BULLETIN DE LA S. M. F.

NGAIMING MOK

An embedding theorem of complete Kähler manifolds of positive bisectional curvature onto affine algebraic varieties

Bulletin de la S. M. F., tome 112 (1984), p. 197-258
<http://www.numdam.org/item?id=BSMF_1984__112__197_0>

Bull. Soc. math. France,
112, 1984, p. 197-258.

AN EMBEDDING THEOREM OF COMPLETE KÄHLER MANIFOLDS OF POSITIVE BISECTIONAL CURVATURE ONTO AFFINE ALGEBRAIC VARIETIES

BY

NGAIMING MOK (*)

RÉSUMÉ. — Nous prouvons qu'une variété complète kählérienne non compacte X de courbure bisectionnelle positive satisfaisant quelques conditions quantitatives géométriques est biholomorphiquement isomorphe a une variété affine algébrique. Si X est une surface complexe de courbure riemannienne positive satisfaisant les mêmes conditions quantitatives, nous démontrons que X est en fait biholomorphiquement isomorphe a C^2.

ABSTRACT. — We prove that a complete noncompact Kähler manifold X of positive bisectional curvature satisfying suitable growth conditions can be biholomorphically embedded onto an affine algebraic variety. In case X is a complex surface of positive Riemannian sectional curvature satisfying the same growth conditions, we show that X is biholomorphic to C^2.

The following conjectures concerning the complex structure of non-compact complete Kähler manifolds of positive curvature, formulated by GREENE and WU [9], SIU [22] and WU [32] are central to the study of such manifolds.

Conjecture I

A non-compact complete Kähler manifold of positive sectional curvature is biholomorphic to $\mathbb{C}^n$.

(*) Texte reçu le 4 juin 1983, révisé le 24 février 1984.

N. MOK, Princeton University, Department of Mathematics, Fine Hall, Box 37, Princeton, N.Y. 08544 U.S.A.

Conjecture II

A non-compact complete Kähler manifold of positive bisectional curvature is a Stein manifold.

The geometric basis of Conjecture I is the following structure theorem on complete Riemannian manifolds of positive sectional curvature.

THEOREM (GROMOLL-MEYER [30], CHEEGER-GROMOLL [6] and POOR [31]). — *A non-compact complete Riemannian manifold of positive sectional curvature is diffeomorphic to* $\mathbb{R}^n$.

By using the above theorem GREENE and WU [10] proved that a non-compact complete Kähler manifold of positive sectional curvature is a Stein manifold. Nonetheless, in the case of positive bisectional curvature, the Busemann functions of CHEEGER-GROMOLL [5] do not immediately give rise to an exhaustion function because one does not have a geometric comparison theorem for geodesic distances as in the case of positive sectional curvature (the theorem of Toponogov). This consideration motivated Conjecture II.

The analogue of Conjecture I for negative (or non-positive) sectional curvature and for manifolds with a pole have been formulated and proved (SIU-YAU [22], GREENE-WU [10] and MOK-SIU-YAU [17]). There it was necessary to assume conditions on the decay of the curvature tensor to make sure first of all that the manifold is parabolic.

From standard examples of complete Kähler metrics of positive bisectional curvature on $\mathbb{C}^n$ it appears also appropriate to assume certain geometric growth conditions on the curvature tensor and the volume of geodesic balls. With Conjectures I and II in mind, we studied the Poincaré-Lelong equation on complete Kähler manifolds (MOK-SIU-YAU [17]). We obtained, among other things, the following pinching theorem on complete Kähler manifolds of nonnegative bisectional curvature.

THEOREM (MOK-SIU-YAU [17]). — *Suppose X is a complete Kähler manifold of complex dimension $n \geqslant 2$. Suppose X is a Stein manifold and the holomorphic bisecture curvature is non-negative. Moreover, assume*

(i) *Volume* $(B(x_0; r)) \geqslant cr^{2n}$.

(ii) $0 \leqslant$ *scalar curvature* $\leqslant C_0/d^{2+\varepsilon}(x_0; x)$ *where* $B(x_0; r)$ *and* $d(x_0; x)$ *denote respectively geodesic balls and geodesic distances,* $c > 0$, $C_0 \geqslant 0$ *and* ε *is an arbitrarily small positive constant. Then, X is isometrically biholomorphic to $\mathbb{C}^n$ with the flat metric.*

In this article we study the problem of compactifying complete Kähler manifolds of positive bisectional curvature. We have the following principal result.

MAIN THEOREM. — *Let X be a complete non-compact n-dimensional Kähler manifold of positive holomorphic bisectional curvature. Suppose for some positive constants c, C*

(i) *Volume* $(B(x_0; r)) \geq cr^{2n}$.

(ii) $0 <$ *scalar curvature* $\leq C/d^2(x_0; x)$. *Then, X is biholomorphic to an affine algebraic variety.*

From the Main Theorem and a theorem of RAMANUJAM [19] in affine algebraic geometry, we obtain the following corollary related to Conjecture I of Siu stated above.

COROLLARY. — *In case of dimension* $n=2$, *if the Kähler manifold X in the Main Theorem is actually of positive Riemannian sectional curvature, then X is biholomorphic to* $\mathbf{C}^2$.

For the proof of the Main Theorem we consider the algebra $P(X)$ of holomorphic functions of polynomial growth. We assume for the following discussion that $n \geq 2$. In [17] we obtained a special plurisubharmonic function u of logarithmic growth by solving the equation $i\partial\bar{\partial}u = $ Ricci form. The existence of non-trivial functions in the algebra $P(X)$ then follows readily from the L^2-estimates on complete Kähler manifolds of ANDREOTTI-VESENTINI [1] and HÖRMANDER [12]. Such an approach was already implicit in SIU-YAU [24]. There the exponential map plays an essential role in the estimates. In particular, it enables them to estimate volume growths of subvarieties by geometric comparison theorems. This was crucial in obtaining "minimal degree functions" defining a biholomorphic map onto $\mathbf{C}^n$. In our case the failure of exponential mappings to give global coordinates present serious difficulties. In particular, we do not have direct uniform estimates for the algebra $P(X)$ (for example, a uniform bound on the degree of $f_{x,1}, \ldots, f_{x,n} \in P(X)$ which give local holomorphic coordinates at x, for an arbitrary point $x \in X$) to show that $P(X)$ is finitely generated. To resolve this difficulty, we prove a series of finiteness theorems related to the algebra $P(X)$ by first passing to the quotient field $R(X)$ of "rational" functions, leading finally to the existence of a proper embedding. As the first step we prove a Siegel's theorem on the transcendence degree of $R(X)$. More precisely, we show that $R(X)$ is a finite extension field of $\mathbf{C}(f_1, \ldots, f_n)$, where $f_1, \ldots, f_n \in P(X)$ are algebraically independent.

Let mult $([V]; x_0)$ be the multiplicity of the zero divisor $[V]$ of a holomorphic function $f \in P(X)$ at $x_0 \in V$, and $\deg(f)$ be the degree of f measured in terms of geodesic distances. Basic to our estimate is the inequality $\operatorname{mult}([V]; x_0) \leqslant C \deg(f)$ for some $C > 0$. This is obtained by comparing both quantities to the volume growths of V over geodesic balls using the classical inequality of Bishop-Lelong and estimates of the Green kernel. Unlike the classical inequality, the multiplicity will now be bounded by some global weighted average of volumes of V over a family of "ringed" domains. From the proof the inequality

$$\operatorname{mult}([V]; x_0) \leqslant C \deg(f)$$

is actually valid for a complete Kähler manifold of positive Ricci curvature satisfying the same growth conditions. However, the holomorphic bisectional curvature enters when we prove existence theorems for $P(X)$. Moreover, results of [17] on the $\partial\bar{\partial}$-equation, which are only valid in case of nonnegative holomorphic bisectional curvature, imply by an application of the proof of the basic inequality that X is Stein.

From the basic inequality $\operatorname{mult}([V]; x_0) \leqslant C \deg(f)$, an existence theorem for $P(X)$ and a classical argument of Poincaré-Siegel, we prove immediately that the field $R(X)$ of "rational" functions is a finite extension field of some $\mathbb{C}(f_1, \ldots, f_n)$, $R(X) = \mathbb{C}(f_1, \ldots, f_n, g/h)$, $f_i, g, h \in P(X)$, such that $f_1, \ldots, f_n$ are algebraically independent. This theorem, which we call the Siegel's Theorem on X, does not imply that $P(X)$ is finitely generated. However, the Siegel's Theorem on X implies that the mapping $F : X \to \mathbb{C}^{n+2}$ given by $F = (f_1, \ldots, f_n, g, h)$ defines, in an appropriate sense, a birational equivalence between X and an irreducible affine algebraic subvariety Z of $\mathbb{C}^{n+2}$ of dimension n. We shall obtain an embedding by desingularizing F. This will involve a number of finiteness theorems.

First, we show that $F : X \to Z$ is almost surjective in the sense that it can miss at most a finite number of possibly singular hypersurfaces of Z. We show this by solving an ideal problem for each point $z \in Z$ missed by F, except for a certain algebraic subvariety T_0 of Z containing the singularities. By using the L^2-estimates of Skoda [25], we show that each such point $z \notin F(X) \cup T_0$ gives rise to some $f_z \in P(X)$ of degree bounded independent of z, which is the pull-back under F of some rational function whose pole set passes through z. By an intermediate result in the proof of the Siegel's Theorem on X, the dimension of the vector space of $f \in P(X)$ such that $\deg(f) \leqslant C$ is finite. If F were not almost surjective, one could select an infinite family of linearly independent f_z's, giving a contradiction.

The next step in proving the Main Theorem is to show that the mapping $F: X \to Z$ can be desingularized by adjoining a *finite* number of holomorphic functions of polynomial growth. First we show that the mapping defined by lifting F to an affine algebraic normalization $\tilde{Z}$ of Z is still defined by functions in $P(X)$. We call such a lifting $\tilde{F}: X \to \tilde{Z}$ a normalization of F. Let $\tilde{V}$ be the branching locus of $\tilde{F}$. By the previous finiteness theorem $\tilde{F}(\tilde{V})$, which we call the image set of indeterminancy, must lie in the union of a finite number of irreducible algebraic subvarieties S_i of codimension 2. However, it is not apparent that only a finite number of irreducible components of $\tilde{V}$ are mapped into S_i. In general $\tilde{F}$ "blows down" branches of $\tilde{V}$, which may have irreducible branches of codimension $\geqslant 2$. In order to show that $F: X \to Z$ can be desingularized by adjoining a finite number of functions in $P(X)$, one would like to show that $\tilde{V}$ must have only a finite number of irreducible components. For branches of codimension one we can prove this by establishing a uniform version of the basic inequality $\operatorname{mult}([V]; x_0) \leqslant C \deg(f)$ with a constant *independent of f and x_0* for regular points x_0 of the zero-divisor V of $f \in P(X)$. We prove this by geometric comparison theorems and the integral formula of LELONG [13], applied to geodesic balls. This involves a useful estimate on the exponential mapping on large Euclidean balls in the tangent spaces (Proposition (7.2)). We remark that the basic inequality with a fixed base point and the uniform version are obtained under different curvature conditions and that the uniform estimate does not apply to all singular points.

Our previous argument is not strong enough when the branching locus $\tilde{V}$ contains branches of codimension $\geqslant 2$. Fortunately, the uniform version of the basic inequality, suitably modified, is sufficient for showing that $\tilde{F}: X \to \tilde{Z}$ can be desingularized in a finite number of steps. Essentially, we show that through each irreducible branch W of $\tilde{V}$ (of positive dimension), there exists an "algebraic" curve C intersecting W at isolated points such that C is defined by $g_i \in P(X)$ of degree *bounded independent of W*. Then, we prove that there are only a finite number of W's by inverting $\tilde{F}$ along slices of algebraic curves on $\tilde{Z}$. (It is essential to reduce the problem to algebraic curves because of indeterminacies of meromorphic functions on higher-dimensional varieties.)

After desingularization, we obtain a biholomorphism of X onto some $Y-W$, where Y is an affine algebraic variety, possibly singular, and W is an algebraic subvariety of pure codimension one. In general such a variety $Y-W$ may fail to be affine algebraic because of the examples of Zariski (GOODMAN [11]). By a somewhat devious application of the vanishing theorem of Serre in algebraic geometry, we show in general that $Y-W$ is biregular to an affine algebraic variety if and only if it is rationally convex. In our case this follows from the fact that X is convex with respect to $P(X)$.

The proof of the basic inequality yields an improvement of a pinching theorem of MOK-SIU-YAU [17] in the case of nonnegative holomorphic bisectional curvature. This is contained in § 4 on applications of the proof of the basic inequality.

A significant part of this article depends on results of [17]. In § 1 we collect basic results of [17] that we shall need. Also, for the sake of completeness, we have included in § 7 certain standard estimates about exponential mappings on complete Riemannian manifolds using geometric comparison theorems. For the proofs of standard comparison theorems, we refer the reader to CHEEGER-EBIN [4] and SIU-YAU [24] (especially for estimates involving the complex Hessian).

A summary of the results of the present article, together with a sketch of the proofs, has appeared in MOK [15]. Related results and problems on non-compact complete Kähler manifolds of positive curvature can be found in the survey article MOK [16].

I want to thank Professor R. Gunning, Professor J. J. Kohn, Professor Y.-T. Siu and Professor S.-T. Yau for their encouragement and help during the course of the research. Professor Nils Øvrelid has given me invaluable help by arranging my summer stay in Oslo University, during which a substantial portion of the present article was worked out and written up. In June 1982, a preliminary version of the results was presented in Seminaire Lelong-Skoda in l'Institute Poincaré. I want to thank Professor P. Dolbeault, Professor P. Lelong and Professor H. Skoda for inviting me to the seminar and for their most encouraging enthusiasm in my research work. Finally, I would like to thank Professor D. Mumford, who kindly pointed out some important examples of Zariski in affine algebraic geometry relevant to my work. It motivated the final stroke (§ 9, on passing from an embedding to a proper embedding) completing the proof of the Main Theorem.

Added in proof. — The author would like to thank the referee for suggestions which improved the exposition of Proposition 7.2 and for correcting a number of inaccuracies on bibliographical references.

Table of Contents

1. Estimates of the Laplace-Beltrami operator and the Poincaré-Lelong equation

We collect in this section relevant estimates taken from § 1 of Mok-Siu-Yau [17] on the Laplace-Beltrami operator and the Poincaré-Lelong equation on complete Kähler manifolds of nonnegative holomorphic bisectional curvature.

(1.1) Estimates of the Green kernel

Proposition (Mok-Siu-Yau [17; p. 190]). — *Let X be a complete m-dimensional complete Riemannian manifold of nonnegative Ricci curvature, $m \geqslant 3$, such that for some fixed base point x_0, the volume of geodesic balls $B(x_0; R)$ satisfies*

$$\text{Volume}\,(B(x_0;R)) \geqslant c\,R^m, \text{ for some } c>0.$$

Then, the Green kernel $(G(x, y)$ exists on X and satisfies the estimates

$$\frac{A}{d(x,y)^{m-2}} \leqslant G(x,y) \leqslant \frac{B}{d(x,y)^{m-2}}$$

for some positive constants A and B independent of x. Moreover,

$$\|\nabla_y G(x, y)\| \leqslant \frac{C}{d(x, y)^{m-1}}$$

for some C independent of x.

From the proof of Proposition (1.1) we can choose the constants A and B so that the estimate

$$A\left(\frac{1}{d(x, y)^{m-2}} - \frac{1}{R^{m-2}}\right) \leqslant G_R(x, y) \leqslant B\left(\frac{1}{d(x, y)^{m-2}} - \frac{1}{R^{m-2}}\right)$$

is valid for the Green kernel G_R on $B(x_0; R)$, whenever $d(x_0; x)$, $d(x_0; y) < R/2$. A similar estimate holds for the gradient of $G_R(x, .)$.

The zero-order estimates of Green kernels are obtained by using the Sobolev inequality of CROKE [8] and the iteration technique of Di Giorgi-Nash-Moser (MOSER [18], BOMBIERI-GIUSTI [3]). The gradient estimates are obtained from the Harnack inequality of YAU [28] and CHENG-YAU [7]. For the sake of reference we also include here the latter version on geodesic balls of Riemannian manifolds of nonnegative Ricci curvature only for the case of harmonic functions.

THEOREM (Harnack inequality, CHENG-YAU [7]). — *Let X be a Riemannian manifold of nonnegative Ricci curvature. Suppose h is a positive harmonic function on a relatively compact geodesic ball $B(p; R)$ centered at p of radius R, then there exists a constant $C<0$ such that*

$$\|\nabla h(x)\| \leqslant \frac{C}{a^2 - r^2} \sup h(x), \qquad r(x) = d(p, x)$$

being the geodesic distance. Moreover, the constant C depends only on the dimension of the Riemannian manifolds X.

(1.2) ESTIMATES OF THE POINCARÉ-LELONG EQUATION

THEOREM (MOK-SIU-YAU [17]; Theorem 1.1, p. 187). — *Let X be a complete Kähler manifold of nonnegative holomorphic bisectional curvature of dimension $n \geqslant 2$. Suppose the scalar curvature is bounded by C/r^2 and Volume $(B(x_0\, pr)) \geqslant cr^{2n}$ for some fixed base point x_0 and some $c>0$. Suppose ρ is a closed (1.1) form $\|\rho\| \leqslant C_1/r^2$, measured in terms of* *norms of the given Kähler metric. Then, there exists a solution u of $1/2\,\Delta u = \text{trace}\,(\rho)$ such that u is of order $0(\log r)$ and satisfies automatically $i\,\partial\bar\partial u = \rho$.*

The theorem above is obtained from a variation of a Bochner inequality due originally to BISHOP-GOLDBERG [2] and applying estimates of the Laplace-Beltrami operator.

2. L^2-estimates of $\bar\partial$ of Andreotti-Vesentini, Hörmander and Skoda

We present here the well-known L^2-estimates of $\bar\partial$ for the sake of reference. We will only use (2.1) in the case of holomorphic line bundles. The basic estimates here are those of ANDREOTTI-VESENTINI [1] and HÖRMANDER [12]. We will also need an adaptation of Skoda's estimates for solving the ideal problem in the context of complete Kähler manifolds.

(2.1) THEOREM (L^2-estimates of $\bar\partial$ on a complete Kähler manifold, ANDREOTTI-VESENTINI [1] and HORMANDER [12]). — *Let X be a complete Kähler manifold and denote by Ric the Ricci curvature form of X. Let φ be a smooth function such that, in terms of the given Kähler metric,*

$$(*)\qquad \langle \partial\bar\partial\varphi + \text{Ric}, \eta \wedge \bar\eta \rangle \geqslant c\,(x)\,\|\eta\|^2$$

for tangent vectors η of type (1.0) at x and for some positive continuous function $c(x)$. Suppose f is a $\bar\partial$-closed smooth (0.1) form on X such that

$$\int_X \frac{\|f\|^2}{c} e^{-\varphi} < \infty.$$

Then, there exists a solution u of $\bar\partial u = f$ such that

$$\int_X |u|^2 e^{-\varphi} \leqslant \int_X \frac{\|f\|^2}{c} e^{-\varphi}$$

For hermitian holomorphic vector bundles V with curvature form $\Theta(\xi, \xi'; \eta, \bar\eta')$, where ξ, ξ' are vectors of V and η, η' are complex tangent vectors of X, both of type (1.0), the inequality $(*)$ should be replaced by :

$$\langle \partial\bar\partial\varphi, \eta \wedge \bar\eta \rangle + \Theta(\xi, \xi; \eta, \bar\eta) \geqslant c(x)\,\|\eta\|^2$$

for all ξ of unit length.

Remark. — (1) ξ, ξ' can also be regarded as complex tangent vectors of type (1.0) of V in the fiber direction. In the case V = tangent bundle T_X, $\Theta(\xi, \bar{\xi}; \eta, \bar{\eta})$ gives the holomorphic bisectional curvature in the directions ξ, η if ξ, η are of unit length. In the case of $V=\Lambda^n T_X$, the anticanonical line bundle, $\Theta(\xi, \bar{\xi}; \eta, \bar{\eta})$ gives the Ricci curvature in the direction of η when ξ is of unit length.

(2) In order to obtain the same estimate when φ is not necessarily smooth one needs to approximate φ by appropriate smooth functions. This can be done whenever X is a Stein manifold.

