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AN EMBEDDING-BASED COMPLETENESS PROOF FOR NELSON'S PARACONSISTENT LOGIC

Abstract

It is known that a syntactical embedding theorem of Nelson's paraconsistent logic N4 into the positive intuitionistic logic LJ is useful to show the cut-elimination and decidability theorems for N4. In this paper, a semantical embedding theorem of N4 into LJ is shown. An alternative proof of the Kripke-completeness theorem for N4 is obtained by combining both the syntactical and semantical embedding theorems. Thus, the completeness, cut-elimination and decidability theorems can uniformly be obtained from these embedding theorems. A single-consequence Kripke semantics for N4 is also addressed based on a modification of the semantical embedding theorem.

1. Syntactical embedding: A review

Nelson's paraconsistent (four-valued) logic N4 (or equivalently N$^-$) [12], which is a paraconsistent variant of *Nelson's constructive (three-valued) logic* N3 (or equivalently N) [6], has been studied by many researchers (see e.g. [2,13] for detailed information on Nelson's logics and their variations). It was shown by Odintsov [7] that N3 is embeddable into N4. Cut-free Gentzen-type sequent calculi for Nelson's logics have been studied (e.g. [8,13,4,5]), and Kripke semantics for Nelson's logics have also been studied (e.g. [10,11,13]). A translation function from N3 into intuitionistic logic has been proposed and studied by Gurevich [3], Rautenberg [9] and Vorob'ev [12]. A similar translation function from N4 into positive intuitionistic logic (called here LJ) can also be obtained. By using such a translation function,

a *syntactical embedding theorem* of N4 into LJ and the cut-elimination and decidability theorems for N4 can easily be obtained (see e.g. [4,5,14] for some related works).

In this section, the standard sequent calculi for the underlying logics are presented, and the syntactical embedding theorem and its consequences (i.e. cut-elimination and decidability) are reviewed. In the next section, the standard Kripke semantics for the underlying logics are presented, and a *semantical embedding theorem*, which is a new result of this paper, is shown. An alternative embedding-based proof of the Kripke-completeness theorem for N4 is obtained by combining both the syntactical and semantical embedding theorems. Thus, the completeness, cut-elimination and decidability theorems can uniformly be obtained from these embedding theorems. In the final section, a new single-consequence Kripke semantics for N4 is introduced, and the equivalence between such a semantics and the standard dual-consequence semantics is proved by modifying the proof of the semantical embedding theorem.

Prior to the detailed discussion, the language used in this paper is introduced below. The usual propositional language with the strong negation connective $\sim$ and without falsity and truth constants is used in this paper. Lower-case letters $p, q, r, ...$ are used to denote propositional variables, Greek lower-case letters $\alpha, \beta, \gamma, ...$ are used to denote formulas, and Greek capital letters $\Gamma, \Delta, ...$ are used to represent finite (possibly empty) sets of formulas. A *sequent* is an expression of the form $\Gamma \Rightarrow \gamma$. If a sequent S is provable in a system L, then such a fact is denoted as $L \vdash S$.

DEFINITION 1 (LJ). *The initial sequents of* LJ *are of the form:* $p \Rightarrow p$ *for any propositional variable p.*

The structural inference rules of LJ *are of the form:*

$$\frac{\Gamma \Rightarrow \alpha \quad \alpha, \Sigma \Rightarrow \gamma}{\Gamma, \Sigma \Rightarrow \gamma} \ (\text{cut}) \quad \frac{\Gamma \Rightarrow \gamma}{\alpha, \Gamma \Rightarrow \gamma} \ (\text{w}-\text{l}).$$

