exist by Theorem 3.9. Let E' and E'' be the effective divisors (F_2) and (F_3) of Y', respectively. Clearly, $3E'$ and $2E''$ are linearly equivalent.

LEMMA 4.4. *E' and E'' are disjoint.*

PROOF. Let $\pi_0: Y'_0 \to Y_0$ be the natural morphism. We denote by $\bar{E}'_0$ and $\bar{E}''_0$ the images by π_0 of the divisors $E'_0 = E' \cap Y'_0$ and $E''_0 = E'' \cap Y'_0$, respectively.

By the definition of E' and E'' and by Theorem 3.9, $\bar{E}'_0$ and $\bar{E}''_0$ correspond to the $\bar{\Gamma}_2$-invariant polynomials $f_2 = \phi_4^2 + \phi_3\phi_5$ and $f_3 = \phi_3^4 + \phi_3\phi_4\phi_5 + \phi_5\phi_7 + \phi_6^2$, respectively. Let $\tilde{E}'_0$ and $\tilde{E}''_0$ be the pull-backs of $\bar{E}'_0$ and $\bar{E}''_0$, respectively, by the natural surjective morphism $h: B \to Y_0$. By Tables 1 and 2, the restrictions of $\tilde{E}'_0$ and $\tilde{E}''_0$ to the rational curve $C(1, 0, 0) \subset B$ is defined by $X_1^2 + X_1X_2 + X_2^2$ and $X_1X_2(X_1 + X_2)$, respectively. In particular, they do not intersect each other on the curve. Since G acts transitively on the set of double curves of V_0, and since B is isomorphic to the component B_1 of V_0, $\tilde{E}'_0$ and $\tilde{E}''_0$ do not intersect each other on the fourteen rational curves in Figure 1 in Section 1. Since the complement of the union of the curves in B is an affine open set, $\tilde{E}'_0$ and $\tilde{E}''_0$ have no common components. E'_0 and E''_0 also have no common components, since they do not contain L_i for $i = 1, 2, 3$ by Remark 3.10, and since E'_0 does not have any zero on the other exceptional curves of π_0.

On the other hand, f_2 and f_3 have zeros of multiplicities 2 and 3, respectively, at the seven F_2-rational points of $P^2_{F_2}$. Since B is the blowing-up of $P^2_{F_2}$ at the seven F_2-rational points, the intersection number $\tilde{E}'_0 \cdot \tilde{E}''_0$ is $\deg f_2 \cdot \deg f_3 - 7 \cdot 2 \cdot 3 = 96 - 42 = 54$. Since $Y_0 \setminus h(C(1, 0, 0))$ is smooth except at the cyclic quotient singularity $\bar{q}(0)$, we can consider the intersection number $\bar{E}'_0 \cdot \bar{E}''_0 = 54/21 = 18/7$, since h is of degree 21. As in the proof of Proposition 4.1, we have

$$\pi_0^* \bar{E}'_0 - E'_0 = 2((3/7)L_1 + (2/7)L_2 + (1/7)L_3)\,,$$
$$\pi_0^* \bar{E}''_0 - E''_0 = 3((3/7)L_1 + (2/7)L_2 + (1/7)L_3) \quad \text{and}$$
$$\bar{E}'_0 \cdot \bar{E}''_0 - E'_0 \cdot E''_0 = 2 \cdot 3 \cdot 3/7 = 18/7.$$

Hence $E'_0 \cdot E''_0 = 0$. We have $E'_0 \cap E''_0 = \emptyset$, since they have no common components. This implies $E' \cap E'' = \emptyset$. q.e.d.

Let $\kappa: Y' \to P^1_{Z_2}$ be the morphism defined by (F_2^3, F_3^2).

PROPOSITION 4.5. *The induced morphism $\kappa^1_{\bar{Q}_2}: \tilde{X} \to P^1_{\bar{Q}_2}$ of the geometric fibers is the elliptic fibration of $\tilde{X}$. It has just two multiple fibers $3E'_{\bar{Q}_2}$ and $2E''_{\bar{Q}_2}$, where $E'_{\bar{Q}_2}$ and $E''_{\bar{Q}_2}$ are the restrictions of E' and E'' to $\tilde{X}$, respectively.*