(2.2). The following theorem is an adaptation of Skoda's estimates for solving the ideal problem in the context of complete Kähler manifolds. The precise constants appearing in Skoda [25] will not be needed.

Theorem (adaptation to complete Kähler manifolds from Skoda [25]). — *Let X be a complete Kähler manifold, φ a smooth function such that $\partial\bar{\partial}\varphi+\mathrm{Ric}$ is a semi-positive $(1,1)$ form, where* Ric *stands for the Ricci curvature form. Let $f_1, \ldots, f_p$, h be holomorphic functions on X, k be a positive constant, $\alpha>1$ arbitrarily such that:*

$$\int_X \frac{|h|^2}{(\sum_{i=1}^{p}|f_i|^2)^{\alpha n+1}} e^{-k\varphi} < \infty.$$

Then, there exists a solution $(g_1, \ldots, g_p)$ of $\sum_{i=1}^{p} f_i g_i = h$ satisfying the estimate:

$$\int \frac{|g_j|^2}{(\sum_{i=1}^{p}|f_i|^2)^{\alpha n}} e^{-k\varphi} \leqslant C_\alpha \int \frac{|h|^2}{(\sum_{i=1}^{p}|f_i|^2)^{\alpha n+1}} e^{-k\varphi}$$

for each j, $1 \leqslant j \leqslant p$.

Since the proof of the above theorem in case of bounded pseudoconvex domains is obtained by applying the L^2-estimates of Hörmander [12] to the weight functions constant $\log(|g_1|^2+\ldots+|g_p|^2)$, application of Theorem (2.1) immediately gives (2.2) in case $g_1, \ldots, g_p$ have no common zero (i. e., $\log(|g_1|^2+\ldots+|g_p|^2)$ is smooth). In the general case smoothing can be obtained by taking $\log(|g_1|^2+\ldots+|g_p|^2+\varepsilon)$, which decreases monotonically to $\log(|g_1|^2+\ldots+|g_p|^2)$ as $\varepsilon \to 0$.

3. **The basic inequality** mult $([V]; x_0) \leq C \deg(f)$.

(3.1) From now on we shall assume that X is an n-dimensional complete Kähler manifold of positive holomorphic bisectional curvature such that the scalar curvature is bounded by $C_0/d(x_0; x)^2$ and Volume $(B(x_0; r)) \geq cr^{2n}$, as in the hypothesis of the Main Theorem. The following basic inequality is the starting point of our study of the algebra $P(X)$ of holomorphic functions of polynomial growth. It relates the degree of such functions to the multiplicity of their zero divisors at some arbitrary but fixed base point. We prove here the inequality in the case of manifolds of nonnegative Ricci curvature. In case of dimension 1 a Riemann surface admitting a complete metric of positive Ricci curvature must be biholomorphic to the complex plane by the classical theorem of BLANC-FIALA [1]. For the following theorem and the rest of this article we consider therefore only dimensions $n \geq 2$.

THEOREM (3.1) (The basic inequality). — *Let X be an n-dimensional complete Kähler manifold of positive Ricci curvature, $n \geq 2$, such that for some fixed base point x_0*

(i) *Scalar curvature* $\leq C_0/d(x_0; x)^2$, $C_0 \geq 0$.

(ii) *Volume* $(B(x_0; r)) \geq cr^{2n}$, $c > 0$.

Let f be a holomorphic function on X of polynomial growth, i.e., $|f(x)| \leq C'(d(x_0; x)^p + 1)$ for some $p \geq 0$, $C' \geq 0$, and let $[V] = i/2\pi\, \partial\bar\partial \log |f|^2$ be the zero divisor, counting multiplicity, determined by f. Then, there exists a positive constant C independent of f such that:

$$\text{mult}\,([V];\, x_0) \leq C \deg(f),$$

where $\deg(f)$ *is defined to be the infimum of all p for which the following estimate holds:*

$$|f(x)| \leq C'(p)(d(x_0; x)^p + 1).$$

The multiplicity here is taken to be the usual multiplicity defined algebraically. By a theorem of THIE [27] this agrees with the Lelong number of the positive (1, 1) current $[V]$ at x_0. We will not distinguish between the zero divisor of f and the positive (1, 1) integral current it represents. Also, holomorphic n-forms and n-vector fields of polynomial growth and their degrees will be defined analogously.

Theorem (3.1) will be a consequence of more precise estimates of both the multiplicity and the degree, in terms of the volume growth of $[V]$ over geodesic balls. Unlike the classical inequality of Bishop-Lelong, the multiplicity will be estimated in terms of a weighted average of volumes of $[V]$ inside geodesic balls over the entire manifold. We remark that a more direct proof can be given in the case under consideration in the Main Theorem, namely, when the holomorphic bisectional curvatures are positive, where we have results of [17] for solving the Poincaré-Lelong equation.

We fix a base point x_0 and some $R_0>0$. For $R\geqslant R_0$, we subdivide X into regions $D_\nu(R_0)$, $\nu\geqslant 0$ defined by:

$$\begin{cases} D_0(R)=B(x_0;\,2R) \\ D_\nu(R)=B(x_0;\,2^{\nu+1}R)-B(x_0;\,2^\nu R) \end{cases}$$

We define a weighted volume of V over $D_\nu(R)$ by:

$$A_\nu(R)=\frac{1}{(2^\nu R)^{2n-2}}\int_{D_\nu(R)}\Delta\log|f|^2$$

With these notations we have the following estimates.

Proposition (3.1.1). — *Let X be a complete Kähler manifold,* $\mathrm{Ric}(X)>0$ *satisfying geometric growth conditions as in the hypothesis of Theorem* (3.1). *Then there exist an $R_0>0$ such that for all $R>R_0$ and for all $f\in P(X)$*

$$\operatorname{mult}([V];\,x_0)\leqslant C\sum_{\nu=0}^{\infty}\frac{A_\nu(R)}{2^\nu},$$

where the constant C is independent of f.

Proposition (3.1.2). — *Hypothesis as in Theorem* (3.1) *and Proposition* (3.1.1), *there exists a constant C_1 independent of f and R and constants $C_2(f)$ depending on f such that for any $R\geqslant R_0>0$ (with R_0 fixed as before),*

$$\mu\deg(f)\geqslant C_1\left(\sum_{\nu=0}^{\mu-1}A_\nu(R)\right)-C_2(f).$$

The next two paragraphs will be devoted to proving the preceding propositions.

(3.2) AN UPPER BOUND FOR THE MULTIPLICITY

In this paragraph we will prove Proposition (3.1.1). Let $x_0 \in X$ and $R_0 > 0$ be fixed as above. Let B be a coordinate neighborhood of x_0 biholomorphic to a Euclidean ball with x_0 corresponding to the center such that $B \subset B(x_0; R_0/2)$. Since the Ricci form on X is positive definite, we have, writing Ric for the Ricci form,

$$\operatorname{mult}([V]; x_0) \leqslant \text{Const.} \int_B \frac{i}{2\pi} \partial\bar{\partial} \log |f|^2 \wedge \operatorname{Ric}^{n-1}$$

$$\leqslant \text{Const.} \int_{B(R)} \frac{i}{2\pi} \partial\bar{\partial} \log |f|^2 \wedge \operatorname{Ric}^{n-1} \text{ for } R \geqslant R_0/2.$$

Here the first inequality could be obtained by applying the inequality of Bishop-Lelong to the a coordinate ball B' with $x_0 \in B' \subset \subset B$. The constant is independent of f, but depends on the choice of B and the smallest (positive) eigenvalue of the Ricci curvature form. Hence, for each $R \geqslant R_0/2$ we obtain by integration by parts:

$$\operatorname{mult}([V]; x_0) \leqslant \text{Const.} \int_{B(R)} \frac{i}{2\pi} \partial\bar{\partial} \log |f|^2 \wedge \operatorname{Ric}^{n-1}$$

$$= \text{Const.} \int_{\partial B(R)} \frac{i}{2\pi} \bar{\partial} \log |f|^2 \wedge \operatorname{Ric}^{n-1}$$

$$\leqslant \frac{\text{Const.}}{R^{2n-2}} \int_{\partial B(R)} \|\nabla \log |f|^2\| \left(\text{scalar curvature} \leqslant \frac{C_0}{R^2}\right).$$

Integrating from $R/2$ to R, we have, for $R \geqslant R_0$:

$$\operatorname{mult}([V]; x_0) \leqslant \frac{\text{Const.}}{R^{2n-1}} \int_{B(R)-B(R/2)} \|\nabla \log |f|^2\|.$$

In order to relate the latter integral with the volume growth of $[V]$, counting multiplicities, we need to represent $\log |f|^2$ as an integral over V, using the following lemma on Riesz representation.

LEMMA (Riesz representation). — *Let $f \in P(X)$ and $\tilde{x}_0$ be a point close to x_0 such that $f(\tilde{x}_0) \neq 0$. Then, on X:*

$$\log |f(x)|^2 = \lim_{R \to \infty} \int_{B(R)} [G_R(\tilde{x}_0; y) - G_R(x; y)] \Delta \log |f(y)|^2 \, dy$$
$$+ \log |f(\tilde{x}_0)|^2,$$

where G_R denotes the Green kernel on $B(R)=B(x_0;\ R)$, taken to be positive, and the limit is in the sense of uniform convergence on compact subsets.

Proof of lemma. — Let:

$$u_R(x)=\int_{B(R)}[G_R(\tilde{x}_0;\ y)-G_R(x;\ y)]\Delta\log|f(y)|^2\,dy+\log|f(\tilde{x}_0)|^2$$

on the geodesic ball $B(R)$. Then on $B(R)$:

$$\log|f|^2=u_R+h_R,$$

where h_R is harmonic, $h_R(\tilde{x}_0)=0$ and on $\partial B(R)$:

$$h_R\leqslant\sup_{\partial B(R)}\log|f|^2-\log|f(\tilde{x}_0)|^2.$$

By the maximum principle the above inequality is also valid on $B(R)$. By the Harnack inequality of CHENG-YAU [7] (*cf.* (1.1)) applied to $(\sup_{B(R)}h_R-h_R)$ we have on $B(R/2)$:

$$\|\nabla h_r\|\leqslant\frac{C}{R}|(\sup_{\partial B(R)}\log|f|^2)-\log|f(\tilde{x}_0)|^2|.$$

Since by assumption f is of polynomial growth,

$$\sup_{\partial B(R)}\log|f|^2\leqslant q\log R+\text{Const.}$$

for R large enough, giving:

$$\|\nabla h_R\|\leqslant\frac{C\,q\,\log R+\text{Const.}}{R}\ \textit{on}\ B(R/2).$$

Recall that $h_R(\tilde{x}_0)=0$. Taking limits we obtain $\lim_{R\to\infty}h_R=0$ uniformly on compact subsets of X, thus proving the lemma.

From the inequality:

$$\text{mult}([V];\ x_0)\leqslant\frac{\text{Const.}}{R^{2n-1}}\int_{B(R)-B(R/2)}\|\nabla\log|f|^2\|$$

we shall obtain the desired upper bound for $\text{mult}([V];\ x_0)$. From the lemma (Riesz representation) one has the gradient estimate (given by (1.1)):

$$\|\nabla \log |f(x)|^2\| \leqslant \int_X \|\nabla_y G(x, y)\| \Delta \log |f(y)|^2 \, dy$$

$$\leqslant \int_X \frac{\text{Const.}}{d(x, y)^{2n-1}} \Delta \log |f(y)|^2 \, dy$$

Hence:

$$\int_{B(R)-B(R/2)} \|\nabla \log |f(x)|^2\|$$

$$\leqslant \int_X \left(\int_{B(R)-B(R/2)} \frac{\text{Const.}}{d(x, y)^{2n-1}} dx \right) \Delta \log |f(y)|^2 \, dy.$$

Since X has nonnegative Ricci curvature, the exponential map (in normal geodesic coordinates) is volume-decreasing. It follows that volume of geodesic spheres $\partial B(x, R)$ is of order $0(d(x, y)^{2n-1})$. Hence, for $R \geqslant R_0$:

$$\int_{B(R)-B(R/2)} \|\nabla \log |f(x)|^2\|$$

$$\leqslant \int_{B(2R)} \left(\int_{B(R)-B(R/2)} \frac{\text{Const.}}{d(x, y)^{2n-1}} dx \right) \Delta \log |f(y)|^2 \, dy$$

$$+ \int_{X-B(2R)} \left(\int_{B(R)-B(R/2)} \frac{\text{Const.}}{d(x, y)^{2n-1}} dx \right) \Delta \log |f(y)^2| \, dy$$

$$\leqslant \text{Const. } R \int_{D_0(R)} \Delta \log |f(y)|^2 \, dy$$

$$+ \sum_{\nu=1}^{\infty} \frac{\text{Const.}}{(2^\nu R)^{2n-1}} \text{Volume}(B(R)) \int_{D_\nu(R)} \Delta \log |f(y)|^2 \, dy$$

Here we subdivide $X-B(2R)$ into $D_\nu(R)=B(x_0; 2^{\nu+1} R)-B(x_0; 2^\nu R)$ and put $D_0(R)=B(2R)$. From the estimate Volume $(B(R)) \leqslant$ Const. R^{2n}, which holds because X has nonnegative Ricci curvature, we obtain (all constants being independent of R)

$$\operatorname{mult}([V];\, x_0) \leqslant \frac{\text{Const.}}{R^{2n-1}} \int_{B(R)-B(R/2)} \|\nabla \log |f|^2\|$$

$$\leqslant \frac{\text{Const.}}{R^{2n-2}} \int_{D_0(R)} \Delta \log |f(y)|^2\, dy$$

$$+\sum_{\nu=1}^{\infty} \frac{1}{2^\nu} \frac{\text{Const.}}{(2^\nu R)^{2n-2}} \int_{D_\nu(R)} \Delta \log |f(y)|^2\, dy \leqslant \text{Const.} \sum_{\nu=0}^{\infty} \frac{A_\nu(R)}{2^\nu}.$$

(3.3) A LOWER BOUND FOR THE DEGREE

The lower bound $\mu \deg(f) \geqslant C_1 (\sum_{\nu=0}^{\mu-1} A_\nu) - C_2(f)$, C_1 independent of f, is in fact an immediate consequence of the lemma on Riesz representation proved in (3.2). To see this, recall that $\tilde{x}_0$ is a point close to x_0 such that $f(\tilde{x}_0) \neq 0$ and let:

$$v_R(x) = \int_{B(R)} -G_R(x, y)\, \Delta \log |f(y)|^2\, dy$$

where $B(R) = B(x_0;\, R)$. The function $\log |f|^2 - v_R$ is harmonic on $B(R)$. Since f is of polynomial growth, given any $\delta > 0$, there is a constant $C'(\delta)$ such that:

$$|f(x)| \leqslant C'(\delta)\,(d(\tilde{x}_0;\, x)^{\deg(f)+\delta} + 1).$$

From the maximum principle for any $R > 0$:

$$\log |f(\tilde{x}_0)|^2 - v_R(\tilde{x}_0) \leqslant 2(\deg(f) + \delta) \sup_{x \in \partial B(R)} (d(\tilde{x}_0;\, x) + 1) + C''(\delta)$$

On the other hand, from estimates of $G_R(x;\, y)$ in (1.1):

$$-v_{2^\mu R}(\tilde{x}_0) \geqslant \sum_{\nu=0}^{\mu-1} \frac{\text{Const.}}{(2^\nu R)^{2n-2}} \int_{D_\nu(R)} \Delta \log |f(y)|^2\, dy$$

where the constant is independent of f. Combining the two inequalities gives the desired inequality when δ is small enough.

(3.4) CONCLUSION OF THE PROOF OF THE BASIC INEQUALITY

Finally, we prove the inequality $\mathrm{mult}([V]; x_0) \leqslant C \deg(f)$ from the inequalities (i) and (ii) for $R \geqslant R_0 > 0$, with x_0 fixed

(i) $$\mathrm{mult}([V]; x_0) \leqslant C \sum_{\nu=0}^{\infty} \frac{A_\nu(R)}{2^\nu};$$

(ii) $$\mu \deg(f) \geqslant C_1 \left(\sum_{\nu=0}^{\mu-1} A_\nu(R)\right) - C_2(f).$$

Recall that:

$$D_0(R) = B(x_0; 2R),$$
$$D_\nu(R) = B(x_0; 2^{\nu+1} R) - B(x_0; 2^\nu R),$$

and that the weighted volume $A_\nu(R)$ is defined by:

$$A_\nu(R) = \frac{1}{(2^\nu R)^{2n-2}} \int_{D_\nu(R)} \Delta \log |f|^2.$$

We consider the inequality (i) for $R = 2^s R_0$, adding up the inequalities:

$$A_0(2^s R_0) + \frac{A_1(2^s R_0)}{2} + \ldots + \frac{A_\nu(2^s R_0)}{2^\nu} + \ldots \geqslant \frac{1}{C} \mathrm{mult}([V]; x_0)$$

Now for $\nu \geqslant 1$, $s \geqslant 0$

$$A_\nu(2^s R_0) = \frac{1}{(2^{\nu+s} R_0)^{2n-2}} \int_{D_{\nu+s}(R_0)} \Delta \log |f|^2 = A_{\nu+s}(R_0).$$

The term $A_0(2^s R_0)$ can be decomposed by regarding $D_0(2^s R_0) = B(x_0; 2^{s+1} R_0)$ as the disjoint union of $B(x_0; 2R_0) = D_0(R_0)$, $D_1(R_0), \ldots D_s(R_0)$. Then,

$$\begin{aligned} A_0(2^s R_0) &= \frac{1}{(2^s R_0)^{2n-2}} \sum_{\nu=0}^{s} \int_{D_\nu(R_0)} \Delta \log |f|^2 \\ &= \frac{1}{2^{s(2n-2)}} A_0(R_0) + \frac{1}{2^{(s-1)(2n-2)}} A_1(R_0) \\ &\qquad + \ldots + \frac{1}{2^{2n-2}} A_{s-1}(R_0) + A_s(R_0), \end{aligned}$$

giving:

$$A_0(R_0)+\frac{A_1(R_0)}{2}+\ldots+\frac{A_\nu(R_0)}{2^\nu}+\ldots\geqslant\frac{1}{C}\operatorname{mult}([V];\,x_0),$$

$$\frac{A_0(R_0)}{2^{2n-2}}+A_1(R_0)+\frac{A_2(R_0)}{2}+\ldots+\frac{A_\nu(R_0)}{2^{\nu-1}}+\ldots\geqslant\frac{1}{C}\operatorname{mult}([V];\,x_0),$$

$$\frac{A_0(R_0)}{4^{2n-2}}+\frac{A_1(R_0)}{2^{2n-2}}+A_2(R_0)+\frac{A_3(R_0)}{2}+\ldots+\frac{A_\nu(R_0)}{2^{\nu-2}}+\ldots\geqslant\frac{1}{C}\operatorname{mult}([V];\,x_0).$$

Adding the first (s) inequalities, we have:

$$a_0 A_0(R_0)+a_1 A_1(R_0)+\ldots+a_s A_s(R_0)+\ldots\geqslant\frac{s}{C}\operatorname{mult}([V];\,x_0)$$

where for $0\leqslant r\leqslant s-1$

$$1\leqslant a_r\leqslant 1+\sum_{\nu=1}^{\infty}\frac{1}{2^2}+\sum_{\nu=1}^{\infty}\frac{1}{2^{(2n-2)\nu}}\leqslant K<\infty,$$

K independent of s; and for $r\geqslant s$

$$a_r=\frac{1}{2^{r-s+1}}\left(1+\frac{1}{2}+\ldots\frac{1}{2^{-1}}\right)<\frac{1}{2^{r-s}}.$$

This gives

$$(\#)\quad K(A_0(R_0)+A_1(R_0)+\ldots A_{s-1}(R_0))+\Bigg(A_s(R_0)+\frac{A_{s+1}(R_0)}{2}+\ldots+\frac{A_{s+\nu}(R_0)}{2^\nu}+\ldots\Bigg)\geqslant\frac{s}{C}\operatorname{mult}([V];\,x_0).$$

Now we use the inequality (ii) to obtain:

$$A_0(R_0)+A_1(R_0)+\ldots+A_{s-1}(R_0)\leqslant\frac{s}{C_1}\deg(f)+\frac{C_2(f)}{C_1}$$

and:

$$\frac{A_{s+\nu}(R_0)}{2^\nu}\leqslant\frac{1}{2^\nu}\,\frac{(s+\nu+1)\deg(f)+C_2(f)}{C_1}.$$

Substituting into the previous inequality (#)

$$\deg(f)\left(\frac{Ks}{C_1}+\sum_{\nu=0}^{\infty}\frac{s+\nu+1}{2^\nu C_1}\right)+(2+K)\frac{C_2(f)}{C_1}\geqslant\frac{s}{C}\operatorname{mult}([V];\,x_0).$$

Since $\sum_{\nu=0}^{\infty}(\nu/2^{\nu})<\infty$ and $\sum_{\nu=0}^{\infty}s/2^{\nu}=2s$ the basic inequality:

$$\operatorname{mult}([V];\, x_0)\leqslant C\deg(f),$$

with a different constant C, follows immediately by letting $s\to\infty$.