The logical inference rules of LJ *are of the form:*

$$\frac{\Gamma \Rightarrow \alpha \quad \beta, \Delta \Rightarrow \gamma}{\alpha \rightarrow \beta, \Gamma, \Delta \Rightarrow \gamma} \ (\rightarrow\text{l}) \quad \frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \rightarrow \beta} \ (\rightarrow\text{r})$$

$$\frac{\alpha, \beta, \Gamma \Rightarrow \gamma}{\alpha \wedge \beta, \Gamma \Rightarrow \gamma} \ (\wedge\text{l}) \quad \frac{\Gamma \Rightarrow \alpha \quad \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \wedge \beta} \ (\wedge\text{r})$$

$$\frac{\alpha, \Gamma \Rightarrow \gamma \quad \beta, \Gamma \Rightarrow \gamma}{\alpha \vee \beta, \Gamma \Rightarrow \gamma} \; (\vee l) \quad \frac{\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \alpha \vee \beta} \; (\vee r1) \quad \frac{\Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \vee \beta} \; (\vee r2).$$

DEFINITION 2 (N4). N4 *is obtained from* LJ *by adding the initial sequents of the form:* $\sim p \Rightarrow \sim p$ *for any propositional variable* p, *and the logical inference rules of the form:*

$$\frac{\alpha, \Gamma \Rightarrow \gamma}{\sim\sim\alpha, \Gamma \Rightarrow \gamma} \; (\sim\sim l) \quad \frac{\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \sim\sim\alpha} \; (\sim\sim r)$$

$$\frac{\alpha, \sim\beta, \Gamma \Rightarrow \gamma}{\sim(\alpha \rightarrow \beta), \Gamma \Rightarrow \gamma} \; (\sim\rightarrow l) \quad \frac{\Gamma \Rightarrow \alpha \quad \Gamma \Rightarrow \sim\beta}{\Gamma \Rightarrow \sim(\alpha \rightarrow \beta)} \; (\sim\rightarrow r)$$

$$\frac{\sim\alpha, \Gamma \Rightarrow \gamma \quad \sim\beta, \Gamma \Rightarrow \gamma}{\sim(\alpha \wedge \beta), \Gamma \Rightarrow \gamma} \; (\sim \wedge l)$$

$$\frac{\Gamma \Rightarrow \sim\alpha}{\Gamma \Rightarrow \sim(\alpha \wedge \beta)} \; (\sim \wedge r1) \quad \frac{\Gamma \Rightarrow \sim\beta}{\Gamma \Rightarrow \sim(\alpha \wedge \beta)} \; (\sim \wedge r2)$$

$$\frac{\sim\alpha, \sim\beta, \Gamma \Rightarrow \gamma}{\sim(\alpha \vee \beta), \Gamma \Rightarrow \gamma} \; (\sim \vee l) \quad \frac{\Gamma \Rightarrow \sim\alpha \quad \Gamma \Rightarrow \sim\beta}{\Gamma \Rightarrow \sim(\alpha \vee \beta)} \; (\sim \vee r).$$

The sequents of the form $\alpha \Rightarrow \alpha$ for any formula α are provable in LJ and N4.

DEFINITION 3. *We fix a set* Φ *of propositional variables and define the set* $\Phi' := \{p' \mid p \in \Phi\}$ *of propositional variables. The language* $\mathcal{L}_{N4}$ *of N4 is defined using* $\Phi, \rightarrow, \wedge, \vee$ *and* $\sim$. *The language* $\mathcal{L}_{LJ}$ *of LJ is obtained from* $\mathcal{L}_{N4}$ *by adding* Φ' *and deleting* $\sim$.

A mapping f *from* $\mathcal{L}_{N4}$ *to* $\mathcal{L}_{LJ}$ *is inductively defined by:*

1. *for any* $p \in \Phi$, $f(p) := p$ *and* $f(\sim p) := p' \in \Phi'$,
2. $f(\alpha \circ \beta) := f(\alpha) \circ f(\beta)$ *where* $\circ \in \{\rightarrow, \wedge, \vee\}$,
3. $f(\sim\sim\alpha) := f(\alpha)$,
4. $f(\sim(\alpha \rightarrow \beta)) := f(\alpha) \wedge f(\sim\beta)$,
5. $f(\sim(\alpha \wedge \beta)) := f(\sim\alpha) \vee f(\sim\beta)$,
6. $f(\sim(\alpha \vee \beta)) := f(\sim\alpha) \wedge f(\sim\beta)$.