PROOF. Let $f: \tilde{X} \to P^1_{\bar{Q}_2}$ be the elliptic fibration, and let $m_1 C_1, \cdots, m_n C_n$ be its multiple fibers. By Kodaira's canonical bundle formula [Ko2, Th. 12], we have

$$K_{\tilde{X}} \sim f^{-1}(-x_0) + \sum_{i=1}^{n} (m_i - 1)C_i\,,$$

where x_0 is a point of $P^1_{\bar{Q}_2}$, since $\deg K_{P^1} + \chi(\mathcal{O}_{\tilde{X}}) = -1$. Since $2K_{\tilde{X}} \sim (n-2)f^{-1}(x_0) + \sum_{i=1}^{n} (m_i - 2)C_i$, we have $\dim |2K_{\tilde{X}}| = n - 2$. Hence $n = 2$ by Theorem 3.9. Since $E'_{\bar{Q}_2}$ is a unique effective bicanonical divisor, we have $E'_{\bar{Q}_2} = (m_1 - 2)C_1 + (m_2 - 2)C_2$. If $m_1, m_2 \geqq 3$, $3K_{\tilde{X}} \sim f^{-1}(x_0) + (m_1 - 3)C_1 + (m_2 - 3)C_2$ and hence $\dim |3K_{\tilde{X}}| = 1$. This contradicts Theorem 3.9. Hence we may assume $m_1 = 2$. Since $(m_2 - 2)C_2 = E'_{\bar{Q}_2}$, we have $m_2 > 2$. Hence $3K_{\tilde{X}} \sim E''_{\bar{Q}_2} = C_1 + (m_2 - 3)C_2$. Since $E'_{\bar{Q}_2} \cap E''_{\bar{Q}_2} = \emptyset$ by Lemma 4.4, we have $m_2 = 3$.

Thus we have $E'_{\bar{Q}_2} = C_2$, $E''_{\bar{Q}_2} = C_1$ and $f^{-1}(x_0) \sim 3E'_{\bar{Q}_2} \sim 2E''_{\bar{Q}_2}$. Hence f is equal to $\kappa_{\bar{Q}_2}$ up to automorphism of $P^1_{\bar{Q}_2}$. q.e.d.

The connected curves $D_2 \cup D_3$, $D_4 \cup D_5$, $D_6 \cup D_7$ and $D_8 \cup D_9$ are unions of (-2)-curves. Hence they are mapped to points in $P^1_{\bar{Q}_2}$ by $\kappa_{\bar{Q}_2}$. We denote $y = \kappa_{\bar{Q}_2}(D_2 \cup D_3)$ and $z_i = \kappa_{\bar{Q}_2}(D_{4+2i} \cup D_{5+2i})$ for $i = 0, 1, 2$.

PROPOSITION 4.6. *$E'_{\bar{Q}_2}$, $E''_{\bar{Q}_2}$, $D_2 \cup D_3$, $D_4 \cup D_5$, $D_6 \cup D_7$ and $D_8 \cup D_9$ are mapped to distinct points in $P^1_{\bar{Q}_2}$ by $\kappa_{\bar{Q}_2}$.*

PROOF. By definition, $\kappa_{\bar{Q}_2}(E'_{\bar{Q}_2}) = (0:1)$ and $\kappa_{\bar{Q}_2}(E''_{\bar{Q}_2}) = (1:0)$. By Remark 3.10, the modulo 2 reduction $L_2 \cup L_3$ of $D_2 \cup D_3$ is contained in neither E'_0 nor E''_0. Hence the specialization of y in $\boldsymbol{P}^1_{F_2}$ is neither (1: 0) nor (0: 1). As we saw immediately before Theorem 2.9, there exists a $\boldsymbol{Z}_2$-morphism $\operatorname{Spec} \boldsymbol{Z}_2[\varepsilon] \to \mathscr{X}$ which is fixed by $\tau\rho$, and the induced $\boldsymbol{Q}_2[\varepsilon]$-valued point in Y splits to p_0, p_1, p_2 in $Y(\bar{\boldsymbol{Q}}_2)$ and the image of the closed point is the triple point P of $\mathscr{X}_0$. As we saw in the proof of Lemma 4.4, the pull-back of $\bar{E}'$ and $\bar{E}''$ to $C(1, 0, 0)$ is defined by $X_1^2 + X_1X_2 + X_2^2$ and $X_1X_2(X_1 + X_2)$, respectively. Hence we have $\bar{P} \in \bar{E}''$ and $\bar{P} \notin \bar{E}'$, where $\bar{P}$ is the image of P in Y. Since $D_4 \cup D_5$, $D_6 \cup D_7$ and $D_8 \cup D_9$ are the exceptional curves of p_0, p_1 and p_2, respectively, the specialization of z_i's are all (1: 0). We get the following diagram after the base extension in Remark 2.10:

$$\begin{array}{ccccccc}
\operatorname{Spec} K[\varepsilon] & \hookrightarrow & \operatorname{Spec} R[\varepsilon] & \to & V'_R & \hookrightarrow & V_R \\
\Big\downarrow{\scriptstyle \mu} & & & & \downarrow & & \downarrow \\
 & & & & Y_R \setminus \bar{E}'_R & \hookrightarrow & Y_R \leftarrow Y'_R \\
 & & & & \downarrow{\scriptstyle F_3^2/F_2^3} & & \downarrow{\scriptstyle \kappa_R} \\
\operatorname{Spec} K[t] & \hookrightarrow & \operatorname{Spec} R[t] & = & \boldsymbol{A}^1_R & \hookrightarrow & \boldsymbol{P}^1_R .
\end{array}$$

Here K is the quotient field of $R = \boldsymbol{Z}_2[\zeta, \omega]$, V'_R a neighborhood of $P_1 \in V_R$, $\bar{E}'_R$ the image of E'_R in Y_R and $\boldsymbol{A}^1_R = \boldsymbol{P}^1_R \setminus \kappa_R(E'_R)$. It suffices to show that the K-homomorphism $\mu^*: K[t] \to K[\varepsilon]$ is surjective, since then the image of μ is a separable point of degree 3 while (1: 0) is the K-rational point $t = 0$. By the notation in Remark 2.10, we get the following sequence of formal completions of local rings:

$$R[\![t]\!] \to R[\![T_0, T_1^3, T_2^3, T_1T_2]\!] \to R[\![T_0, T_1, T_2]\!] \xrightarrow{l} R[\varepsilon] ,$$

where T_0, T_1, T_2 have a relation $T_0^3 + T_1^3 + T_2^3 - 3T_0T_1T_2 = 27\varepsilon^3$. l is given by $l(T_0) = 3\varepsilon$ and $l(T_1) = l(T_2) = 0$. The image of t in $R[\![T_0, T_1, T_2]\!]$ is equal to F_3^2/F_2^3. Since Y is a Gorenstein scheme and since F_2 and F_3 are sections of $\omega_Y^{\otimes m}$ for $m = 2, 3$, respectively, we may regard F_2 and F_3 as elements of $R[\![T_0, T_1^3, T_2^3, T_1T_2]\!]$. By the restriction of the polynomials f_2 and f_3 to $C(1, 0, 0) \subset B$, we see that $F_3 \in (T_0, T_1, T_2) \setminus (T_0, T_1, T_2)^2$ and F_2 is a unit. Hence F_3 has a unit coefficient for T_0, and hence F_3^2/F_2^3 has a unit coefficient for T_0^2. This implies that the image of t in $R[\varepsilon]$ is outside R. Hence μ^* is surjective. q.e.d.

Now we can determine the types of the singular fibers:

THEOREM 4.7. *The elliptic fibration* $\kappa_{\bar{Q}_2}: \tilde{X} \to \boldsymbol{P}^1{}_{\bar{Q}_2}$ *has singular fibers at* $\{(1:0), (0:1), y, z_0, z_1, z_2\} \subset \boldsymbol{P}^1{}_{\bar{Q}_2}$ *and smooth elsewhere. The singular* *fibers over* z_0, z_1, z_2 *and* y *are not multiple and are of type* I_3 *in the notation of* [Ko1, Th. 6.2]. *The fibers over* (1: 0) *and* (0: 1) *are* $2E''_{\bar{Q}_2}$ *and* $3E'_{\bar{Q}_2}$, *respectively, where* $E''_{\bar{Q}_2}$ *and* $E'_{\bar{Q}_2}$ *are smooth elliptic curves.*

PROOF. Each of the fibers over z_0, z_1, z_2 and y contains a union of two (-2)-curves intersecting each other transversally at one point. Hence they are not of type II nor III. Hence the Euler number of the non-elliptic fiber is at least three and is equal to three if and only if it is of type I_3. Now we apply Kodaira's formula for the second Betti number of an elliptic surface [Ko1, Th. 12.2]. Since $c_2(\tilde{X}) = 12$ by Proposition 4.1, all these fibers are of type I_3 and the other fibers are elliptic curves. The multiple fibers are only $2E''_{\bar{Q}_2}$ and $3E'_{\bar{Q}_2}$ by Proposition 4.5. q.e.d.

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