4. Steinness of X and an improvement of a pinching theorem for nonnegative bisectional curvature

(4.1) Recall that the complete n-simensional Kähler manifold X in the Main Theorem satisfies

(i) holomorphic bisectional curvature>0;

(ii) scalar curvature$<C/d(x_0;\, x)^2$;

(iii) Volume $(B(x_0;\, r))\geqslant cr^{2n}$, $c>0$.

We assume here $n\geqslant 2$. Using the solution of the Poincaré-Lelong equation developed in [17], we showed in the same article (Theorem 1.2 (2), p. 200) that under the stronger assumption

(ii)′ $C'(d(x_0,\, x)+1)^2<$scalar curvature$<C/d(x_0,\, x)^2$, X is a Stein manifold because the solution φ of $i\partial\bar{\partial}u=$Ricci curvature form (reduced to $1/2\,\Delta u=$scalar curvature) is an exhaustion function. Using intermediate estimates of the basic inequality (Theorem (3.1)) we shall prove the same thing under the weaker assumption (ii) with only an upper bound on the scalar curvature. For the sake of completeness we shall recall the arguments used in [17, Theorem 1.2(2), p. 200]. From now on X will be the complete Kähler manifold satisfying the hypothesis of the Main Theorem and x_0 will be a fixed base point. We formulate our result in the following more general form.

PROPOSITION (4.1). — *Let ρ be a closed positive* (1, 1) *form on X such that $\|\rho\|\leqslant C/d(x_0;\, x)^2$. Then there is a solution of $1/2\,\Delta u=$trace* (ρ) *of order* $0(\log d(x_0;\, x))$ *which satisfies automatically the Poincaré-Lelong equation $i\partial\bar{\partial}u=\rho$. Moreover, either $\rho\equiv 0$ or u actually satisfies the more precise estimates* $C'(\log d(x_0;\, x)\leqslant u\leqslant C''\log d(x_0;\, x)$, $C',\, C''>0$ *for* $d(x_0;\, x)$ *large enough.*

Proof. — The lower estimate in (3.2) of volume growth of hypersurfaces on geodesic balls is clearly also valid in a modified form for closed positive

(1, 1) forms ρ such that $\|\rho\| \leqslant C/d(x_0; x)^2$. We use the same notations as in (3.2). We define the regions:

$$\begin{cases} D_0(R_0) = B(2R_0) \\ D_\nu(R_0) = B(2^{\nu+1}R_0) - B(2^\nu R_0) \qquad \text{for} \quad \nu \geqslant 1 \end{cases}$$

and the weighted averages:

$$A_\nu(\rho; R_0) = \frac{1}{(2^\nu R_0)^{2n-2}} \int_{D_\nu(R_0)} \operatorname{trace}(\rho).$$

From the arguments of (3.2) we have:

$$\sum_{\nu=0}^{\infty} \frac{1}{2^\nu} A_\nu(\rho; 2^\mu R_0) \geqslant c > 0,$$

for any integer $\mu \geqslant 0$, where c is independent of μ. Write trace $(\rho) = h$ and define $h_p = \chi_{D_p(R_0)} h$, $h = \sum_{p=0}^{\infty} h_p$. Let u_p be the solution on X of $1/2\,\Delta u_p = h_p$ obtained by solving the Dirichlet boundary value problem and normalizing at x_0 so that $u_p(x_0) = 0$. Thus $1/2\,\Delta u_{p,k} = h_p$, on $B(2^k R_0)$, $k \geqslant p$, $u_{p,k} \equiv$ constant on $\partial B(x_0, k)$ and $u_{p,k}(x_0) = 0$, then u_p is the uniform limit of $u_{p,k}$ on compact subsets. Let x be such that $2^q R_0 \leqslant d(x_0; x) \leqslant 2^{q+1} R_0$ and write:

$$u(x) = \sum_{p \leqslant q} u_p(x) + \sum_{p \geqslant q+1} u_p(x).$$

We estimate the two terms

$$w_1(x) = \sum_{p \leqslant q} u_p(x) \qquad \text{and} \qquad w_2(x) = \sum_{p \geqslant q+1} u_p(x)$$

separately. We do w_2 first. On $B(x_0, 2^q R_0)$ u_p is harmonic for $p \geqslant q+1$. Since estimates of the solution $\Delta v = \chi_{B(R)}$ by solving the Dirichlet boundary problem (*cf.* [17, Theorem 1]) give $|v| \leqslant C_0 R^2$, we obtain from $h_p \leqslant C_0/(2^p R_0)^2 \chi_{B(2^p R_0)}$ and the gradient estimate of harmonic functions (YAU [28]):

$$\|\nabla u_p\| \leqslant \frac{C_1}{2^p R_0} \quad \text{on } B(x_p; 2^{p-1} R_0)$$

giving:

$$w_2(x) \geqslant -\sum_{p \geqslant q+2} \frac{C_1}{(2^p R_0)} d(x_p; x) + \inf_X u_q \geqslant -\sum_{i=0}^{\infty} \frac{C_1}{2^i} - CC_0 = -C_2$$

where C is a constant depending only on the geometry of X. Clearly, C_2 is independent of x. Now we estimate $w_1(x)$. Fix $p \leqslant q$. u_p is harmonic on $X-B(2^p R_0)$. We have:

$$u_p(z)=\int_{D_p(R_0)} -h(y)\,G(z;y)+\int_{D_p(R_0)} h(y)\,G(x_0;y).$$

For all p, the first term is larger than CC_0 on X. Let:

$$\alpha_p=\int_{D_p(R_0)} h(y)\,G(x_0;y)\,dy.$$

Then, at x, by comparing to harmonic measures, we have:

$$u_p(x)\geqslant -CC_0+(CC_0)\left(1-\frac{C_3\,2^{p(2n-2)}R_0^{(2n-2)}}{d(x_0;x)^{2n-2}}\right)+\alpha_p.$$

There exists an integer m such that for $q\geqslant p+m$, we have:

$$1-\frac{C_3\,2^{p(2n-2)}R_0^{(2n-2)}}{d(x_0;x)^{2n-2}}\geqslant 1-\beta^{q-p}, \text{ with some } \beta<1.$$

Thus,

$$\begin{aligned} w_1(x) &\geqslant -m\,CC_0+\textstyle\sum_{p\leqslant q-m} -CC_0+(CC_0)(1-\beta^{q-p})+\alpha_p \\ &= -m\,CC_0-CC_0\textstyle\sum_{p\leqslant q-m}\beta^{q-p}+\sum_{p\leqslant q-m}\alpha_p \\ &\geqslant -CC_0\left(m+\frac{1}{1-\beta}\right)+\textstyle\sum_{p\leqslant q-m}\alpha_p. \end{aligned}$$

From:

$$w(x)=w_1(x)+w_2(x)\geqslant -C_2-CC_0\left(m+\frac{1}{1-\beta}\right)+\textstyle\sum_{p\leqslant q-m}\alpha_p$$

to prove the proposition it remains to estimate $\sum_{p\leqslant q-m}\alpha_p$.

Recall that:

$$\textstyle\sum_{\nu=0}^{\infty}\frac{1}{2^\nu}A_\nu(\rho;2^\nu R_0)\geqslant c>0,$$

where:

$$A_\nu(\rho, R)=\frac{1}{(2^\nu R)^{2n-2}}\int_{D_\nu(R_0)} \operatorname{trace}(\rho).$$

Clear $\alpha_p \geqslant A_p(\rho; R_0)$ because $G(x_0; y)\geqslant 1/(2^p R_0)^{2n-2}$ for $y\in D_p(R_0)$.

From the inequality (#) of (3.4) adapted to the weighted total variations $A_\nu(\rho; R_0)$ we obtain, writing $A_\nu = A_\nu(\rho; R_0)$

$$K(A_0+A_1+\ldots+A_{s-1})+\left(A_s+\frac{A_{s+1}}{2}+\frac{A_{s+2}}{4}+\ldots\right)\geqslant sc$$

for some constant $c>0$.

From the proof of Proposition (3.1.2) we know that:

$$\frac{A_{s+\nu}}{2^\nu}\leqslant\frac{C_1(s+\nu+1)}{2^\nu}+\frac{C_2}{2^\nu}.$$

There is an integer $k\geqslant 0$, independent of x, such that:

$$\sum_{\nu\geqslant k}^{\infty}\frac{s+\nu+1}{2^\nu}\leqslant\frac{sc}{2C_1}+C_4.$$

Then:

$$K(A_0+A_1+\ldots+A_{s-1})+\left(A_s+\ldots+\frac{A_{s+k-1}}{2^{k-1}}\right)\geqslant\frac{sc}{2}-C_1C_4,$$

$$A_0+A_1+\ldots+A_s+\ldots+A_{s+k-1}\geqslant\frac{1}{K}\left(\frac{sc}{2}-C_1C_4\right).$$

Combining:

$$\alpha_0+\alpha_1+\ldots+\alpha_{q-m}\geqslant A_0+A_1+\ldots+A_{q-m}$$

with the last inequality, we obtain the desired estimate:

$$u(x)\geqslant -C_2-CC_0\left(m+\frac{1}{1-\beta}\right)+\sum_{p\leqslant q-m}\alpha_p\geqslant C_5 q-C_6,\ C_5>0.$$

Recall that x is a point on $D_q(R_0)$, i.e., $2^q R_0\leqslant d(x_0; x)\leqslant 2^{q+1}R_0$, and that $u(x)$ is the solution to $i\partial\bar\partial u=\text{Ric}$ obtained by reduction to $1/2\,\Delta u=$ scalar curvature. The estimate $u(x)\geqslant C_5 q-C_6$ gives the lower bound:

$$u\geqslant \underline{C}'\log d(x_0; x).$$

The upper bound:

$$u \leqslant C'' \log d(x_0; x)$$

was already contained in [17, Theorem 1.1].

From Proposition (4.1) and solving $i\partial\bar{\partial}u = \text{Ricci}$ curvature form using techniques of [17] we conclude immediately:

(4.2) PROPOSITION. — *Let X be an n-dimensional complete Kähler manifold of positive holomorphic bisectional curvature. Suppose for a fixed base point:*

(i) *scalar curvature* $< C/d(x_0; x)^2$;

(ii) *Volume* $(B(x_0, r)) \geqslant cr^{2n}$;

for some C, $c > 0$. Then X is a Stein manifold.

Proof. — The case of dimension $n=1$ is trivial. From Proposition (4.1), for $n \geqslant 2$, the solution of $i\partial\bar{\partial}u = \text{Ricci}$ curvature form is a strictly plurisubharmonic exhaustion function. X is Stein by Grauert's solution of the Levi problem.

Another consequence of Proposition (4.1), combined with results of MOK-SIU-YAU [17] is an improvement of the pinching theorem in [17, Theorem 1.2, p. 194] in case of nonnegative holomorphic bisectional curvature. The improvement here is simply that we drop the assumption that X is Stein.

THEOREM. — *Let X be a complete Kähler manifold of nonnegative holomorphic bisectional curvature of dimension $n \geqslant 2$. Suppose for a fixed base point x_0:*

(i) *scalar curvature* $< C/d(x_0; x)^{2+\varepsilon}$

(ii) *Volume* $(B(x_0; r)) \geqslant cr^{2n}$

for some C, $c > 0$ and for an arbitrarily small positive constant ε. Then, X is isometrically biholomorphic to $\mathbb{C}^n$ with the flat metric.

Proof. — The solution of $i\partial\bar{\partial}u = \text{Ricci}$ curvature form obtained in [17, Theorem 1.2, p. 194] is bounded. From Proposition (4.1) u is identically zero. Since X has nonnegative holomorphic bisectional curvature, it follows from the vanishing of the Ricci form that X is flat. X is therefore covered by $\mathbb{C}^n$ such that the covering transformations are unitary. The volume growth condition (ii) then forces $\pi_1(X)$ to be finite. Since any finite fixed-point free group of diffeomorphisms of $\mathbb{R}^m$ is trivial, it follows that X is isometrically biholomorphic to $\mathbb{C}^n$ with the flat metric.

5. Siegel's Theorem on the field of rational functions

(5.1) Recall that $P(X)$ stands for the algebra of holomorphic functions of polynomial growth on X. We shall call the quotient field of $P(X)$ the field of rational functions, denoted by $R(X)$. The main result of this section can be summarized in the following analogue of the classical theorem of Siegel. The proof is obtained from the basic inequality of § 3, L^2-estimates of $\bar\partial$ and the classical arguments of Poincaré and Siegel.

PROPOSITION (5.1) (Siegel's Theorem on the field of rational functions). — *Let X be a complete Kähler manifold of positive bisectional curvature satisfying the geometric hypothesis of the Main Theorem. Then, the field $R(X)$ of rational functions on X is a finite extension field over $\mathbb{C}(f_1, \ldots, f_n)$ for some algebraically independent holomorphic functions $f_1, \ldots, f_n$ in $P(X)$.*

Proof. — In this section we denote as before by u the solution of $i\partial\bar\partial u=$ Ricci form obtained in [17]. First we find $f_1, \ldots, f_n$ in $P(X)$ algebraically independent over $\mathbb{C}$. Let $x \in X$, and let $z_1, \ldots, z_n$; $\sum_{i=1}^n |z_i|^2<1$; be local holomorphic coordinates at x such that $z_1(x)=\ldots=z_n(x)=0$. Let ρ be a smooth cut-off function on $\mathbb{C}^n$ such that $\operatorname{Supp}\rho \subset\subset B^n(1)$ and $\rho\equiv 1$ on the ball $B^n(1/2)$. The function $\rho \log|z|=\rho(z_1(x)\ldots z_n(x)\log(\sum|z_i(x)|^2)^{1/2}$ is globally defined on X and is smooth except for the logarithmic singularity at X. Furthermore, the $(1, 1)$ form $\partial\bar\partial\rho\log|z|$ dominates a negative multiple of the Kähler form on X. Choose now a positive constant C such that

$$\nu=C u+\rho((2n+2)\log|z|)$$

is plurisubharmonic on X ($\partial\bar\partial u$ being positive definite). Then, for the plurisubharmonic weight function ν, and any non-zero tangent vector v of type $(1, 0)$ on X:

$$\langle \partial\bar\partial\nu+\mathrm{Ric}, v \wedge \bar v\rangle >0.$$

Now $\bar\partial(\rho z_i)$ is a $\bar\partial$-closed $(0, 1)$-form on the complete Kähler manifold X. Using the standard L^2-estimates of $\bar\partial$ (*cf.* § 2), there exists a smooth function u_i such that $\bar\partial u_i=\bar\partial(\rho z_i)$ and:

$$\int_X |u_i|^2 e^{-\nu}\leqslant\frac{1}{c}\int_X \|\bar\partial(\rho z_i)\|^2 e^{-\nu}$$

where $\langle \partial\bar{\partial}v+\mathrm{Ric},\ v\wedge\bar{v}\rangle \geqslant c\|v\|^2$ whenever v is a tangent vector on Supp ρ. Because of the singularity $(2n+2)\log|z|$, u_i and its first order derivatives have to vanish at x. Moreover, the functions $f_i=u_i-\rho z_i$ are then holomorphic ($\bar{\partial} f_i\equiv 0$). They define a local biholomorphism at x. Suppose $P(f_1, \ldots, f_n)\equiv 0$ for some polynomial P in n variables. Differentiating P at x_0 shows that $df_1(x) \ldots df_1(x)$ would be linearly dependent, contradicting with the fact $\partial f_i/\partial z_j(x_0)=\delta_{ij}$.

Let now d_p denote the dimension of all functions in $P(X)$ of degree $\leqslant p$. We claim that $d_p\leqslant C'p^n$ for some $C'>0$. To show this consider the mapping $\Phi_m : P(X)\to \mathbb{C}^{q(m)}$ defined by taking all partial derivatives of $f\in P(X)$ of order $\leqslant m$ at the point x. There exists a constant $k>0$, $k=k(n)$, such that $q(m)<km^n$. Recall that by § 3 there exists a constant $C>0$ such that $\mathrm{mult}([V];\ x_0)\leqslant C\deg(f)$. We can assume that C is an integer. Now choose C' such that $C'>k\,C^n$. To show that $d_p\leqslant C'p^n$ we argue by contradiction. If $d_p>C'p^n$ we would have

$$(\#)\qquad d_p>k\,(C_p)^n>q\,(C_p)$$

($q(m)$ being the number of coefficients in the Taylor expansion at x_0 of terms of degree $\leqslant m$). Choose a basis over $\mathbb{C}$ of the vector space V_p of polynomials in $P(X)$ of degree p, denoted by $\{g_1, \ldots, g_{d_p}\}$ and considerer $\Phi_{C_p} : V_p\to\mathbb{C}^{q(C_p)}$. From (#) it follows that some non-zero linear combination $\sum_{i=1}^{d_p} a_i g_i=g$ would be mapped to zero by Φ_{C_p}. g has degree $\leqslant p$ and vanishes at the point x_0 with multiplicity $\geqslant C_p+1$. This contradicts with the inequality $\mathrm{mult}([V];\ x_0)\leqslant C\deg(f)$, proving $d_p\leqslant C'p^n$ by contradiction.

From $d_p\leqslant C'p^n$ it will follow that the field $R(X)$ of rational functions on X is a finite extension field of $\mathbb{C}(f_1, \ldots, f_n)$. First we observe from the algebraic independence of $f_1, \ldots, f_n$ that for p large enough:

$$\dim\{l\in\mathbb{C}[f_1, \ldots, f_n] : \deg l\leqslant p\}\geqslant cp^n \quad \text{for some } c>0.$$

Let $k_1, \ldots, k_s$ be rational functions linearly independent over the field $\mathbb{C}(f_1, \ldots, f_n)$. Write $k_i=h_i/g_i$, g_i, $h_i\in P(X)$. By assumption $\sum_{i=1}^{s} l_i k_i$, $l_i\in\mathbb{C}(f_1, \ldots, f_n)$ are all distinct. By taking $l_i\in\mathbb{C}[f_1, \ldots, f_n]$ of degree $\leqslant p$, p sufficiently large, it follows from

$$\dim\{l\in\mathbb{C}[f_1, \ldots, f_n] : \deg l\leqslant p\}\geqslant cp^n$$

that:

$$d_{p+t} \geqq cp^n . s \text{ for } p \text{ large enough}$$

where $t = \max\{\deg(h_i)\}$. Hence $cp^n . s \leqq C'(p+t)^n$, and

$$s \leqq \lim_{p \to \infty} \frac{C'}{c} \frac{(p+t)^n}{p^n} = \frac{C'}{c},$$

proving the assertion that $R(X)$ is a finite extension field over $\mathbb{C}(f_1, \ldots, f_n)$ where $f_1, \ldots, f_n$ are holomorphic functions of polynomial growth which together define a local holomorphism at x_0. One can write $R(X) = \mathbb{C}(f_1, \ldots, f_n, g/h)$ for some $g/h = k \in R(X)$, $g, h \in P(X)$.

(5.2) In the following sections we consider the mapping $F : X \to \mathbb{C}^{n+2}$ defined by $F = (f_1, \ldots, f_{n+2})$, $f_{n+1} = g$, $f_{n+2} = h$. Since $R(X)$ is a finite extension field of $\mathbb{C}(f_1, \ldots, f_n)$ both g and h satisfy equalities of the form

$$f_k^{v_k} + \sum_{j=1}^{v_k - 1} R_j^k(f_1, \ldots, f_n) f_k^j = 0, \qquad k = n+1, \qquad n+2,$$

where $R_j(w_1, \ldots, w_n)$ are rational functions in $w_1, \ldots, w_n$. Let Z_0 be the subvariety of $\mathbb{C}^{n+2}$ *defined by*

$$Z_0 = \{w_1, \ldots, w_{n+2}) : w_k^{v_k} + \sum_{j=1}^{v_k - 1} R_j^k(w_1, \ldots, w_n) w_k^j = 0, \quad k = n+1,\ n+2\}.$$

Outside the union of the pole sets of R_j^k, $1 \leqq j \leqq v_k - 1$, $k = n+1$, $n+2$, the projection mapping $Z_0 \to \mathbb{C}^n$ given by the first n coordinates is a finite mapping. It follows that Z_0 is a subvariety of maximal dimension n. Let Z be the connected component of Z_0 containing $F(X)$.