An expression $f(\Gamma)$ denotes the result of replacing every occurrence of a formula α in Γ by an occurrence of $f(\alpha)$.

THEOREM 4 (Syntactical embedding). *Let Γ be a set of formulas in $\mathcal{L}_{N4}$, γ be a formula in $\mathcal{L}_{N4}$, and f be the mapping defined in Definition 3. Then:*

1. *N4 $\vdash \Gamma \Rightarrow \gamma$ iff LJ $\vdash f(\Gamma) \Rightarrow f(\gamma)$,*
2. *N4 $-$ (cut) $\vdash \Gamma \Rightarrow \gamma$ iff LJ $-$ (cut) $\vdash f(\Gamma) \Rightarrow f(\gamma)$.*

COROLLARY 5 (Cut-elimination). *The rule* (cut) *is admissible in cut-free N4.*

PROOF. Suppose N4 $\vdash \Gamma \Rightarrow \gamma$. Then, we have LJ $\vdash f(\Gamma) \Rightarrow f(\gamma)$ by Theorem 4 (1), and hence LJ $-$ (cut) $\vdash f(\Gamma) \Rightarrow f(\gamma)$ by the cut-elimination theorem for LJ. By Theorem 4 (2), we obtain the required fact N4 $-$ (cut) $\vdash \Gamma \Rightarrow \gamma$. Q.E.D.

COROLLARY 6 (Decidability). *N4 is decidable.*

PROOF. By decidability of LJ, for each α, it is possible to decide if $f(\alpha)$ is provable in LJ. Then, by Theorem 4, N4 is decidable. Q.E.D.

2.   Semantical embedding

DEFINITION 7. *A Kripke frame is a structure $\langle M, R \rangle$ satisfying the following conditions:*

1. *M is a nonempty set,*
2. *R is a reflexive and transitive binary relation on M.*

DEFINITION 8. *A valuation $\models$ on a Kripke frame $\langle M, R \rangle$ is a mapping from the set Φ of propositional variables to the power set 2^M of M such that for any $p \in \Phi$ and any $x, y \in M$, if $x \in \models (p)$ and xRy, then $y \in \models (p)$. We will write $x \models p$ for $x \in \models (p)$. This valuation $\models$ is extended to a mapping from the set of all formulas to 2^M by:*

1. *$x \models \alpha \rightarrow \beta$ iff $\forall y \in M$ [xRy and $y \models \alpha$ imply $y \models \beta$],*
2. *$x \models \alpha \wedge \beta$ iff $x \models \alpha$ and $x \models \beta$,*
3. *$x \models \alpha \vee \beta$ iff $x \models \alpha$ or $x \models \beta$.*

The following *hereditary condition* holds for $\models$: for any formula α and any $x, y \in M$, if $x \models \alpha$ and xRy, then $y \models \alpha$.

DEFINITION 9. *A Kripke model is a structure $\langle M, R, \models \rangle$ such that*

1. *$\langle M, R \rangle$ is a Kripke frame,*
2. *$\models$ is a valuation on $\langle M, R \rangle$.*

A formula α is true in a Kripke model $\langle M, R, \models \rangle$ if $x \models \alpha$ for any $x \in M$, and is LJ-valid in a Kripke frame $\langle M, R \rangle$ if it is true for every valuation $\models$ on the Kripke frame.

DEFINITION 10. *Paraconsistent valuations $\models^+$ and $\models^-$ on a Kripke frame $\langle M, R \rangle$ are mappings from the set Φ of propositional variables to the power set 2^M of M such that for any $\star \in \{+, -\}$, any $p \in \Phi$ and any $x, y \in M$, if $x \in \models^\star (p)$ and xRy, then $y \in \models^\star (p)$. We will write $x \models^\star p$ for $x \in \models^\star (p)$. These paraconsistent valuations $\models^+$ and $\models^-$ are extended to mappings from the set of all formulas to 2^M by:*