6. An ideal problem on X and the existence of a "quasi-embedding" into an affine algebraic variety

(6.1) Suppose there is a holomorphic embedding of X onto some affine algebraic variety given by functions of polynomial growth on X, then the algebra $P(X)$ of such functions would be finitely generated. But the Siegels's Theorem we proved in § 5, in particular the fact that the quotient field $R(X)$ of rational functions is finitely generated, in général does not imply the finite generation of $P(X)$. Recall that

$$R(X) \cong \mathbb{C}(f_1, \ldots, f_n, g/h),$$

where $f_1, \ldots, f_n, g, h \in P(X)$ and $df_1 \wedge \ldots \wedge df_n$ is non-zero at a point x_0. We have therefore obtained a holomorphic mapping $F=(f_1, \ldots, f_n, g, h) : X \to \mathbb{C}^{n+2}$. We write $f_{n+1}=g$, $f_{n+2}=h$. We bypass the difficulty of proving directly that $P(X)$ is finitely generated. Instead, we examine how far $F : X \to \mathbb{C}^{n+2}$ is from an embedding and complete F to a proper embedding by adjoining a finite number of functions of polynomial growth. As stated in the introduction, this process involves, among other things, two finiteness theorems, one on the number of irreducible hypersurfaces missed by the mapping F, the other on the number of blow-ups necessary to resolve the singularities of the mapping. As before, we shall always assume X to be of dimension $n \geqslant 2$.

The example $F_0=(z_1, z_1 z_2 - 1) : \mathbb{C}^2 \to \mathbb{C}^2$ illustrates the type of degenerary of the mapping $F: X \to \mathbb{C}^{n+2}$. The field $R(X)$ of rational functions on $\mathbb{C}^2$, can be generated by $\{z_1, z_1 z_2 - 1\}$. F_0 is an affine blow-down which maps the entire Z_2-axis to the point $(0, -1)$; it is otherwise injective and gives a biholomorphism $\mathbb{C}^2 - (z_2 - \text{axis}) \to \mathbb{C}^2 - (w_2 - \text{axis})$. F_0 misses precisely $(w_2 - \text{axis}) - \{(0, -1)\}$. Another type of degenerary can be seen from the ampping $F_1 : \mathbb{C} \to \mathbb{C}^2$ below. The field of rational functions on $\mathbb{C}$, $R(\mathbb{C})$, can be generated by $\{z^2, z^3\}$. The mapping $F_1 : \mathbb{C} \to \mathbb{C}^2$ defined by $F_1(z)=(z^2, z^3)$ is an injective holomorphic mapping from $\mathbb{C}$ onto the subvariety Z of $\mathbb{C}^2$ defined by $Z=\{(w_1, w_2)=w_1^3=w_2^2\}$ with an isolated singularity. F_1 is degenerate at the single point $(0, 0)$.

In the above examples, one can adjoin polynomials to complete the given mapping to a proper embedding. In the case of

$$F_0=(f_1, f_2)=(z_1, z_1 z_2 - 1),$$

one adjoins $f_3=z_2=(f_2+1)/f_1$. This extra function f_3 can be recovered in the following way. F_0 misses the origin. One can solve the equation $f_1 g_1 + f_2 g_2 = 1$ from the Nullstellensatz. An explicit solution is given by:

$$z_1(z_2)+(z_1 z_2 - 1)(-1)=1$$

$f_3=z_2=(f_2+1)/f_1$ is given as one of the g_i's. As a function of the coordinates w_1, w_2 of the target space f_3 is a function whose pole set almost lies outside $F(X)$. It intersects $F_0(\mathbb{C}^2)$ at the single point $(0, -1)$. The mapping $F_1 : \mathbb{C} \to \mathbb{C}^2$, $F_1=(f_1, f_2)=(z^2, z^3)$ can on the other hand be completed to a proper embedding by adjoining $f_3=z=f_2/f_1$ to "smooth out" the isolated singularity $(0, 0)$ of Z.

Of course, one can argue in case of such examples by adjoining step-by-step polynomials to obtain a mapping of rank n everywhere. But this involves the knowledge that every algebraic subvariety of $\mathbf{C}^n$ has only a finite number of irreducible branches. We shall prove the analogous statement for our Kähler manifold X in § 7 and § 8 and use it to desingularize our mapping F.

In this section we shall first prove that $F: X \to \mathbf{C}^{n+2}$ is in a certain sense "almost injective" and "almost surjective".

(6.2) Almost injectivity of $F: X \to \mathbf{C}^{n+2}$

Proposition. — *Let $F=(f_1, \ldots, f_{n+2}): X \to \mathbf{C}^{n+2}$ be the holomorphic mapping defined above and let Z be the connected component of the subvariety defined by $(f_1, \ldots, f_{n+2})$ as in (5.2). Then, there exists a subvariety V_0 of such that $F|_{X-V_0}: X \to Z$ is an injective locally biholomorphic mapping. Moreover, Z is irreducible.*

Proof. — The arguments in the proof of the Siegel's Theorem (5.1) show that the algebra $P(X)$ separates points on X. Let $x_j \in X$, $j=1, 2$ be different points such that $F(x_i)$ are smooth of Z and $df_{i_1} \wedge \ldots \wedge df_{i_n}(x_j) \neq 0$, $1 \leqslant i_1 < \ldots < i_n \leqslant n+2$. Then, locally at x_1, x_2, holomorphic functions on X can be given by holomorphic functions of $w_1, \ldots, w_n$, where $(w_1, \ldots, w_{n+2})$ are coordinates of the target space $\mathbf{C}^{n+2}$. It follows from the fact that $P(X)$ separates points that $F(x_1) \neq F(x_2)$. Let V_0 be the union of the branching locus of F and $F^{-1}(\mathrm{Sing}(Z))$. Then, F is injective and locally biholomorphic on $X-V_0$. Since $F(X) \subset \overline{F(X-V_0)}$, $F(X)$ lies in an irreducible component. By definition (in (5.2)) Z is irreducible.

(6.3) Almost surjectivity of F—an ideal problem on X

In order to show that F can miss at most a finite number of irreducible branches of Z we proceed as in the examples to solve ideal problems on X. The solutions of analogues of $f_1 g_1 + f_2 g_2 = 1$ (as in the example in (6.1)) are then rational functions in the image coordinates $(w_1, \ldots, w_{n+2})$. But since they are holomorphic functions on X the pole set must be disjoint from $F(X-V_0)$. In practice we shall only be able to solve the ideal problem at points of $Z-F(X)$ outside a subvariety S containing the singularities of Z. We use an adaptation of Skoda's estimates on the ideal problem [25] to complete Kähler manifolds. Continuing this way we shall recover hypersurfaces H_j of Z which almost lie outside $F(X)$ in the sense that

$$H_j \cap F(X) = H_j \cap F(V_0).$$

That such a process must terminate in a finite number of steps would follow from estimates of degrees of g_i's and the fact that the dimension of $f \in P(X)$ of degree $\leqslant$ const. is finite.

In order to solve ideal problems using estimates of SKODA [25] it would be necessary to establish the following estimate which gives a lower bound of the proximity of $F(x)$ to a point b outside $F(X)$ on Z, in terms of geodesic distances on X. It would be necessary to assume that b lies outside some subvariety S of Z.

PROPOSITION (6.3.1). — *On Z there exists an algebraic subvariety S such that for all $b \in Z - S - F(X)$*

$$\operatorname{dist}(F(x), b) > \frac{C(b)}{R^k} R(x) = d(x_0; x)$$

where dist denotes the Euclidean distance in $\mathbb{C}^{n+2}$, $C(b)$ is a constant depending on b and k is a constant depending only on the sum of degrees of $f_1, \ldots, f_{n+2}$.

Proof. — We prove the proposition by solving for the equation $F(y) = z$ for points z on Z sufficiently close to $F(x)$. This amounts to estimating the vector fields obtained by inverting $df_{i_1}, \ldots, df_{i_n}$ for some $(i_1, \ldots, i_n)$, $1 \leqslant i_1 < \ldots < i_n \leqslant n+2$. The algebraic subvariety S, which contains all singular points of Z, will be determined later. Let first $b \in Z - F(X)$ be a smooth point of Z and choose $(i_1, \ldots, i_n)$, $1 \leqslant i_1 < \ldots < i_n \leqslant n+2$ such that $dz_{i_1} \wedge \ldots \wedge dz_{i_n}(b) \neq 0$. Let $x \in X$, $d(x_0, x) = R$ and

$$F(x) = (f_1(x), \ldots, f_{n+2}(x))$$

be inside a fixed open neighborhood N of b in Z such that the mapping $G = (f_{i_1}, \ldots, f_{i_n}) : X \to \mathbb{C}^n$ is a biholomorphism on a neighborhood of $F^{-1}(\bar{N})$. In order to prove the proposition it suffices to solve the equation:

$$G(y) = (z_{i_1}, \ldots, z_{i_n})$$

with:

$$0<\left|G(x)-(z_{i_1}, \ldots, z_{i_n})\right| \leqslant C(b) R^{-k}$$

for some constant $C(b)$, $k>0$.

Let $W_1, \ldots, W_n$ be meromorphic vector fields on X defined by:

$$\langle W_p, df_{i_q}\rangle=\delta_{pq}.$$

At a point $x \in X$ where $df_{i_1} \wedge \ldots \wedge df_{i_n} \neq 0$, W_p is simply $\partial/\partial f_{i_p}$ when $(f_{i_1}, \ldots, f_{i_n})$ is regarded as a local system of holomorphic coordinates. One can invert the holomorphic mapping G in a neighborhood of $G(x)$ by tracing integral curves of real and imaginary parts of the vector fields W_p, which are holomorphic in a neighborhood of x.

By the Cramer's rule have:

$$\| W_p \| \leqslant \frac{1}{\| df_{i_1} \wedge \ldots \wedge df_{i_n} \|} \cdot n \times (\sup_{1 \leqslant q \leqslant n} \| df_{i_1} \wedge \ldots \wedge \widehat{df_{i_q}} \wedge \ldots \wedge df_{i_n} \|)$$

where all norms are measured in terms of the given Kähler metric on X and $\widehat{df_{i_p}}$ means the omission of df_{i_p} in taking wedge products. By the gradient estimate of harmonic functions $\| df_{i_1} \wedge \ldots \wedge \widehat{df_{i_q}} \wedge \ldots \wedge df_{i_n} \|$ grows at most polynomially. Let η be a holomorphic n-vector field on X obtained by solving $\bar\partial$ with L^2-estimates using the weight function ku, $k>0$ sufficiently large. Since $i\,\partial\bar\partial u = \mathrm{Ric}$, and:

$$i\,\partial\bar\partial \log \| \eta \| \geqslant -\mathrm{Ric},$$

we have:

$$i\,\partial\bar\partial\,(\log \| \eta \| + u) \geqslant 0.$$

From the estimates of u and the sub-mean value inequality, it follows that $\log \| \eta \|$ grows at most logarithmically. The holomorphic function $h_{i_1, \ldots, i_n}$ defined by:

$$h_{i_1, \ldots, i_n} = \langle df_{i_1} \wedge \ldots \wedge df_{i_n}, \eta \rangle$$

is of polynomial growth. We have the estimate:

$$\| W_p \| \leqslant \frac{\| \eta \|}{| h_{i_1}, \ldots, {}_{i_n} |} \cdot n$$
$$\times (\sup_{1 \leqslant q \leqslant n} \| df_{i_1} \wedge \ldots \wedge df_{i_q} \wedge \ldots df_{i_n} \|)$$
$$\leqslant \frac{C R^{k_0}}{| h_{i_1}, \ldots, {}_{i_n} |} \quad \text{for some } C,\ k_0 > 0.$$

From the Siegel's theorem for $R(X)$ $h_{i_1}, \ldots, {}_{i_n}$ can be expressed on Z as a rational function, i. e.,

$$h_{i_1}, \ldots, {}_{i_n}(x) = H_{i_1}, \ldots, {}_{i_n}(F(x))$$

for some rational $H_{i_1}, \ldots, {}_{i_n}$ on Z and for all $x \in X$ outside a subvariety. Now we define S to be the union of zero-sets of all $H_{i_1}, \ldots, {}_{i_n}$ and $\mathrm{Sing}(Z)$. Then, for $F(x)$ sufficiently close to b, we have the estimate:

$$\| W_p \| \leqslant C_0(b) R^{k_0},$$

for some constant $C_0(b)$ depending on $b \in Z - S$.

(Here, of course, W_p is defined by a specific choice of $(f_{i_1}, \ldots, f_{i_n})$.) Consider the real vector field:

$$v = \sum_{p=1}^{n} \alpha_p (2 \operatorname{Re}(W_p) + \sum_{p=1}^{n} \beta_p (2 \operatorname{Im}(W_p))$$

where:

$$\sum_{p=1}^{n} (|\alpha_p|^2 + |\beta_p|^2) = 1.$$

v is the pull-back under G of the real vector field

$$\sum_{p=1}^{n} (\alpha_p \partial / \partial x_{i_p} + \beta_p \partial / \partial y_{i_p})$$

defined on C^n.

To solve for:

$$G(y) = (z_{i_1}, \ldots, z_{i_n})$$

with:

$$0 \leqslant |G(x)-(z_{i_1}, \ldots, z_{i_n})| \leqslant C(b) R^{-k}$$

for some constant $C(b)$, $k>0$ it suffices to show that the integral curves $\gamma_v(t)$ defined by v with initial values $\gamma_v(0)=x$ can be defined for $0 \leqslant t \leqslant C(b) R^{-k}$. Consider the ordinary differential equation:

$$\begin{cases} \dfrac{dr}{dt} = C_0(b) r^{k_0} \\ r(0) = R(x). \end{cases}$$

This equation admits a finite solution for $0 \leqslant t \leqslant c(x)$ with:

$$c(x) = \frac{k_0 - 1}{C_0(b)} R(x)^{-k_0+1} \quad \text{assuming } k_0 > 1.$$

By comparing $R(\gamma_v(t))$ with r we complete the proof of Proposition (6.3.1).

Now we are ready to formulate and prove the "almost surjectivity" of the holomorphic mapping $F: X \to \mathbb{C}^{n+2}$.

PROPOSITION (6.3.2). — *There exists an algebraic subvariety T of Z such that $F(X)$ contains $Z-T$. Furthermore one can choose T such that F maps $X-V$ biholomorphically onto $Z-T$, for some subvariety V of X containing the branching locus of F.*

Proof. — Recall first there exists an algebraic subvariety S containing the singularities of Z such that the estimate of (6.2.1) hold Let now b be a point of $Z-S$ lying outside the image of F. For each such point there exists $(i_1, \ldots, i_n)$, $1 \leqslant i_1 < \ldots < i_n \leqslant n+2$ such that the projection map $Z \to \mathbb{C}^n$ given by $(z_1, \ldots, z_{n+2}) \to (z_{i_1}, \ldots, z_{i_n})$ is non-degenerate at b. We shall denote this projection map by π_I, $I=(i_1, \ldots, i_n)$. $\pi_I^{-1}(\pi_I(b))$ consists of $\leqslant M$ points, for some M independent of b and I. Let h_b be a holomorphic function on $\mathbb{C}^{n+2}$ such that $h_b(b)=1$, $h_b(w)=1$ for $w \in \pi_I^{-1}(\pi_I(b)) - \{b\}$. By interpolation such a polynomial can be chosen of degree $\leqslant M$. We now solve on X the ideal problem

$$(f_{i_1} - b_{i_1}) g_1 + \ldots + (f_{i_n} - b_{i_n}) g_n = (h_b \circ F)^{n+2}$$

where $b=(b_1, \ldots, b_n)$.

By Proposition (6.2.1) there exists a constant $k>0$ independent of b such that

$$\operatorname{dist}(F(x), b) \geq \frac{C(b)}{R^k}$$

with $C(b)$ possibly depending on b. Recall that u is the solution of the Poincaré-Lelong equation $i\partial\bar{\partial}u = \text{Ricci}$ form obtained by the method of [17], which satisfies by an intermediate estimate of the basic inequality

$$u(x) \geq \text{const.} \log d(x_0; x) \quad \text{for} \quad d(x_0; x) \textit{ sufficiently large}$$

The opposite inequality

$$u(x) \leq \text{const.} \log d(x_0; x) \quad \text{for} \quad d(x_0; x) \text{ sufficiently large}$$

is a consequence of estimates of the Green kernel.

The function $h_b(F(x))$ is a holomorphic function on X is of polynomial growth and of degree $\leq M \max_{1 \leq i \leq n+2} \deg(f_i)$. Since $h_b(w)=0$ on $\pi_I^{-1}(\pi_I(b)) - \{b\}$

$$\frac{|h_b|^{2n+4}}{(\sum_{k=1}^{n} |z_{i_k} - b_{i_k}|^2)^{\alpha n+1}}$$

is bounded near $\pi_I^{-1}\pi_I(b) - \{b\}$ for $\alpha>1$ small enough. From $\operatorname{dist}(F(x), b) \geq C(b)/R^k$ it follows that there exists some positive constant K_1 such that

$$\int_X \frac{|h_b \circ F|^{2n+4}}{(\sum_{1 \leq k \leq n} |f_{i_k} - b_{i_k}|^2)^{\alpha n+1}} e^{-K_1 u} < \infty \text{ for some } \alpha > 1$$

By the estimate of Skoda, adapted to complete Kähler manifolds, there exists a solution of $(g_i, \ldots, g_n)$

$$(f_{i_1} - b_{i_1}) g_1 + \ldots + (f_{i_n} - b_{i_n}) g_n = (h_b \circ F)^{n+2}$$

such that for some $\alpha > 0$ fixed and independent of b,

$$\int_X \frac{\sum_{1<k<n} |g_k|^2}{(\sum_{1 \leq k \leq n} |f_{i_k} - b_{i_k}|^2)^{\alpha n}} e^{-K_1 u} \leq C_\alpha \int_X \frac{|h_b . F|^{2n+4}}{(\sum_{1 \leq k \leq n} |f_{i_k} - b_{i_k}|^2)^{\alpha n+1}} e^{-K_1 u} < \infty.$$

Since f_i are of polynomial growth, by the upper bound of u it follows that there exists a constant $K_2 > 0$ such that

$$\int_X (\sum_{1 \leq k \leq n} |g_k|^2)\,(d\,(x_0;\,x)+1)^{-K_2} < \infty.$$

By the sub-mean value inequality, and the fact that $\sum_{1 \leq k \leq n} |g_k|^2$ is plurisubharmonic (hence subharmonic in any Kähler metric), it follows that for $1 \leq k \leq n$, g_k is a function of polynomial growth on X of degree $\leq K$ independent of b. Consider the equation

$$(f_{i_1} - b_{i_1})g_1 + \ldots + (f_{i_n} - b_{i_n})g_n = (h_b \circ F)^{n+2}$$

as an equation on the subvariety Z. By the Siegel Theorem of § 4, g_k, $1 \leq k \leq n$ are rational functions of $f_1, \ldots, f_{n+2}$. Considered as rational functions of the coordinates $(w_1, \ldots, w_{n+2})$ of the target space $\mathbf{C}^{n+2}$, at least one of g_k must have a pole at b, otherwise

$$(w_{i_1} - b_{i_1})g_1 + \ldots + (w_{i_n} - b_{i_n})g_n = h_b^{n+2}$$

would yield a contradiction at b, since $h_b(b) = 1$. Let now T_1 be the union of the pole sets of $g_1, \ldots, g_n$ on Z. If F misses any point outside $T_1 \cup S$ we can proceed by choosing $b' \in Z - (T_1 \cup S \cup F(X))$ to construct holomorphic functions $g'_1, \ldots, g'_n$ on X of degree $\leq K$. At least one of g'_k must have a pole at b'. Proceeding this way we obtain holomorphic functions $p_j \in P(X)$ of degree $\leq K$ such that p_j, when considered as rational functions of $w_1, \ldots, w_{n+2}$, has a pole at b_j. More precisely, pole set of $p_j \not\subset \bigcup_{k<j}$ pole sets of p_k, and $b_j \in$ pole set of $p_j - (\bigcup_{k<j}$ pole sets of $p_k)$.

Such functions p_j must be linearly independent. In fact, the equation

$$c_1 p_1 + \ldots + c_j p_j = 0, \qquad c_j \neq 0$$

would be contradicted at b_j, where $p_1, \ldots, p_{j-1}$ are regular at b_j and p_j has a pole at b_j (which is a smooth point of Z). But the estimates in the Siegel's Theorem of § 4 gives

$$\dim \{ f \in P(X) = \deg(f) \leq K \} < \infty.$$

It follows that the whole process of locating "exceptional" subvarieties must terminate in a finite number of steps, say at $j = m$. Let $T = S \cup T_1 \cup \ldots \cup T_m$. Then $F(X) \supset Z - T$. Define $V = F^{-1}(T)$.

$F: X-V \to Z-T$ is injective, and hence bijective, by Proposition (6.2) ($V \supset V_0$). Since $Z-T$ is smooth, F maps $X-V$ biholomorphically onto $Z-T$.

Remark. — It is possible to further reduce T so that V can be taken to be V_0 as in Proposition (6.2), i. e., the union of the branching locus of F and F^{-1}(Sing Z). Later we can further reduce V precisely to the branching locus of some $\tilde{F}$ by normalizing the holomorphic mapping F (7.1).

To finish the proof of the Main Theorem, we have first to desingularize the holomorphic map $F: X \to Z$ which is almost a biholomorphism. The difficulty of the problem is to prove that the obvious process of desingularization will come to an end in a finite number of steps. The latter statement would be immediate if we know that every "algebraic" subvariety of X, i. e., one defined by functions in $P(X)$, has necessarily only a finite number of irreducible branches. Geometric difficulty arises in dimension $n \geqslant 3$ because we do not have sufficient control of the geometry of "algebraic" subvarieties of codimension $\geqslant 2$. In order to solve this problem of desingularization, we shall show in § 8 that a uniform bound on the multiplicities of irreducible branches of the zero-divisors of $f \in P(X)$ is sufficient for the finiteness of the desingularization process (affine blow-ups). This uniform bound on multiplicities will first be established in the next section (§ 7) using geometric comparison theorems. Then in the last section (§ 9) we shall show that the resulting holomorphic map, which is a biholomorphism onto some Zariski open subset of an affine algebraic variety, can be completed to a proper embedding. This will involve proving the rational convexity of the image and an application of the vanishing theorem of SERRE [20] in algebraic geometry.

7. A uniform bound on multiplicities of branches of an "algebraic" divisor

(7.1) Let $f \in P(X)$ and $[V] = i/2\pi\, \partial\bar{\partial} \log |f|^2$ be the zero divisor (or closed positive (1,1) integral current), counting multiplicity, defined by f. The basic inequality of § 3 shows that the multiplicity of $[V]$ at each point $x_0 \in V$ is bounded by a constant multiple of the degree of f with a constant possibly depending on x_0. In fact, because we have to insert a coordinate Euclidean ball at x_0 and estimate $[V] \wedge \mathrm{Ric}^{n-1}$, the constant depends on the choice of the ball and a lower bound of the eigenvalues of the Ricci tensor. In this chapter we shall derive the uniform version of the basic inequality. For the derivation we shall need bounds on the sectional curvature. On the other hand, the Ricci curvature tensor is only required to be positive semi-definite.

THEOREM (7.1). – *Let X be a Stein complete Kähler manifold of nonnegative Ricci curvature of dimension $n \geq 2$. Suppose, for a fixed base point $\tilde{x}_0$ and for $R(x) = d(x; \tilde{x}_0)$*

(i) $-(C_0/R^2) \leq$ *sectional curvature* $\leq C_0/R^2$

(ii) *Volume* $B(\tilde{x}_0; r) \geq cr^{2n}$, $c > 0$.

Let f be a holomorphic function on X of polynomial growth and $[V_i]$ be an irreducible non-compact branch of the zero divisor $[V]$, then,

$$\operatorname{mult}([V_i]) \leq C \deg(f),$$

where C is a constant independent of f and the particular branch $[V_i]$.

Remark. – (1) Here the multiplicity of $[V_i]$ is defined as the multiplicity at regular points of V_i. Multiplicities at singular points of V_i would be strictly larger.

(2) With obvious modifications of the proof, the theorem also applies to zero sets of holomorphic n-forms ω of polynomial growth. Morevover, the number of branches V_i are finite in both cases, as would be obvious from the proof.

We shall prove Theorem (7.1) by means of geometric comparison theorems. We shall show that the volume of $[V_i]$ in geodesic balls of radius r (with a fixed center) grows at least like $C' r^{2n-2}$ with a positive constant $C' = C_1$. multiplicity of $[V_i]$, $C_1 > 0$ a universal constant depending only on the geometry of X. A theorem of this nature on simply connected complete Kähler manifolds of nonpositive sectional curvature can be found in SIU-YAU [24]. There they only use the fact that V_i is a minimal subvariety. The proof makes heavy use of the theorem of Cartan-Hadamard, i. e., that the exponential map is a diffeomorphism at each base point. In our case one would need an estimate of the injectivity radius. It would also be necessary to take into account the positive upper bound of the sectional curvature tensor. In the next paragraph we shall prove a proposition on geodesic balls which is weaker than the desirable estimate of the injectivity radius but which is nonetheless sufficient for estimates of volume growths of complex subvarieties.

(7.2) PROPOSITION. — *Let X be a complete Kähler manifold of dimension $n \geqslant 2$ satisfying the hypothesis of Theorem* (7.1). *Then there exists a positive constant c_1 such that the conjugate radius at p with $d(p; x_0)=R$ is bounded from below by $c_1 R$ and that the local homeomorphism*

$$\exp_p = B_0(0, c_1 R) \to X$$

from the Euclidean ball $B_0(0, c_1 R)$ *in the tangent space $T_p(X)$ into X has at most k sheets, with an integer k independent of p.*

Remarks. — (i) By the conjugate radius we mean the largest possible s such that the exponential map $\exp_x : T_x(X) \to X$ is a local homeomorphism on $B_0(s)=B_0(0; s)$. We shall say that a local homeomorphism has at most k sheets if the preimage of every point is a finite set of at most k points.

(ii) The estimate here on the conjugate radius is standard. It is included for readers not familar with differential-geometric arguments.

Proof. — To prove the first part of Proposition (7.2), we make use of Rauch's Comparison Theorem (*cf.* CHEEGER-EBIN [5]). Let x_0 be a fixed base point, p be an arbitrary point on X such that $d(x_0; p)=R$. By the assumption (i) in the statement of the Main Theorem, on the geodesic ball $B(x; R/2)$ we have the inequality

$$\text{sectional curvature} \leqslant \frac{c_0}{(R/2)^2} = \left(\frac{R}{2\sqrt{c_0}}\right)^{-2}.$$

Hence, by Rauch's Comparison Theorem (comparing with the Euclidean sphere of radius $R/2\sqrt{c_0}$), there is no conjugate point of p along geodesic emanating from p of length $\pi R/2\sqrt{c_0}$ or $R/2$, whichever is smaller, proving the first part of Proposition (7.2) for any

$$c_1 \leqslant \min(1/2, \pi/2\sqrt{c_0}).$$

From now on we shall assume

$$c_1 < \min(1/2, \pi/2\sqrt{c_0})$$

and determine it later. For any point p on X with $d(x_0; p)=R$ let $\sum_{ij} g_{ij}(x)\, dx_i \otimes dx_j$ denote the pull-back of the Kähler metric on X under the exponential map $\exp_p$. We observe that the metric

$\sum_{ij} g_{ij}(x/c_1 R)\, dx_i \otimes dx_j$ is non-degenerate on the Euclidean unit ball and that the corresponding Laplacian operators are uniformly elliptic with elliptic constants independent of the point p.

In order to bound the number of sheets of $\exp_p : B_0(c_1 R) \to X$ for a suitable choice of c_1, we shall compare the Green kernels of $B(p; c_2 R)$, equipped with the Kähler metric on X, and $B_0(c_2 R)$ equipped with $\sum_{i,j} g_{ij}(x)\, dx^i \otimes dx^j$, for c_2 sufficiently small. We write the Green kernels as $G_{c_2 R}(x; y)$ and $\tilde{G}_{c_2 R}(\tilde{x}; \tilde{y})$ respectively. We assert that for c_2 sufficiently small and for $0 < c_1 < c_2/4$ there exists positive constants A and B independent of p such that whenever

$$x, y \in \overline{B(p; 2c_1 R)} \quad \text{and} \quad \tilde{x}, \tilde{y} \in \overline{B_0(2c_1 R)}$$

$$(\#) \qquad \frac{A}{d^{2n-2}(x; y)} \leqslant G_{c_2 R}(x; y) \leqslant \frac{B}{d^{2n-2}(x; y)}$$

$$\left(\frac{A}{\|\tilde{x}-\tilde{y}\|^{2n-2}}\right) \leqslant \tilde{G}_{c_2 R}(\tilde{x}; \tilde{y}) \leqslant \frac{B}{\|\tilde{x}-\tilde{y}\|^{2n-2}}$$

where $\|\tilde{x}-\tilde{y}\|$ is simply the Euclidean distance on $B_0(2c_1 R)$. We shall only need two of the four inequalities.

The first line of $(\#)$ appears in Mok-Siu-Yau [17, § 1.2] except that the constant B depends on the constant c_p appearing in

$$\text{Volume}\,(B\,(p; r)) \geqslant c_p r^{2n}.$$

We assert that c_p can in fact be chosen independent of p. By looking at the exponential map at p and observing that for the volume form $\sqrt{g}\, dx^1 \ldots dx^{2n}$ in normal geodesic coordinates, $\sqrt{g}$ decreases along each geodesic (by a standard comparison theorem and nonnegativity of Ricci curvatures), the ratio Volume $B(p; r)/r^{2n}$ is a decreasing function in r for p fixed. But by comparing very large geodesic balls the inequality

$$\text{Volume}\,(B\,(\tilde{x}_0; r)) \geqslant (c+\varepsilon)\, r^{2n}, \ \varepsilon > 0,$$

for one single base point $\tilde{x}_0$ implies the same inequality for large geodesic balls centered at p. Taking limits one has in fact

$$\text{Volume}\,(B\,(p; r)) \geqslant c r^{2n},$$

for all base points p and for all geodesic balls $B(p; r)$. The second line of $(\#)$ follows easily from the Harnack inequality of Moser for uniformly elliptic operators [18]. A proof of the given estimates can be found in STAMPACCHIA [26] using capacity functions.

We shall now finish the proof of Proposition (7.2). Let $x \in B(p; c_1 R)$, $\exp_p^{-1}(x) \cap B_0(c_1 R) = \{\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_q\}$ with $\tilde{x}_1 \neq \tilde{x}_2 \neq \ldots \neq \tilde{x}_q$. Let y_0 be a point on $\partial B(p; 2c_1 R)$. From the estimates above

$$G_{c_2 R}(x; y_0) \leqslant \frac{B}{d^{2n-2}(x; y)} \leqslant \frac{B}{(c_1 R)^{2n-2}}.$$

Let $\tilde{y}_0$ be a point on $\partial B_0(p; 2c_1 R)$ such that the segment joining the origin in $\tilde{y}$ corresponds to a minimal geodesic between p and y_0 under the exponential map. Then our assertion gives

$$\tilde{G}_{c_2 R}(\tilde{x}_i; \tilde{y}_0) \geqslant \frac{A}{d^{2n-2}(\tilde{x}_i; \tilde{y})} \geqslant \frac{A}{(3c_1 R)^{2n-2}} \geqslant \left(\frac{A}{(c_1 R)^{2n-2}}\right)\left(\frac{1}{3^{2n-2}}\right).$$

We write $\tilde{\Delta}$ for the Laplacian operator of $(B_0(c_2 R), \sum_{i,j} g_{ij}\, dx^i \otimes dx^j)$ which is simply the pull-back of the Laplacian operator Δ on X. Let $\exp_p^{-1}(x) \cap B_0(c_2 R) = \{\tilde{x}_1, \ldots, \tilde{x}_q, \tilde{x}_{q+1}, \ldots, \tilde{x}_r\}$ where $\tilde{x}_{q+1}, \ldots, \tilde{x}_r$ lie outside $B_0(c_1 R)$. Then, for $\tilde{x} = \tilde{x}_i$, $1 \leqslant i \leqslant r$ as above, on the ball $B_0(c_1 R)$,

$$-\tilde{\Delta}_{\tilde{y}} G_{c_2 R}(\exp_p(\tilde{x}), \exp_p(\tilde{y})) = (\delta_{\tilde{x}_1} + \delta_{\tilde{x}_2} \ldots + \delta_{\tilde{x}_q}) + (\delta_{\tilde{x}_{q+1}} + \ldots + \delta_{\tilde{x}_r})$$

where $\delta_{\tilde{x}_i}$ denotes the point mass at $\tilde{x}_i$.

On the other hand

$$-\tilde{\Delta}_{\tilde{y}} \sum_{i=1}^{r} G_{c_2 R}(\tilde{x}_i, \tilde{y}) = (\delta_{\tilde{x}_1} + \delta_{\tilde{x}_2} + \ldots + \delta_{\tilde{x}_q}) + (\delta_{\tilde{x}_{q+1}} + \ldots + \delta_{\tilde{x}_r}).$$

Since $G_{c_2 R}(\exp_p(\tilde{x}), \exp_p(\tilde{y}))$ is positive on $\partial B_0(c_2 R)$, possibly infinite at some points, by the maximum principle we have

$$G_{c_2 R}(\exp_p(\tilde{x}), \exp_p(\tilde{y})) \geqslant \sum_{i=1}^{r} \tilde{G}_{c_2 R}(\tilde{x}_i, \tilde{y}) \geqslant \sum_{i=1}^{q} \tilde{G}_{c_2 R}(\tilde{x}_i, \tilde{y}).$$

In particular, at the point $\tilde{y} = \tilde{y}_0$ (Recall that $\exp_p(\tilde{x}) = x$, $\exp_p(\tilde{y}_0) = y_0$),

$$\frac{B}{(c_1 R)^{2n-2}} \geqslant G_{c_2 R}(x; y_0) \geqslant \sum_{i=1}^{q} \tilde{G}_{c_2 R}(\tilde{x}_i; \tilde{y}_0) \geqslant \frac{qA}{3^{2n-2}(c_1 R)^{2n-2}}$$

which gives the bound

$$q \leqslant 3^{2n-2}\left(\frac{B}{A}\right)$$

on the number of sheets of $\exp_p : B_0(c_1 R) \to X$.

Remarks. – We remark here that the hypothesis in the Main Theorem on the curvature tensor of X, namely bisectional curvature >0, scalar curvature $< C/d^2(x_0; x)$, $C>0$ implies that $(-C')/d^2(x_0; x) <$ sectional curvature $< C'/d^2(x_0; x)$, $C'>0$, because at any point $x \in X$, the Riemannian sectional curvature of a 2 plane $p \wedge q$, p, $q \in T_x(X)$ (the tangent space at x), can be expressed in terms of holomorphic bisectional curvatures. More precisely, if z_i, $1 \leqslant i \leqslant n$ is a local system of coordinates, with $z_i = x_i + \sqrt{-1}\, x_{n+i}$ and R denotes the sectional curvature tensor, then in terms of the basis $\partial/\partial x_i$, $1 \leqslant i \leqslant 2n$, of $T_x(X)$, for $1 \leqslant i, j, k, l \leqslant n$

$$R_{ijkl} = \left\langle R\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)\frac{\partial}{\partial x_l}, \frac{\partial}{\partial x_k}\right\rangle$$

can be expressed in terms of $R_{i\bar{j}k\bar{l}}$, $1 \leqslant i, j, k, l \leqslant n$ by putting

$$\frac{\partial}{\partial x_i} = \frac{\partial}{\partial z_i} + \frac{\partial}{\partial \bar{z}_i}$$

$$\frac{\partial}{\partial x_{n+i}} = -\sqrt{-1}\left(\frac{\partial}{\partial z_i} - \frac{\partial}{\partial \bar{z}_i}\right), \text{ both for } 1 \leqslant i \leqslant n$$

Proposition (7.2) and hence Theorem (7.1) can be sharpened by dropping the negative lower bound for Riemannian sectional curvatures. To do this, we apply the arguments of [17, (1.1)] to the ball $B_0(c_2 R)$ with the metric given by the Kähler form $\exp_p^* \omega$, where $\omega =$ Kähler form of X. Namely, the inequality

$$\frac{A}{\|\tilde{x}-\tilde{y}\|^{2n-2}} \leqslant \tilde{G}_{c_2 R}(\tilde{x}; \tilde{y}) \leqslant \frac{B}{\|\tilde{x}-\tilde{y}\|^{2n-2}}$$

can be obtained by using comparison theorems for positive Ricci curvature, the isoperimetric inequality of Croke [8] and the iteration technique of Di Giorgi-Nash-Moser. In practice we shall work with $B_0(Kc_2 R)$ with K a large constant, $Kc_2 < \pi/\sqrt{C_0}$. One has, however, to be careful in obtaining the inequality $\Delta r_{\tilde{y}}^2 \leqslant 4n$, where $r_{\tilde{y}}(\tilde{x}) = d(\tilde{x}; \tilde{y})$ denotes metric distance on the incomplete Riemannian manifold $(B_0(Kc_2 R), (g_{ij}))$. This can always be done by choosing c_2 small enough, as long as

(i) there is a minimal geodesic in the incomplete metric $(B_0(Kc_2 R), (g_{ij}))$ realizing the infimum of lengths of curves joining $\tilde{x}$ and $\tilde{y}$ for $\tilde{x}$, $\tilde{y} \in B_0(c_2 R)$,

(ii) all such minimal geodesics lie within $B_0(Kc_2/2R)$.

The property (ii) guarantees that one can use the arguments of Cheeger-Gromoll [5] at points where there is more than one minimal geodesic.

(7.3) *Proof of Theorem* (7.1). – In order to prove Theorem (7.1) we argue as in § 3 by comparing both mult $([V_i], x)$ and $\deg f$ with the volume growth of V_i. For the lower bound of $\deg(f)$ we shall still use Proposition (3.1.2). In order to obtain the inequality

$$\operatorname{mult}([V_i], x) \leqslant C \deg(f)$$

it is sufficient to show that for each irreducible branch V_i of the zero set V of f, and for some $R_0 > 0$ fixed and for a fixed base point $\tilde{x}_0$

$$\text{Volume}\,([B(\tilde{x}_0; 2^{\nu+1} R_0) - B(\tilde{x}_0; 2^{\nu} R_0)] \cap V_i \geqslant C_0 (2^{\nu} R_0)^{2n-2},$$

for ν large enough with a constant C_0 independent of the holomorphic function f and the particular branch V_i.

In fact, using the notations of § 3 with

$$A_\nu([V_i]) = \frac{1}{(2^\nu R_0)^{2n-2}} \int_{D_\nu(R_0)} [V_i] \wedge \omega^{n-1}, \omega = \text{Kähler form on } X,$$

$D_\nu(R_0) = B(2R_0)$, and $D_\nu(R_0) = B(2^{\nu+1} R_0) - B(2^\nu R_0)$ for $\nu \geqslant 1$, the above estimate gives, for ν sufficiently large

$$A_\nu([V_i]) \geqslant C_0 \operatorname{mult}([V_i]).$$

But since $[V_i]$ is only part of $[V] = i/2\pi\, \partial\bar{\partial} \log |f|^2$, clearly Proposition (3.1.2) implies that, when we choose $\tilde{x}_0$ such that $f(\tilde{x}_0) \neq 0$,

$$\textstyle\sum_{\nu=0}^{\mu} A_\nu([V_i]) \leqslant C_3 \mu \deg(f) + C_4(f).$$

The inequality $A_\nu([V_i]) \geqslant C_0 \,\mathrm{mult}([V_i])$ implies immediately the desired bound

$$\mathrm{mult}([V_i]) \leqslant C \deg(f).$$

Suppose now $D_{\nu_0}(R_0) \cap V_i \neq \varnothing$. Since X is Stein, each V_i is noncompact (and connected). It must intersect $\partial B(\tilde{x}_0; R)$ whenever $R \geqslant 2^{\nu_0+1} R_0$.