1. *$x \models^+ \alpha {\rightarrow} \beta$ iff $\forall y \in M$ [xRy and $y \models^+ \alpha$ imply $y \models^+ \beta$],*
2. *$x \models^+ \alpha \wedge \beta$ iff $x \models^+ \alpha$ and $x \models^+ \beta$,*
3. *$x \models^+ \alpha \vee \beta$ iff $x \models^+ \alpha$ or $x \models^+ \beta$,*
4. *$x \models^+ {\sim}\alpha$ iff $x \models^- \alpha$,*
5. *$x \models^- {\sim}\alpha$ iff $x \models^+ \alpha$,*
6. *$x \models^- \alpha {\rightarrow} \beta$ iff $x \models^+ \alpha$ and $x \models^- \beta$,*
7. *$x \models^- \alpha \wedge \beta$ iff $x \models^- \alpha$ or $x \models^- \beta$,*
8. *$x \models^- \alpha \vee \beta$ iff $x \models^- \alpha$ and $x \models^- \beta$.*

   The hereditary condition holds for $\models^+$ and $\models^-$.

DEFINITION 11. *A paraconsistent Kripke model is a structure $\langle M, R, \models^+, \models^- \rangle$ such that*

1. *$\langle M, R \rangle$ is a Kripke frame,*
2. *$\models^+$ and $\models^-$ are paraconsistent valuations on $\langle M, R \rangle$.*

A formula α is true in a paraconsistent Kripke model $\langle M, R, \models^+, \models^- \rangle$ if $x \models^+ \alpha$ for any $x \in M$, and is N4-valid in a Kripke frame $\langle M, R \rangle$ if it is true for every paraconsistent valuations $\models^+$ and $\models^-$ on the Kripke frame.

LEMMA 12. *Let f be the mapping defined in Definition 3. For any paraconsistent Kripke model $\langle M, R, \models^+, \models^- \rangle$, there exists a Kripke model $\langle M, R, \models \rangle$ such that for any formula α and any $x \in M$,*

1. $x \models^+ \alpha$ *iff* $x \models f(\alpha)$,
2. $x \models^- \alpha$ *iff* $x \models f(\sim\alpha)$.

PROOF. Let Φ be a set of propositional variables and Φ' be the set $\{p' \mid p \in \Phi\}$ of propositional variables. Suppose that $\langle M, R, \models^+, \models^- \rangle$ is a paraconsistent Kripke model where $\models^+$ and $\models^-$ are mappings from Φ to the power set 2^M of M, and that the hereditary condition w.r.t. $p \in \Phi$ holds for $\models^+$ and $\models^-$. Suppose that $\langle M, R, \models \rangle$ is a Kripke model where $\models$ is a mapping from $\Phi \cup \Phi'$ to 2^M, and that the hereditary condition w.r.t. $p \in \Phi \cup \Phi'$ holds for $\models$. Suppose moreover that these models satisfy the following conditions: for any $x \in M$ and any $p \in \Phi$,

1. $x \models^+ p$ iff $x \models p$,
2. $x \models^- p$ iff $x \models p'$.

Then, the lemma is proved by (simultaneous) induction on the complexity of α.

• Base step:

Case $\alpha \equiv p \in \Phi$: For (1), we obtain: $x \models^+ p$ iff $x \models p$ iff $x \models f(p)$ (by the definition of f). For (2), we obtain: $x \models^- p$ iff $x \models p'$ iff $x \models f(\sim p)$ (by the definition of f).

• Induction step:

Case $\alpha \equiv \beta \wedge \gamma$: For (1), we obtain: $x \models^+ \beta \wedge \gamma$ iff $x \models^+ \beta$ and $x \models^+ \gamma$ iff $x \models f(\beta)$ and $x \models f(\gamma)$ (by induction hypothesis for 1) iff $x \models f(\beta) \wedge f(\gamma)$ iff $x \models f(\beta \wedge \gamma)$ (by the definition of f). For (2), we obtain: $x \models^- \beta \wedge \gamma$ iff $x \models^- \beta$ or $x \models^- \gamma$ iff $x \models f(\sim\beta)$ or $x \models f(\sim\gamma)$ (by induction hypothesis for 2) iff $x \models f(\sim\beta) \vee f(\sim\gamma)$ iff $x \models f(\sim(\beta \wedge \gamma))$ (by the definition of f).