Let now p_ν be a point of V_i lying on $\partial B(\tilde{x}_0, 3.2^{\nu-1} R_0)$. Such a point exists when ν is sufficiently large. $p_\nu \in D_\nu(R_0)$ and is mid-way between the boundaries. Let $c_1 < \min(1/2, \pi/\sqrt{C_2})$ be small enough that $B(p_\nu, c_1 R_\nu)$ lies in $D_\nu(R_0)$, where $R_\nu = d(\tilde{x}_0; p_\nu)$. Write $\exp_{p_\nu} = \pi_\nu$. Let $\tilde{V}_{i,\nu}$ be the subvariety of $B_0(c_1 R_\nu)$ defined by $\tilde{V}_{i,\nu} = \pi_\nu^{-1}(V_i) \cap B_0(c_1 R_\nu)$.

Note that here the Euclidean ball $B_0(c_1 R_\nu)$ is a spread over X (via the local homeomorphism π_ν) and hence inherits a complex structure. $\tilde{V}_{i,\nu}$ is then a complex subvariety of $B_0(c_1 R_\nu)$ with this complex structure. Since there is a uniform bound on the number of sheets of $exp_{p_\nu} = B_0(c_1 R_\nu) \to X$, we have

$$\mathrm{Volume}(B(p_\nu; c_1 R_\nu) \cap D_\nu(R_0)) \leqslant \frac{1}{k} \mathrm{Volume}(\tilde{V}_{i,\nu}),$$

where $\tilde{V}_{i,\nu}$ is measured in terms of $\pi_\nu^* \omega$, ω = Kähler form of X. To complete the proof of Theorem (7.1) we shall now use geometric comparison theorems to show

$$\mathrm{Volume}(\tilde{V}_{i,\nu}) \geqslant C_5 R_\nu^{2n-2}.$$

In order to have a convenient comparison for a lower bound of volumes of subvarieties, we use a model which is a piece of $\mathbb{P}^n$ with a multiple of the Fubini-Study metric. Precisely let $B \subset\subset \mathbb{C}^n \subset \mathbb{P}^n$ be a Euclidean ball with center at the origin, equipped with the Fubini-Study metric suitable normalized so that the Riemannian sectional curvatures are bounded from below by C_2. Let $z_1, \ldots, z_n$ be the usual complex coordinates on $\mathbb{C}^n$. Write $s^2 = \sum_{i=1}^n |z_i|^2$, and denote by μ the normalized Fubini-Study Kähler form on B. There exists positive constants C_6, C_7 such that

$$C_6 \mu \leqslant i \partial\bar{\partial} s^2 \leqslant C_7 \mu.$$

Let $\rho/2 < c_1 < \min(1/2, \pi/\sqrt{C_2})$ and $(B_0(\rho), \sum_{i,j} l_{ij} dx^i \otimes dx^j)$ be the metric (B, μ) in normal geodesic coordinates at the origin. Consider now the model $(B_0(\rho R_\nu/2), \sum_{i,j} g_{ij}\,dx^i \otimes dx^j)$ i. e., with the metric $\pi_\nu^*\omega$. The function $R_\nu(s(2x/R_\nu))^2$ is strictly plurisubharmonic on $B_0(\rho R_\nu/2)$ with the usual complex structure. By a standard comparison theorem for the complex Hessian (cf. SIU-YAU [24, (1.1)]), on

$$\left(B_0\left(\frac{\rho R_\nu}{2}\right), \sum_{i,j} g_{ij}\,dx^i \otimes dx^j\right),$$

with $\pi_\nu : B_0(\rho R_\nu) \to X$ a local biholomorphism, we have

$$C_8' \pi_\nu^* \omega \leqslant i\partial\bar\partial R_\nu^2\left(s\left(\frac{2x}{R_\nu}\right)\right) \leqslant C_8 \pi_\nu^* \omega$$

where C_8, C_8' are positive constants independent of ν. By definition the volume of $\tilde V_{i,\nu}$ is given by

$$\text{Volume}(\tilde V_{i,\nu}) = \int_{B_0(c_1 R_\nu)} [\tilde V_{i,\nu}] \wedge (\pi_\nu^*\omega)^{n-1}$$

$$\geqslant \int_{B_0(\rho R_\nu/2)} [\tilde V_{i,\nu}] \wedge \left(\frac{i}{C_8}\partial\bar\partial R_\nu^2\left(s\left(\frac{2x}{R_\nu}\right)\right)\right)^{n-1}.$$

We can now apply the integral formula of LELONG [13] to the function $R_\nu^2(s(2x/R_\nu))^2 = \varphi_\nu^2$. On $\partial B(\rho R_\nu/2)$, $\varphi_\nu = \alpha R_\nu$, with $\alpha > 0$ fixed.

Moreover, φ_ν is equal to 0 only at the origin in $B_0(\rho R_0/2)$. Hence, from the integral formula of LELONG,

$$\frac{1}{(\alpha R_\nu)^{2n-2}} \int_{B_0(\rho R_\nu/2)} [\tilde V_{i,\nu}] \wedge \left(\frac{i}{C_8}\partial\bar\partial R_\nu^2\left(s\left(\frac{2x}{R_\nu}\right)\right)^{n-1}\right.$$

$$= \lim_{r\to 0} \frac{1}{r^{2n-2}} \int_{\{\varphi_\nu < r\}} [\tilde V_{i,\nu}] \wedge \left(\frac{i}{C_8}\partial\bar\partial R_\nu^2\left(s\left(\frac{2x}{R_\nu}\right)\right)\right)^{n-1}$$

$$+ \int_{B_0(\rho R_\nu/2)} [\tilde V_{i,\nu}] \wedge \left(i\partial\bar\partial \log\left(s\left(\frac{2x}{R_\nu}\right)\right)\right)^{n-1}.$$

The second term corresponds to the "projectivized volume" of LELONG [13]. It is nonnegative because by the geometric comparison theorem for the complex Hessian (SIU-YAU [24, (1.1)]) the function $\log s^2(2x/R_\nu)$ is plurisubharmonic on the Riemann domain $B_0(\rho R_\nu/2)$ over X (with the spread given by $\pi_\nu = \exp_{p_\nu}$). The limit term is analogous to the density number of LELONG [13]. It equals C_9 mult$([\tilde{V}_{i,v}], 0) = C_9$ mult$([V_i])$ as can be seen by locally comparing with the standard potentials. More precisely, one can use the biholomorphic invariance of Lelong numbers (LELONG [14], SIU [23]), diagonalize the (1, 1) form $i\partial\bar{\partial} R_v^2 s^2(2x/R_v)$ at 0 and approximate level sets by Euclidean balls after appropriate linear coordinate transformations. With this we have completed the proof of the inequality

$$\text{Volume}(\tilde{V}_{i,v}) \geqslant C_5 R_v^{2n-2}$$

or

$$\text{Volume}([\tilde{V}_{i,v}]) \geqslant C_5 \,\text{mult}([V_i])\, R_v^{2n-2}$$

where the volume of $[\tilde{V}_{i,v}]$ has the obvious meaning of counting multiplicities. This completes the proof of Theorem (7.1). The analogous statement for holomorphic n-forms ω is obtained by integrating the Pincaré-Lelong equation $i/2\pi\,\partial\bar{\partial} \log\|\omega\|^2 = [V] + (1/2\pi)$ (Ricci form), where $[V]$ is the zero-divisor of ω.

Results of this chapter will now be applied in § 8 to show that the quasi-embedding $F: X \to Z$ into an affine algebraic variety constructed in § 6 can be desingularized in a finite number of steps to yield an embedding into some affine algebraic variety (which in general may not be surjective). We remark here that the proof of Theorem (7.1) clearly implies that V can only have a finite number of branches. However, the weaker statement in Thorem (7.1) together with a modified version of it (8.3), (#) 4) will be sufficient. We also remark here that the estimates of this chapter also yield lower estimates of complex analytic subvarieties of X of any dimension. But this does not prove that the number of branches of an "algebraic" subvariety is finite because we did not establish corresponding estimates of Green kernels on "algebraic" subvarieties.

8. Desingularization of the quasi-embedding by affine blow-ups

(8.1) In § 6 we constructed a holomorphic map $F: X \to Z$ into an affine algebraic variety which maps $X-U$ biholomorphically onto $Z-T$ for some subvariety U of X and some affine algebraic subvariety T of Z. U contains the branching locus of F but may in general be larger because there can be self-intersections (and also subvarieties on which the solution of the ideal problem fails (Proposition (6.3.2)). By normalizing the affine algebraic variety Z we remove the self-intersections. We show that the map $\tilde{F}$ resulting from the normalization is still defined by functions of polynomial growth on X by using the lemma on Riesz representation in § 3. From the existence theorem of $P(X)$ in § 2 it is clear how one can reduce the branching locus on X by adjoining additional functions of polynomial growth. In § 8.2 we shall prove a finiteness theorem on affine blow-ups by using a strengthened version of Theorem (7.1).

PROPOSITION (8.1). — *Normalization of an "algebraic" holomorphic map into affine algebraic varieties.*

Let $F : X \to Z$ be a holomorphic mapping of X into an affine algebraic variety defined by functions of polynomial growth. Let $\tilde{Z}$ be the affine algebraic normalization of Z. Then, the lifting $\tilde{F}$ of F into $\tilde{Z}$ is again defined by holomorphic functions of polynomial growth.

Proof. — Let $\mathrm{Reg}(Z)$ denote the Zariski dense subset of Z consisting of regular points. It is well known that the normalization $\tilde{Z}$ of Z (which one proves easily to be affine algebraic) can be obtained by taking $\tilde{Z}$ to be the closure of the graph of $\{Q_1, \ldots, Q_m\}$ on $\mathrm{Reg}(Z)$ where Q_i is a rational function which is holomorphic (or regular in the terminology of algebraic geometry) on $\mathrm{Reg}(Z)$. The lifting of $F : X \to Z$ to $\tilde{F} : X \to \tilde{Z}$ is then defined by $(f_1, \ldots, f_N, Q_1 \circ F, \ldots, Q_m \circ F)$ where $F = (f_1, \ldots, f_N)$ and $Q_i \circ F$ denotes the holomorphic extension of $Q_i \circ F$ on $F^{-1}(\mathrm{Reg}(Z))$ to the whole manifold X. To prove Proposition (8.1) it suffices to establish the following statement.

(*) Let h_1, h_2 be holomorphic functions on X of polynomial growth, $h_2 \not\equiv 0$. Suppose the function $g = h_1/h_2$ on ($X-$ zero set of h_2) can be extended to a holomorphic function on X, also denoted by g. Then, g is a holomorphic function of polynomial growth on X.

Proof of (*). — $\Delta \log |g|^2 = \Delta \log |h_1|^2 - \Delta \log |h_2|^2$ as measures. The trace

$$\frac{1}{4\pi} \Delta \log |g|^2 = \frac{i}{2\pi} \partial\bar{\partial} \log |g|^2 \wedge \omega^{n-1},$$

$\omega =$ Kähler form of X, is simply the integral measure on the zero set of g, counting multiplicity. Both $\log |h_1|^2$ and $\log |h_2|^2$ can be obtained by

Riesz representation as in (3.2), lemma, in the sense that with $h_1(\tilde{x}_0)$, $h_2(\tilde{x}_0)\neq 0$

$$\log|h_i(x)|^2=\lim_{R\to\infty}\int_{B(\tilde{x}_0\,;\,R)}[G_R(\tilde{x}_0;\,y)$$
$$-G_R(x;\,y)]\,\Delta\log|h_i(y)|^2\,dy+\log|h_i(\tilde{x}_0)|^2,$$
$$i=1,2,$$

with G_R=Green kernel on the geodesic ball $B(\tilde{x}_0;\,R)$.

It follows from the above that $\log|g|^2=\log|h_1|^2-\log|h_2|^2$ can be represented by the same integral formula. By Proposition (3.1.2)

$$\mu\deg(h_1)\geqslant C_1\left(\textstyle\sum_{v=0}^{\mu-1}A_v(h_1;\,R)-C_2(h_1)\right)$$

where $A_v(h_1,\,R)$ is the area of $[H_1]$=zero divisor of h_1, counting multiplicities. The inequality

$$\mu\deg(h_1)\geqslant C_1\left(\textstyle\sum_{v=0}^{\mu-1}A_v(g;\,R)-C_2(h_1)\right)$$

(with $A_v(g;\,R)$ defined as $A_v(h_1,\,R)$) holds since $\Delta\log|g|^2\leqslant\Delta\log|h_1|^2$ in the sense of measures. It follows readily from the Riesz representation formula for $\log|g|^2$ and the last inequality that $\log|g(x)|^2$ grows at most like const. $\log d(\tilde{x}_0;\,x)$, so that g has polynomial growth.

(8.2) Affine blow-ups of "algebraic" holomorphic maps into $\mathbb{C}^N$. — In this section we shall write $F_0:X\to Z_0$ for the quasi-embedding defined in § 6 ($F_0=F$) and denote by $\tilde{F}_0:X\to\tilde{Z}_0$ a normalization of F_0. If x is a point on the branching locus $\tilde{V}_0$ of $\tilde{F}_0$ we claim that $\tilde{F}_0$ must blow down a complex curve passing through x to a point. Otherwise $\tilde{F}_0$ would be a finite proper mapping from an open neighborhood of x to an open neighborhood of $\tilde{F}_0(x)$. Since $\tilde{F}_0$ is one-to-one on $X-\tilde{V}_0$, $\tilde{F}_0$ can be inverted because $\tilde{Z}_0$ is normal. From the existence theorem for $P(X)$ in § 2 one can adjoin a finite number of holomorphic functions of polynomial growth to get a map $F_1:X\to Z_1$ for some irreducible affine algebraic variety Z_1 of dimension $n=\dim_{\mathbb{C}}X\geqslant 2$, such that F_1 is locally biholomorphic at x. Hence, the locus of ramification V_1 of F_1 is strictly smaller than $\tilde{V}_0$. We crite $\tilde{F}_1:X\to\tilde{Z}_1$ for a normalization of F_1 and continue this way to get holomorphic mappings $F_i : X \to Z_i$, and their normalizations $\tilde{F}_i : X \to \tilde{Z}_i$ such that

$$\tilde{V}_0 \supsetneqq \tilde{V}_1 \supsetneqq \ldots \supsetneqq \tilde{V}_i \supsetneqq \ldots$$

where $\tilde{V}_i$ = locus of ramification of $\tilde{F}_i$.

If after a finite number of steps $\tilde{V}_m = \varnothing$, then $\tilde{F}_m : \tilde{X} \to Z_m$ is a local biholomorphism. Moreover, it is sufficient to show that $\tilde{V}_m$ consists at most of isolated points, in which case it must be empty because $\tilde{F}_m$ can be inverted by the normality of $\tilde{Z}_m$. By using the existence theorem for $P(X)$ as done in § 6, Proposition (6.2.2), $\tilde{F}_m$ is actually a biholomorphism of X onto its image. Arguments of Proposition (6.2.2) show that the image of $\tilde{F}_m$ can miss at most a finite number of irreducible subvarieties of $\tilde{Z}_m$, say $\tilde{T}_1^{(m)}, \ldots, \tilde{T}_q^{(m)}$. If $\tilde{F}_m(X) \cap \tilde{T}_i^{(m)} \neq \varnothing$, then it must interest $\tilde{T}_i^{(m)}$ in a non-empty open set because $\tilde{F}_m$ is open. We arrange $\tilde{T}_i^{(m)}$ so that $\tilde{F}_m(X) \cap \tilde{T}_i^{(m)} = \varnothing$ for $1 \leqslant i \leqslant p$ and $\tilde{F}_m(X) \cap \tilde{T}_i^{(m)} \neq \varnothing$ for $p+1 \leqslant i \leqslant q$. $\tilde{F}_m(X)$ is a Stein subset of $\tilde{Z}_m$ because X is Stein (§ 4) and $\tilde{F}_m$ maps X biholomorphically onto its image. By Hartog's extension theorem of holomorphic functions every holomorphic function on $\tilde{Z}_m - \bigcup_{i \leqslant q} \tilde{T}_i^{(m)}$ extends to $\tilde{Z}_m - \bigcup_{i \leqslant p} \tilde{T}_i^{(m)}$ (extension phenomenon of Thullen type). By Steinness of $\tilde{F}_m(X)$ we obtain $\tilde{F}_m(X) = \tilde{Z}_m - \bigcup_{i \leqslant p} \tilde{T}_i^{(m)}$. Moreover each $\tilde{T}_i^{(m)}$ must be of codimension one.

The difficulty of completing $F_0 : X \to Z_0$ to an embedding onto a Zariski dense open subset of some affine algebraic variety is therefore to show that, if done appropriately, the descending chain

$$\tilde{V}_0 \supsetneqq \tilde{V}_1 \supsetneqq \ldots \supsetneqq \tilde{V}_i \supsetneqq \ldots$$

must stop in a finite number of steps. We solve this difficulty by considering the vanishing order of holomorphic functions and n-forms of polynomial growth.

We state the main result of this chapter in the following proposition.

PROPOSITION (8.2). — *Let $F : X \to Z$ be the quasi-embedding of X into an affine algebraic variety Z defined as in Proposition* (6.2.2), *$F = (f_1, \ldots, f_N)$. Then, there exists a finite number of holomorphic functions f_i of polynomial growth, $N+1 \leqslant i \leqslant \hat{N}$, such that the holomorphic mapping $\hat{F} = (f_1, \ldots, f_N, f_{N+1}, \ldots, f_{\hat{N}}) : X \to \mathbb{C}^{\hat{N}}$ defines a biholomorphism of X onto some $\hat{Z} - \hat{T}$, where $\hat{Z}$ is an irreducible affine algebraic variety (possibly singular) and $\hat{T}$ is an algebraic subvariety of $\hat{Z}$ of pure codimension one.*

By the arguments given above, Proposition (8.2) is a consequence of the following proposition.

PROPOSITION (8.2)′. — *Let $F : X \to Z$ be the quasi-embedding of X into an affine algebraic variety Z defined as in Proposition* (6.2.2), *$F=(f_1, \ldots, f_N)$. Let $F' : X \to \mathbf{C}^{N'}$, $F'=(f_1, \ldots, f_N, f_{N+1}, \ldots, f_{N'})$ be obtained by adjoining a finite number of f holomorphic functions of polynomial growth. Then, the branching locus of F' has only a finite number of irreducible branches of positive dimension.*

Remarks. — We observe first that the proof of Theorem (7.1) shows immediately that V' has only a finite number of irreducible branches of codimension one (by considering upper and lower bounds on the volume growth of the zero set of $df_{i_1} \wedge \ldots \wedge df_{i_n}$). This settles Proposition (8.2)′ for dimension $n=2$.

Since the upper estimate on the volume growth of such hypersurfaces depends on estimates of the Green kernel on X, a direct generalization of the arguments of Theorem (7.1) to subvarieties of higher codimension would necessitate estimates of Green kernels on "algebraic" subvarieties of X. We will bypass this difficulty on Bezout estimates. Instead, all we need is a strengthened version of Theorem (7.1) which can be applied to most singular points of the branches V_i in the theorem.

For the proof of Proposition (8.2)′ in general we need to introduce some terminology. We say that a family of irreducible subvarieties E_i of dimension p is of bounded degree if each E_i is an irreducible component of the zero set of $n-p$ holomorphic functions $f \in P(X)$, of degree bounded independent of i. With this terminology we can formulate an essential step in the proof of Proposition (8.2)′.

PROPOSITION (8.2)″. — *Let $F' : X' \to \mathbf{C}^{N'}$ be as in Proposition* (8.2)′ *and let $V' = \bigcup_{i \in I} V_i$ be the decomposition of the branching locus V' of F' into irreducible components. Let I_0 be the set of all indices i for which V_i is of positive dimension. Then, there exists a family of irreducible analytic curves C_i, $i \in I_0$, of bounded degree such that for each $i \in I_0$, C_i intersects V_i at isolated points in $V_i - \bigcup_{j \neq i} V_j$.*

In order to give a better picture of the arguments we shall first give a proof of Proposition (8.2)″ for dimension 2 which can be generalized to higher dimensions with some technical modifications.

Proof of Proposition (8.2)″. — *Case of dimension* 2. It is sufficient to prove that proposition is valid for the set of all irreducible curves $\tilde{V}_i\,(i\in I)$, belonging to the branching locus $\tilde{V}$ of $\tilde{F}: X\to\tilde{Z}\subset \mathbf{C}^{\tilde{N}}$, which is a normalization of the quasi-embedding $F: X\to Z$ of Proposition (6.2.2). (Recall that $\tilde{V}$ contains no isolated points by the normality of $\tilde{Z}$). For each $i\in I$ pick a point x_i of $\tilde{V}_i$ which does not lie on other branches. We assert that $(*)_1$ Each f_j, $1\leqslant j\leqslant\tilde{N}$, must be constant on each branch $\tilde{V}_i$. To prove $(*)_1$ by contradiction, suppose $\tilde{V}_i$ is a branch on which some f_j, $1\leqslant j\leqslant \tilde{N}$, is not constant. Choose a neighborhood U of x_i relatively compact in X such that $\tilde{F}(\partial U\cap\tilde{V}_i)$ is disjoint from $\tilde{F}(x_i)$. By choosing U small enough, one can even assume that $\tilde{F}(\partial U)$ is disjoint from $\tilde{F}(x_i)$ since $\tilde{F}(X-\tilde{V})\cap\tilde{F}(\tilde{V})=\varnothing$. Hence, $\tilde{F}(x_i)$ is an interior point of the compact set $\tilde{F}(\bar{U})=K$. $\tilde{F}$ maps $U\cap\tilde{F}^{-1}(\operatorname{Int}K)$ onto Int K. Since $\tilde{F}$ can be inverted on $\tilde{Z}-\tilde{T}$, by the normality of $\tilde{Z}$, $\tilde{F}$ can be inverted on Int K. (Recall that U is relatively compact in X). This shows that $\tilde{F}$ is biholomorphic at x_i, contradicting with the fact that $x_i\in\tilde{V}$.

Let $f_1\equiv a_i$ on $\tilde{V}_i$, $i\in I$. By Theorem (7.1) the vanishing order m_i of f_1-a_i at x_i is bounded independent of i. We will further assume that x_i does not belong to another branch of the zero-set of f_1. For each $i\in I$, let w_i be an m_i-th root of f_1-a_i in a neighborhood U_i of i. Then, consider the expansion.

$$f_2=\textstyle\sum_{\nu=0}^{p_i} b_{i,\,\nu}\,w_1^{\nu}+b_{i,\,p_i+1}(x)\,w_1^{p_i+1}$$

where $b_{i,\,p_i+1}(x)$ is defined on U_i and non-constant on $U_i\cap\tilde{V}_i$ and $b_{i,\,\nu}$ are constants for $1\leqslant\nu\leqslant p_i$. We claim that $(*)_2\,p_i$ is bounded independent of $i\in I$.

$(*)_2$ is immediate consequence of Theorem (7.1). In fact, on U_i

$$df_1\wedge df_2=w_1^{p_i+1}\,df_1\wedge db_{i,\,p_i+1}$$

so that the vanishing order is at least p_i+1. But by Theorem (7.1) applied to holomorphic n-forms the vanishing order $df_1\wedge df_2$ at x_i is uniformly independent of i, proving the assertion $(*)_2$.

Now let $\zeta=b_{i,\,p_i+1}(x_i)$. The holomorphic function

$$\left(\frac{f_2-\sum_{\nu=0}^{p_i} b_{i,\,\nu}\,w_1^{\nu}-\zeta\,w_1^{p_i+1}}{w_1^{p_i+1}}\right),$$

is defined near x_i, vanishes at x_i but not at points on $\tilde{V}_i$ near x_i. It follows that the zero set C_i' must intersect $\tilde{V}_i$ at isolated points. Now consider the zero set D_i of the multi-valent function

$$(\#) \qquad z_2 - \sum_{\nu=0}^{p_i} b_{i,\nu}(z_1 - a_i)^{\nu/m_i} - \zeta(z_1 - a_1)^{(p_i+1)/m_i}$$

defined on the affine algebraic variety $\tilde{Z}$. By passing to a finite branched covering, it is easy to see that D_i is contained in the zero set of some polynomial $P_i(z_1, z_2)$ of degree bounded independent of i. Hence, there exists an irreducible branch C_i of $P_i(f_1, f_2)=0$ which contains x_i and intersects V_i at isolated points.

Remarks. — (i) By the zero-set of a multivalent function φ we mean the set of all points at which some branch of φ vanishes.

(ii) It is also apparent how one can construct polynomials $P_i(z_1, z_2)$ on $\mathbb{C}^2$ whose zero set contains D_i, with degrees bounded independent of i, directly from the multivalent functions in (#).

The technical complication in higher dimensions comes from branches V_i of codimension$\geqslant 2$. For the proof of Proposition (8.2)″ for higher dimensions we need the following modified version of Theorem (7.1).

Strengthened version of Theorem (7.1). — Notations as in Theorem (7.1) the estimate $\mathrm{mult}([V_i]) \leqslant C \deg(f)$ can be replaced by the estimate

$$\mathrm{mult}([V]; x) \leqslant C \deg(f)$$

for all points $x \in X$ except possibly for a discrete sequence of points $\{x_\nu\}$ on V, where the constant C is independent of x and f. The analogous estimate is also valid for holomorphic n-forms of polynomial growth.

Proof. — In the lower estimate of volume growth in Theorem (7.1), we consider the intersection of V_i, an irreducible component of V, with suitably large geodesic balls. In order to prove the strengthened statement for $x \in V$ it is sufficient to show that on each ringed domain $D_\nu = B(x; 2^{\nu+1} R_0) - B(x; 2^\nu R_0)$ there exists a point y_ν such that

$$\mathrm{mult}([V]; y_\nu) \geqslant \mathrm{mult}([V]; x).$$

Recall that the set of all points $y \in V$ such that

$$\mathrm{mult}([V]; y) \geqslant \mathrm{mult}([V]; x)$$

is an analytic subvariety E_x. (A purely analytic proof can be given by Siu's Theorem on Lelong numbers [23] and Thie's result [27]). Either x

is an isolated point of E_x or there is a positive dimensional branch E'_x of E_x passing through x. In the latter case E'_x must be non-compact since X is Stein, so that E'_x intersects each D_ν. The above inequality is then satisfied for any $y_\nu \in D_\nu \cap E'_x$. If x is an isolated point of E_x then we assert that E_z must be positive-dimensional for y sufficiently near x. In fact, if $\{z_\mu\}$ is a sequence of points on V such that z_μ is an isolated point of E_{z_μ} and $\mathrm{mult}([V], z_\mu)=c>0$ for all μ, then $\{z_\mu\} \subset V$ is a closed subvariety of $E(c)=\{z \in V : \mathrm{mult}([V]; z) \geqslant c\}$, so that $\{z_\mu\}$ must be discrete, from which our assertion follows easily. Hence the set of all $x \in V$ for which the estimate

$$\mathrm{mult}([V]; x) \leqslant C \deg(f)$$

can possibly fail is at most a discrete set.

With this we can continue with the proof of Proposition (8.2)″ in higher dimensions.

Proof of (8.2)″ *continued.* — Let $\tilde{V}_i$ be a k-dimensional branch of $\tilde{V}$, $0<k\leqslant n-1$. Let $x_i \in \tilde{V}_i$ be a regular point. In local coordinates $(w_1, \ldots, w_n)$ suppose $(w_1(x_i), \ldots, w_n(x_i))=0$ and $\tilde{V}_i$ be defined by $w_1=\ldots=w_{n-k}=0$. Consider the σ-process defined in local coordinates by

$$\Phi(u_1, \ldots, u_n)=(u_1, u_1 u_2, \ldots, u_1 u_{n-k}, u_{n-k+1}, \ldots, u_n),$$

Φ mapping some open neighborhood U of $0 \in \mathbf{C}^n$ into X, $\Phi(0)=x_i$. Let η be a holomorphic n-vector field on X of polynomial growth and consider the holomorphic function $f=\langle df_1 \wedge \ldots \wedge df_n, \eta \rangle$, where $f_1, \ldots, f_n$ are the first n-components of $\tilde{F}$, assumed to be of rank n. Then f vanishes identically on $\tilde{V}$, in particular on $\tilde{V}_i$. Denote the zero-divisor of f by W. By suitably choosing the values of η and its first derivatives at one point, one can certainly arrange $df \wedge df_1 \wedge \ldots \wedge df_{n-1}$ to be non-trivial. Let $P \in U$ be such that $u_1(P)=0$ and P is a regular point of the zero set of $f \circ \Phi$. Assume now $x_i \in \tilde{V}_i$ has been chosen such that the estimate

$$\mathrm{mult}([W]; x_i) \leqslant C \deg(f)$$

applies. By comparing the Taylor expansion of f and $f \circ \Phi$ at x_i (in the w-coordinates) and at 0 respectively, we see immediately that

$$\mathrm{mult}(\text{zero-divisor of } f \circ \Phi; 0) \leqslant 2\, \mathrm{mult}([W]; x_i) \leqslant 2C \deg(f).$$

In particular the vanishing order of $f \circ \Phi$ at P is at most $2C \deg(f)$. Now consider the n-tuple $(f_1 \circ \Phi, \ldots, f_n \circ \Phi)$ of functions on $\{u \in U : u_1=0\}$.

Suppose they have rank p (=rank of $(f_1, \ldots, f_n)$ on $\tilde{V}$ and that the rank at P is also equal to p, without loss of generality. Then, there is an open neighborhood U_1 of P in U such that the common zero set of $(f_1 \circ \Phi - \zeta_1, \ldots, f_n \circ \Phi - \zeta_n)$ intersects $\tilde{V}_1$ transversally. Call this level set $E(\zeta_1, \ldots, \zeta_n)$ (as a subset of U_1). On $E(\zeta_1, \ldots, \zeta_n)$ the multiplicity of $f \circ \Phi$ on $E_1(\zeta_1, \ldots, \zeta_n) \cap \{u_1 = 0\}$ is the same as the multiplicity of $f \circ \Phi$ (as a function on U) on the u_1-axis. We shall write $\hat{f}$ for $f \circ \Phi$ and $\hat{f}_i$ for $f_i \circ \Phi$ for short. Let the multiplicity of $\hat{f}$ on $\{u_1 = 0\}$ be m and let $\hat{f}^{1/m}$ denote some m-th root of $\hat{f}$ in a neighborhood of P in U. Suppose, after renumbering, $(f_1, \ldots, f_p)$ have rank p at P. Define $\zeta_i(u_1, \ldots, u_n) = \hat{f}_i(0, u_2, \ldots, u_n)$ on U_i for $1 \leqslant i \leqslant p$ and suppose it is possible to write $\hat{f}_1$ in the expansion

$$(\#) \quad \hat{f}_1 = \zeta_1 + \xi_1(\zeta_1, \ldots, \zeta_p)\hat{f}^{1/m} + \ldots + \xi_q(\zeta_1, \ldots, \zeta_p)\hat{f}^{q/m} + g_{q+1}\hat{f}^{(q+1)/m}$$

where $\xi_1, \ldots, \xi_q, g_{q+1}$ are holomorphic functions on U_1 and the functions ξ_i depends only on values of $\hat{f}_1, \ldots, \hat{f}_p$. Then clearly the vanishing order of $d\hat{f} \wedge d\hat{f}_1 \wedge \ldots \wedge d\hat{f}_{n-1}$ at P is at least q because $d\xi_i(\zeta_1, \ldots, \zeta_p) \wedge d\hat{f}_1 \wedge \ldots \wedge d\hat{f}_{n-1} \equiv 0$. However, by choosing P such that $\Phi(P) \in \tilde{V}_i \subset X$ does not belong to the bad set of $df \wedge df_1 \wedge \ldots \wedge df_{n-1}$ (the discrete point set for which the strengthened version of Theorem (7.1) does not apply), and by comparing the Taylor expansions at P and $\tilde{F}(P)$ respectively, we have: vanishing order of $d\hat{f} \wedge d\hat{f}_1 \wedge \ldots \wedge d\hat{f}_{n-1}$ at P = vanishing order of $df \wedge df_1 \wedge \ldots \wedge df_{n-1}$ at $\tilde{F}(P) + (n-k-1)$, where the extra constant $(n-k-1)$ comes from the Jacobian determinant of Φ. This means that with this choice of P the exponent q in the expansion (#) is bounded independent of $\tilde{V}_i$. The expansion (#) is obtained simply as follows. Let g_1 be the extension of $(\hat{f}_1 - \zeta_1)/\hat{f}^{1/m}$ to U_1. Suppose g is constant on each $E(\zeta_1, \ldots, \zeta_n) \cap \{u_1 = 0\}$. For U_1 sufficiently small this means that $g_1|_{E(\zeta_1, \ldots, \zeta_n) \cap \{u_1 = 0\}} = \xi_1(\zeta_1, \ldots, \zeta_p)$. Then one continues by letting g_2 be the extension of $(\hat{f}_1 - \zeta_1 - \xi_1(\zeta_1, \ldots, \zeta_p)\hat{f}^{1/m})/\hat{f}^{2/m}$ to U_1 and so on.

Now choose q such that g_{q+1} is not identically constant on generic level sets $E(\zeta_1, \ldots, \zeta_n) \cap \{u_1 = 0\}$. We write $E(\zeta_1, \ldots, \zeta_n; \eta_1)$ for the zero set of the extended holomorphic function

$$\frac{\hat{f}_1 - \zeta_1 - \sum_{i=1}^{q} \xi_i(\zeta_1, \ldots, \zeta_p)\hat{f}^{i/m}}{\hat{f}^{(q+1)/m}} - \eta_1.$$

Then for suitable choice of $\zeta_1, \ldots, \zeta_n$ and η_1, $E(\zeta_1, \ldots, \zeta_n; \eta_1)$ intersects $E(\zeta_1, \ldots, \zeta_n)$ transversally at a point P_1 sufficiently close to P. $E(\zeta_1, \ldots, \zeta_n; \eta_1)$ is of dimension $n-p-1$. For generic choices of P_1 the vanishing order of $f_2|_{E(\zeta_1, \ldots, \zeta_n; \eta_1)}$ at P_1 is the same as that of f_2 at P_1. Then, one performs the same process on $E(\zeta_1, \ldots, \zeta_n; \eta_1)$ for the function f_2 and so on until one obtains an analytic curve C_i' intersecting $\{u_1=0\}$ transversally, defined as $\overline{W-\{u_1=0\}}$, where W is the simultaneous zero set of holomorphic functions

$$\hat{f}_j-\zeta_j-\sum_{v=1}^{q_j} \xi_v^{(j)} \hat{f}^{v/m}-\eta_j \hat{f}^{(q_j+1)/m}$$

for some constants ζ_j, η_j and $\xi_v^{(j)}$, $1 \leqslant v \leqslant q_j$ and for $q_j \leqslant$const. independent of i for $1 \leqslant j \leqslant p-1$. The local analytic set $\Phi(C_i')$ passes through some $y_i \in \tilde{V}_i - \bigcup_{j \neq i} \tilde{V}_j$ (not necessarily the initial point x_i, but chosen as to avoid the bad set of $df \wedge df_1 \wedge \ldots \wedge df_{n-1}$ for applying the strengthened version of Theorem (7.1)).

Finally the last argument in case of dimension 2 can easily be adapted to show that $\Phi(C_i')$ is contained in some curve C_i defined by global holomorphic functions of polynomial growth, with a degree bounded independent of i. This completes the proof of Proposition (8.2)‘.

We are now ready to prove Proposition (8.2)′ and hence Proposition (8.2), the main result of this chapter.

Proof of Proposition (8.2)′. – Let $\tilde{F}: X \to \tilde{Z}$ be a normalization of the initial quasi-embedding $F: X \to Z$, and $F_p: X \to \tilde{Z}_p$ be obtained from $\tilde{F}$ by adjoining a finite number of $f \in P(X)$ and then by normalization. It suffices to prove that the branching locus $\tilde{V}^{(p)}$ of $\tilde{F}_p$ has only a finite number of irreducible branches $\tilde{V}_1^{(p)}, \ldots, \tilde{V}_k^{(p)}$ ($\tilde{V}^{(p)}$ contains no isolated points). We prove this by inverting the mapping $\tilde{F}_p$ along slices of algebraic curves of bounded degrees. Recall that by Proposition (8.2)‘, there exists an s such that for each $\tilde{V}_i^{(p)}$, $1 \leqslant i \leqslant p$, there exists a curve C_i which is an irreducible component of the common zero-set of $(n-1)$ $f \in P(X)$ of degree$\leqslant s$, such that C_i interesects $\tilde{V}_i^{(p)}$, passing through some point $y_i \in \tilde{V}_i^{(p)} - \bigcup_{j \neq i} \tilde{V}_j^{(p)}$. Let $\{f_1, \ldots, f_r\}$ be a basis of the vector space of all $f \in P(X)$ of degree$\leqslant s$, and consider the affine algebraic variety E_0 defined by

$$E^{(0)}=\{(z, \zeta) \in \tilde{Z}_p \times \mathbf{C}^{(n-1)r}: \textstyle\sum_{v=1}^{r} \zeta_{\alpha r+v} f_v(Z)=0 \text{ for all } \alpha,\ 0 \leqslant \alpha \leqslant n-2\}.$$

$E^{(0)}$ is thus a parametrized space of common zero-sets of $(n-1)$ holomorphic functions $f \in P(X)$ of degree $\leqslant s$. By Proposition (8.2)'', if $\tilde{V}^{(p)} \neq 0$ then there exists some $\zeta^{(0)} \in \mathbb{C}^{(n-1)r}$ such that $E^{(0)} \cap (\tilde{Z}_p \times \{\zeta^{(0)}\})$ contains an algebraic curve as a branch. It follows immediately that $E^{(0)}$ must then contain at least one branch of dimension $(n-1)r+1$. (Note that for ζ sufficiently near $\zeta^{(0)}$, $E^{(0)} \cap (\tilde{Z}_p \times \{\zeta\}) \neq \emptyset$). Let E be the union of all $((n-1)r+1)$-dimensional branches E_l of $E^{(0)}$ such that the generic fiber of the projection $E_l \to \mathbb{C}^{(n-1)r+1}$ is of dimension 1.

For each branch $\tilde{V}_i^{(p)}$ of $\tilde{V}^{(p)}$, there is now a point $(\tilde{F}_p(y_i); \zeta^{(i)}) \in E^{(0)}$ such that $y_i \in \tilde{V}_i^{(p)} - \bigcup_{j \neq i} \tilde{V}_j^{(p)}$. It is clear that $(\tilde{F}_p(y_i); \zeta^{(i)})$ belongs to some E_l in E. Let $\sigma : \tilde{E} \to E$ be the normalization of E. We regard σ as a mapping $(\sigma_1, \sigma_2) : \tilde{E} \to \tilde{Z}_p \times \mathbb{C}^{(n-1)r}$. The generic fibers of $\sigma_1 : \tilde{E} \to \tilde{Z}_q$ will now consist of smooth algebraic curves. Let $q \geqslant p$ be an integer and $\tilde{F}_q : X \to \tilde{Z}_q$ be obtained from $\tilde{F}_{q-1}$ by adjunction of $f \in P(X)$ and normalization, as before. We consider all possible directed sets $\{\tilde{F}_q, q \geqslant p\}$ obtained this way. Recall that there is an algebraic hypersurface (possibly singular) $\tilde{T}_p$ of $\tilde{Z}_p$ such that $\tilde{F}_p|_{X-\tilde{V}^{(p)}}$ is a biholomorphism of $X - \tilde{V}^{(p)}$ onto $\tilde{Z}_p - \tilde{T}_p$, and $\tilde{F}_p|_{\tilde{V}^{(p)}}$ is degenerate on each branch $\tilde{V}_i^{(p)}$, mapping it into $\tilde{T}_p$. The mapping $\tilde{F}_p^{-1} \circ \sigma_1$ is well-defined on $\tilde{E} - \sigma^{-1}(\tilde{T}_p \times \mathbb{C}^{(n-1)r})$. For $q \geqslant p$ consider the holomorphic mapping $\Phi_{pq} = \tilde{F}_q \circ \tilde{F}_p^{-1} \circ \sigma_1 : \tilde{E} - \sigma^{-1}(\tilde{T}_p \times \mathbb{C}^{(n-1)r}) \to \tilde{Z}_q$, which clearly extends to an $\tilde{N}_q$-tuple of meromorphic functions ($\tilde{Z}_q \subsetneq \mathbb{C}^{\tilde{N}_q}$). We are interested in the extension of Φ_{pq} across $\sigma^{-1}(\tilde{T}_p \times \mathbb{C}^{(n-1)r})$.