Case $\alpha \equiv \beta \vee \gamma$: Similar to the above case.

Case $\alpha \equiv \beta \rightarrow \gamma$: For (1), we obtain: $x \models^+ \beta \rightarrow \gamma$ iff $\forall y \in M[xRy$ and $y \models^+ \beta$ imply $y \models^+ \gamma]$ iff $\forall y \in M[xRy$ and $y \models f(\beta)$ imply $y \models f(\gamma)]$ (by induction hypothesis for 1) iff $x \models f(\beta) \rightarrow f(\gamma)$ iff $x \models f(\beta \rightarrow \gamma)$ (by the definition of f). For (2), we obtain: $x \models^- \beta \rightarrow \gamma$ iff $x \models^+ \beta$ and $x \models^- \gamma$ iff $x \models f(\beta)$ and $x \models f(\sim\gamma)$ (by induction hypothesis for 1 and 2) iff $x \models f(\beta) \wedge f(\sim\gamma)$ iff $x \models f(\sim(\beta \rightarrow \gamma))$ (by the definition of f).

Case $\alpha \equiv \sim\beta$: For (1), we obtain: $x \models^+ \sim\beta$ iff $x \models^- \beta$ iff $x \models f(\sim\beta)$ (by induction hypothesis for 2). For (2), we obtain: $x \models^- \sim\beta$ iff $x \models^+ \beta$ iff $x \models f(\beta)$ (by induction hypothesis for 1) iff $x \models f(\sim\sim\beta)$ (by the definition of f). Q.E.D.

LEMMA 13. *Let f be the mapping defined in Definition 3. For any Kripke model $\langle M, R, \models \rangle$, there exists a paraconsistent Kripke model $\langle M, R, \models^+, \models^- \rangle$ such that for any formula α and any $x \in M$,*

1. *$x \models f(\alpha)$ iff $x \models^+ \alpha$,*
2. *$x \models f(\sim\alpha)$ iff $x \models^- \alpha$.*

PROOF. Similar to the proof of Lemma 12. Q.E.D.

THEOREM 14 (Semantical embedding). *Let f be the mapping defined in Definition 3. For any formula α, α is N4-valid iff $f(\alpha)$ is LJ-valid.*

PROOF. By Lemmas 12 and 13. Q.E.D.

THEOREM 15 (Completeness). *For any formula α, N4 $\vdash \Rightarrow \alpha$ iff α is N4-valid.*

PROOF. N4 $\vdash \Rightarrow \alpha$ iff LJ $\vdash \Rightarrow f(\alpha)$ (by Theorem 4) iff $f(\alpha)$ is LJ-valid (by the Kripke completeness theorem for LJ) iff α is N4-valid (by Theorem 14). Q.E.D.

3. Single-consequence semantics

Some modifications of Lemmas 12 and 13 give an alternative "single-consequence" semantics for N4, which has a single valuation $\models^*$ instead of the dual valuations $\models^+$ and $\models^-$.

DEFINITION 16. *Let Φ be the set of propositional variables and $\Phi^\sim$ be the set $\{\sim p \mid p \in \Phi\}$. A single paraconsistent valuation $\models^*$ on a Kripke frame $\langle M, R \rangle$ is a mapping from $\Phi \cup \Phi^\sim$ to 2^M such that for any $p \in \Phi \cup \Phi^\sim$ and any $x, y \in M$, if $x \in \models^* (p)$ and xRy, then $y \in \models^* (p)$. We will write $x \models^* p$ for $x \in \models^* (p)$. The single paraconsistent valuation $\models^*$ is extended to a mapping from the set of all formulas to 2^M by:*

1. *$x \models^* \alpha \rightarrow \beta$ iff $\forall y \in M$ [xRy and $y \models^* \alpha$ imply $y \models^* \beta$],*
2. *$x \models^* \alpha \wedge \beta$ iff $x \models^* \alpha$ and $x \models^* \beta$,*
3. *$x \models^* \alpha \vee \beta$ iff $x \models^* \alpha$ or $x \models^* \beta$,*
4. *$x \models^* \sim\sim\alpha$ iff $x \models^* \alpha$,*
5. *$x \models^* \sim(\alpha \rightarrow \beta)$ iff $x \models^* \alpha$ and $x \models^* \sim\beta$,*

6. $x \models^* \sim(\alpha \wedge \beta)$ *iff* $x \models^* \sim\alpha$ *or* $x \models^* \sim\beta$,
7. $x \models^* \sim(\alpha \vee \beta)$ *iff* $x \models^* \sim\alpha$ *and* $x \models^* \sim\beta$.

The hereditary condition holds for $\models^*$.

DEFINITION 17. *A single paraconsistent Kripke model is a structure* $\langle M, R, \models^* \rangle$ *such that*

1. $\langle M, R \rangle$ *is a Kripke frame,*
2. $\models^*$ *is a single paraconsistent valuation on* $\langle M, R \rangle$.

A formula α *is true in a single paraconsistent Kripke model* $\langle M, R, \models^* \rangle$ *if* $x \models^* \alpha$ *for any* $x \in M$, *and is S-valid in a Kripke frame* $\langle M, R \rangle$ *if it is true for every single paraconsistent valuation* $\models^*$ *on the Kripke frame.*

LEMMA 18. *For any paraconsistent Kripke model* $\langle M, R, \models^+, \models^- \rangle$, *there exists a single paraconsistent Kripke model* $\langle M, R, \models^* \rangle$ *such that for any formula* α *and any* $x \in M$,

1. $x \models^+ \alpha$ *iff* $x \models^* \alpha$,
2. $x \models^- \alpha$ *iff* $x \models^* \sim\alpha$.

PROOF. Let Φ be a set of propositional variables and $\Phi^{\sim}$ be the set $\{\sim p \mid p \in \Phi\}$. Suppose that $\langle M, R, \models^+, \models^- \rangle$ is a paraconsistent Kripke model where $\models^+$ and $\models^-$ are mappings from Φ to the power set 2^M of M, and that the hereditary condition w.r.t. $p \in \Phi$ holds for $\models^+$ and $\models^-$. Suppose that $\langle M, R, \models^* \rangle$ is a single paraconsistent Kripke model where $\models^*$ is a mapping from $\Phi \cup \Phi^{\sim}$ to 2^M, and that the hereditary condition w.r.t. $p \in \Phi \cup \Phi^{\sim}$ holds for $\models^*$. Suppose moreover that these models satisfy the following conditions: for any $x \in M$ and any $p \in \Phi$,

1. $x \models^+ p$ iff $x \models^* p$,
2. $x \models^- p$ iff $x \models^* \sim p$.

Then, the lemma is proved by (simultaneous) induction on the complexity of α. The proof is similar to the proof of Lemma 12. Q.E.D.

LEMMA 19. *For any single paraconsistent Kripke model* $\langle M, R, \models^* \rangle$, *there exists a paraconsistent Kripke model* $\langle M, R, \models^+, \models^- \rangle$ *such that for any formula* α *and any* $x \in M$,

1. $x \models^* \alpha$ *iff* $x \models^+ \alpha$,
2. $x \models^* \sim\alpha$ *iff* $x \models^- \alpha$.

PROOF. Similar to the proof of Lemma 18. Q.E.D.

THEOREM 20 (Equivalence). *For any formula α, α is N4-valid iff α is S-valid.*

PROOF. By Lemmas 18 and 19. Q.E.D.

THEOREM 21 (Completeness). *For any formula α, N4 $\vdash \Rightarrow \alpha$ iff α is S-valid.*

PROOF. By Theorems 20 and 15. Q.E.D.

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