For a generic point $\zeta \in \mathbb{C}^{(n-1)r}$, $E \cap (\tilde{T}_p \times \{\zeta\})$ is the intersection of an algebraic curve with $\tilde{T}_p$. Let $\sigma^{-1}(\tilde{T}_p \times \mathbb{C}^{(n-1)r}) = \bigcup_{k=1}^{t} D_k$ be the decomposition of $\sigma^{-1}(\tilde{T}_p \times \mathbb{C}^{(n-1)r})$ into irreducible components.

Two possibilities can happen in the meromorphic extension of $\Phi_{pq} : \tilde{E} - \bigcup_{k=1}^{t} D_k \to \mathbb{C}^{\tilde{N}_q}$ across D_k; $1 \leqslant k \leqslant t$ fixed.

(i) For all possible $\tilde{F}_q : X \to \tilde{Z}_q \subsetneq \mathbb{C}^{\tilde{N}_q}$, $q \geqslant p$, Φ_{pq} extends holomorphic across generic points of D_k.

(ii) There exists some choice of $\tilde{F}_q : X \to \tilde{Z}_q$, $q > p$ such that the pole set of some component of Φ_{pq} contains $D_k \subset \tilde{E}$. (In this case D_k is necessarily of codimension one in $\tilde{E}$.)

In case of (i) actually Φ_{pq} extends holomorphically across $D_k - \bigcup_{l \neq k} D_l$.

We are going to define an algebraic subvariety D' of $\sigma^{-1}(\tilde{T}_p \times \mathbb{C}^{(n-1)r}) = \bigcup_{k=1}^{t} D_k$ by the following procedure: Suppose $\{D_1, \ldots, D_{t_0}\}$, $\{D_{t_0+1}, \ldots, D_t\}$ is the division of $\sigma^{-1}(\tilde{T}_p \times \mathbb{C}^{(n-1)r})$ into classes satisfying (i) and (ii) respectively. For each $k > t_0$ choose some $\tilde{F}_q : X \to \tilde{Z}_q$ such that the pole set of some component h of Φ_{pq} contains D_k. Let D'_k be the set of indeterminancy of h belonging to D_k. Then, define

$$D' = (\cup_{i=t_0+1}^{t} D'_k) \cup \mathrm{Sing}(\sigma^{-1}(\tilde{T}_p \times \mathbb{C}^{(n-1)r})).$$

We claim the following is true:

(#) For all but a finite number of $\tilde{V}_p^{(i)}$, the point $(\tilde{F}_p(y_i); \zeta^{(i)}) \in E$ defined above is such that $\zeta^{(i)} \in \sigma_2(D')$.

Given (#), it is then clear by induction how the proof of Proposition (8.2)′ is completed. To prove (#), we again consider the sets $\{D_1, \ldots, D_{t_0}\}$ and $\{D_{t_0+1}, \ldots, D_t\}$ separately. It suffices to prove, for each k, the statement:

$(\#)_k$ There exists at most one $\tilde{V}_i^{(p)}$ such that for the points $(\tilde{F}_p(y_i); \zeta^{(i)}) \in E$ defined as above, $\sigma_2^{-1}(\zeta^{(i)}) \cap (D_k - D')$ contains a point which corresponds to y_i.

Here, if $\sigma_2^{-1}(\zeta) \subset \tilde{E}$ is a smooth Riemann surface intersecting $\sigma_1^{-1}(\tilde{T}_p)$ at isolated points we say that $\xi \in \sigma_2^{-1}(\zeta) \cap (D_k - D')$ corresponds to $y \in \tilde{V}^{(p)}$ if the mapping Ψ defined on the slice $\sigma_2^{-1}(\zeta) - \sigma_1^{-1}(\tilde{T}_p)$ by $\Psi = \tilde{F}^{-1} \circ \sigma_1 : \sigma_2^{-1}(\zeta) - \sigma_1^{-1}(\tilde{T}_p) \to X$ extends holomorphically across ξ, with the value $\Psi(\xi) = y$. In general, one considers the normalization of one-dimensional branches of $\sigma_2^{-1}(\xi)$ intersecting $\sigma_1^{-1}(\tilde{T}_p)$ at isolated points, in which case the points ξ may correspond to several end-points $y \in \tilde{V}^{(p)}$.

Proof of $(\#)_k$. — (i) Let k be such that $1 \leqslant k \leqslant t_0$. To prove $(\#)_k$ suppose there exists some $\tilde{V}_i^{(p)}$ such that for some $\xi^{(i)}$ with $\sigma_2(\xi^{(i)}) = \zeta^{(i)}$, $\xi^{(i)} \in D_k$. Choose $\tilde{F}_q : X \to \tilde{Z}_q$, $q \geqslant p$ such that $\tilde{F}_q$ is locally biholomorphic at $y_i \in \tilde{V}_i^{(p)} - \cup_{j \neq i} \tilde{V}_j^{(p)}$. Let W_i be the algebraic subvariety of $\tilde{Z}_p$ that corresponds to $\tilde{V}_i^{(p)}$. Then $\Phi_{pq}(\xi^{(i)}) \in W_i$. By the definition of Φ_{pq}, $\Phi_{pq}(D_k - \cup_{l \neq k} D_l) \subset \pi_{pq}^{-1}(\tilde{T}_p)$, where $\pi_{pq} : \tilde{Z}_q \to \tilde{Z}_p$ is the natural projection map induced by $\mathbb{C}^{\tilde{N}_q} \to \mathbb{C}^{\tilde{N}_p}$. W_i is an irreducible component of $\pi_{pq}^{-1}(\tilde{T}_p)$.

It follows that for $\xi \in D_k$, $\Phi_{pq}(\xi) \in W_i$. Since $y_j \notin \tilde{V}_i^{(p)}$ for $j \neq i$, we have proved $(\#)_k$ for $1 \leqslant k \leqslant t_0$.

(ii) Suppose now $t_0 + 1 \leqslant k \leqslant t$. Let $\xi \in \sigma_2^{-1}(\zeta) \cap (D_k - D')$. Φ_{pq} was chosen such that some component h of Φ_{pq} has a pole at ξ. Let

$$\Psi = \tilde{F}^{-1} \circ \sigma_1 \big|_{\sigma_2^{-1}(\zeta) - \sigma_1^{-1}(\tilde{T}_p)}$$

and let Ψ be the mapping defined by lifting to the normalization $\tilde{C}$ of one-dimensional branches of $\sigma_2^{-1}(\zeta)$ intersecting $\sigma_1^{-1}(\tilde{T}_p)$ at isolated points. Obviously ξ cannot correspond to y_i since Ψ cannot be extended holomorphically across any point of $\tilde{C}$ which correspond to ξ under the normalization map, completing the proof of $(\#)_k$.

By an obvious inductive argument based on (#), one can step-by-step reduce the dimension of the parameter space of "algebraic curves" which gives the different components $\tilde{V}_i^{(p)}$ of $\tilde{V}^{(p)}$. In each step, normalization is needed to make sure that the set of indeterminancy is well-defined. This proves Proposition (8.2)′ and hence the main result Proposition (8.2).

9. Completion to a proper embedding onto an affine algebraic variety by techniques of algebraic geometry

(9.1) In the last chapter we proved that for the n-dimensional, $n \geqslant 2$, complete Kähler manifold X of positive bisectional curvature satisfying the geometric growth conditions of the Main Theorem, there exists a holomorphic mapping $G: X \to \mathbf{C}^N$ for some N, defined by holomorphic functions of polynomial growth such that G maps X biholomorphically onto a Zariski dense open subset of some affine algebraic variety Y, possibly singular. Moreover, the complement is of pure codimension 1. We denote here the complement of $G(X)$ by W, i. e., $G: X \to Y - W$ is a biholomorphism. If Y is non-singular, then it follows easily from the vanishing theorem of Serre that $Y - W$ is biregular to an affine algebraic variety (for any algebraic W of codimension one). However, in general this is not true if Y has singularities. There are well-kown examples in affine algebraic geometry due to Zariski (*cf.* GOODMAN [11]) such that the algebra of rational functions regular on $Y - W$, W of pure codimension one, is infinitely generated. Although in our case $Y - W$ is a manifold, there is no guarantee in the way we construct the embedding G that Y is non-singular. In general, $Y - W$ fails to be affine algebraic because divisors defined by W are not locally principle, i. e., W cannot in general be defined locally by a single polynomial. On the positive side, we prove.

THEOREM (9.1). — *Let Y be an affine algebraic variety, possibly singular, and let W be an algebraic subvariety of pure codimension one. Suppose $Y - W$ is rationally convex. Then $Y - W$ is biregular to an affine algebraic variety.*

Here a subset S of $\mathbb{C}^N$ is said to be rationally convex if, given any compact subset K of $\mathbb{C}^N$, there exists a point $z \in S$, a rational function f which is holomorphic on an open neighborhood of S, such that $|f(z)| > \sup_{x \in K} |f(x)|$.

In order to prove Theorem (9.1), we shall need the following proposition obtained by Runge approximation.

PROPOSITION (9.1). — *X is convex with respect to the algebra $P(X)$ of holomorphic functions of polynomial growth, i. e., given a compact subset K of X, there exists a compact subset K' of X containing K such that given $x \in X - K'$, there exists an $f \in P(X)$ such that $|f(x)| > \sup_K |f|$.*

In the case of the biholomorphic mapping $G: X \to Y - W$ this implies that $Y - W$ is rationally convex in Y because $P(X)$ is now precisely the pull-back under G of rational functions on Y which are regular on $Y - W$.

Proof of Proposition (9.1). — Recall that there exists a plurisubharmonic exhaustion function u solving $i\partial\bar{\partial}u = $ Ricci form, such that

$$C_1 \log R \leqslant u \leqslant C_2 \log R$$

for $R(x) = d(x_0; x)$ sufficiently large, with C_1, $C_2 > 0$. Given this, the proof of Proposition (9.1) is a standard application of L^2-techniques of Runge approximation.

Let K be a fixed compact subset of X and let x be a point on X such that $u(x) > \sup_K u$. Let U be a small coordinate open ball centered at x such that $U \cap K = \emptyset$, and let $z_1, \ldots, z_n$ be local holomorphic coordinate functions on U with $z_i(x) = 0$, $1 \leqslant i \leqslant n$. Let φ be a function smooth on X except for a logarithmic singularity at x, with compact support contained in U such that $\varphi = 2n \log(|z_1|^2 + \ldots + |z_n|^2)$ in a neighborhood of x in U. The function $ku + \varphi$ is strictly plurisubharmonic on X for k sufficiently large. Let now χ be a smooth cut-off function supported on U (i. e., $\operatorname{Supp}\chi \subset\subset U$) such that $\chi = 1$ near x. Let c be chosen such that $\sup Ku < c < \inf Uu$. Let v be a solution of $\bar{\partial}V = \bar{\partial}(\alpha\chi)$, $\alpha \neq 0$, satisfying

$$\int_X |\bar{\partial}v|^2 e^{-(ku-kc+\varphi)} \leqslant \int_X \frac{|\bar{\partial}(\alpha\chi)|^2}{C} e^{-(ku-kc+\varphi)} \text{ for some } C > 0.$$

Such a v exists because $\mathrm{Ric} > 0$ and $\bar{\partial}(\alpha\chi)$ has compact support. C depends here on the lower bound of eigenvalues of $\partial\bar{\partial}(ku+\varphi)$. Since $\bar{\partial}(v - \alpha\chi) = 0$ $f = v - \alpha\chi$ is holomorphic on X. By examining the singularity at x, $v(x) = 0$, so that $f(x) = \alpha$. The weight function $k(u-c) + \varphi$ tends to infinity on U when k approaches infinity, but is arbitrarily small on $K \subset \{u<c\}$ (approaching $-\infty$) when k is arbitrarily large. An application of the sub-mean value inequality then shows that for k large enough,

$$\sup_K |v| < |\alpha|.$$

Moreover, f is of polynomial growth from the sub-mean value inequality on X because of the upper estimate $u \leqslant C_2 \log R$ for R large enough.

Proof of Theorem (9.1). — We first give a proof of Theorem (9.1) under the assumption that the subvariety W can locally be defined by a single polynomial. In this case, rational convexity is not needed. More precisely, we assume that, given any $z \in W$, there exists a polynomial P on $\mathbb{C}^N$ such that $P(z)=0$ and that there exists an open neighborhood U of z in Y such that $\{z \in Y : P(z)=0\} \cap U = W \cap U$. (Clearly one can also assume that U is Zariski-open.) Fix $z_1 \in W$ and let P_1 be a polynomial as above. The rational function $1/P_1$ is regular on $Y-W-E_1$, where E_1 is an algebraic subvariety of Y not passing through z. Let now Q be a polynomial in $\mathbb{C}^N$ such that $Q(z_1)=1$ and Q vanishes on E_1. To find such a Q we appeal to the vanishing theorem of Serre: (One can also use the vanishing theorem of Kodaira or the L^2-estimates of $\bar\partial$ of Hörmander on $\mathbb{C}^N$.)

THEOREM (SERRE) [20]. — *Let M be a projective variety and L the positive hyperplane section line bundle on M. Let $\mathscr{F}$ be a coherent algebraic sheaf on M. Then, there exists an integer $m>0$ such that for all $\nu \geqslant 1$*

$$H^\nu(M, \mathscr{F} \otimes L^m)=0.$$

Here the cohomology can be either defined in terms of the algebraic coherent sheaf $\mathscr{F} \otimes L^m$ or the corresponding analytic coherent sheaf, because of GAGA (SERRE [21]).

For an algebraic variety V of $\mathbb{C}^N$, we shall denote by $\bar V$ the (Zariski) closure of V in $\mathbb{P}^N$. Now we prove the existence of a polynomial Q_1 as given above. Let $\mathscr{I}_{\bar E_1}$ denote the ideal sheaf of $\bar E_1$ and m_z be the maximal ideal sheaf at z_1. To find such a polynomial Q_1 it suffices to show that the restriction map of section modules

$$\Gamma(\mathbb{P}^n, L^m) \to \Gamma(\bar E_1 \cup \{z_1\}, \mathcal{O}_{\bar E_1 \cup \{z_1\}} \otimes L^m)$$

is surjective, where $\mathcal{O}_{\bar{E}_1 \cup \{z_1\}}$ denotes the reduced structure sheaf of $\bar{E}_1 \cup \{z\}$. From the short exact sequence on

$$0 \to \mathcal{I}_{\bar{E}_1} \otimes m_{z_1} \otimes L^m \to \mathcal{O}_{\mathbb{P}^n} \otimes L^m \to \mathcal{O}_{\bar{E}_1 \cup \{z_1\}} \otimes L^m \to 0$$

it suffices by passing to long exact sequences to prove that

$$H^1(\mathbb{P}^n, \mathcal{I}_{\bar{E}_1} \otimes m_{z_1} \otimes L^m) = 0,$$

which is valid because of the vanishing theorem of Serre. Clearly the function $g_1 = Q_1^s/P_1$ is regular on $Y - W$ for s large enough. Moreover $g_1 = Q_1^s/P_1(z) \to \infty$ as z approaches $U_1 \cap W$. It is now a standard argument how one can find a finite number of points z_i, $1 \leqslant i \leqslant t$, and corresponding functions $g_i = Q_i^s/P_i$ such that $(z_1, \ldots, z_N, g_1, \ldots, g_t)$ gives a proper embedding of $Y - W$ onto an affine algebraic subvariety of $\mathbb{C}^{N+t}$.

Assume now W is an arbitrary algebraic subvariety of pure codimension one and that $Y - W$ is rationally convex. We claim that there exist rational functions $g_1, \ldots, g_p$ regular on $Y - W$ such that if Y' is the closure of the graph of $(g_1, \ldots, g_p)$ on $Y - W$ (the closure being taken in $\mathbb{C}^{N+p}$), and W' is the part of Y' sitting above W under the projection map $\pi(z_1, \ldots, z_{N+p}) = (z_1, \ldots, z_N)$, then W' can be locally defined by a single polynomial on Y'. We shall first give the argument in the case of dimension 2.

Proof for dimension 2. — Let g_1 be a non-zero polynomial on $\mathbb{C}^N$ which vanishes on W. Let Z be the zero set of g_1 on Y and $Z = \bigcup Z_i$ be the decomposition of Z into irreducible components Z_i. Let Z_i, $1 \leqslant i \leqslant m$ be those branches which do not lie on W. By rational convexity of $Y - W$ it is easy to see (as we did in the proof of Theorem (8.3) for dimension 2) that there exist a finite number of polynomials $g_2, \ldots, g_p$ such that $(z_1, \ldots, z_N, g_1, g_2, \ldots, g_p)$ defines a proper embedding when restricted to each $Z_i - W$, $1 \leqslant i \leqslant m$. Recall that π is the coordinate projection map of $\mathbb{C}^{N+p}$ onto the first N components. Then, the function $z_{N+1} = g_1 \circ \pi$ on Y' vanishes precisely on $\pi^{-1}(Z) \supset W'$. However, for $1 \leqslant i \leqslant m$, $\pi^{-1}(Z_i - W)$ is a closed algebraic curve on Y' so that $\pi^{-1}(Z_i) \cap W' = \emptyset$. It follows that at each $z \in W'$, there is an open neighborhood U' such that $U' \cap W'$ is defined by z_{N+1}. This gives Theorem (9.1) for dimension 2.

Proof for arbitrary dimensions. — In arbitrary dimensions n we need an inductive argument in order to embed $Z_i - W$ properly for $1 \leqslant i \leqslant m$, where Z_i has the same meaning as above. We show now by induction that $(*)_k$

there exists a k-dimensional algebraic subvariety V_k of Y, a finite number of rational functions $(h_{k,1}, \ldots, h_{k,s_k})$ which are regular on $Y-W$, such that $(h_{k,1}, \ldots, h_{k,s_k})$ defines an embedding of $Y-W$ which is a proper embedding when restricted to $V_k - W$. The statement $(*)_n$ gives Theorem (9.1).

Clearly, by adjoining the coordinate functions $(z_1, \ldots, z_N)$, it is sufficient to find $(h_1, \ldots, h_s)$ such that the restriction to V_k is proper.

We define V_k inductively in descending dimensions as follows. V_n is defined to be Y. $V_{n-1} = \bigcup_{i \leq l \leq m} Z_l$, where Z_l have the same meaning as in dimension 2, i. e., for some polynomial g_1 vanishing on W, V_{n-1} is the union of those irreducible components of its zero set on Y which do not lie on W. Suppose V_{k+1} is defined. Let g_{n-k} be a polynomial vanishing on W such that g_{n-k} is not identically zero on any irreducible branch of V_{k+1}. Then, V_k is defined to be the union of those irreducible branches of the zero set of g_{n-k} which do not lie on W.

The arguments in case of dimension two clearly gives $(*)_1$. Suppose $(*)_k$ is true. Let Y_k be the closure of the graph of $(h_{k,1}, \ldots, h_{k,s_k})$ on $Y-W$ and W_k be the part of Y_k sitting above W. Let $\pi_k : Y_k \to Y$ be the natural coordinate projection. Then $\pi_k^{-1}(V_k) \cap W_k = \emptyset$ because $(h_{k,1}, \ldots, h_{k,s_k})$ is proper on V_k. On $\pi_k^{-1}(V_{k+1})$, $\pi_k^{-1}(V_{k+1}) \cap W_k$ can be locally defined by the polynomial $g_{n-k} \circ \pi_k$. Then, the embedding argument in case of dimension 2 using the theorem of Serre immediately yields $(*)_{k+1}$. This completes the proof of Theorem (8.1) and hence the proof pf the Main Theorem.

In case of dimension 2, a theorem of RAMANUJAM [19] in algebraic geometry says that a quasi-projective surface homeomorphic to $\mathbb{R}^4$ is actually biregular to $\mathbb{C}^2$. By the theorem of GROMOLL-MEYER and others stated in the introduction, a complete m-dimensional Riemannian manifold of positive sectional curvature is diffeomorphic to $\mathbb{R}^m$. Combined with the above theorem of Ramanujam, we obtain the corollary to the Main Theorem for non-compact Kähler surfaces of positive sectional curvature.

BIBLIOGRAPHY

[1] ANDREOTTI (A.) and VESENTINI (E.). — Carleman estimates for the Laplace-Beltrami operator on complex manifolds, Publ. Math. Inst. Hautes Études Sci. 25 (1965), 81-130.

[2] BISHOP (R. L.) and GOLDBERG (S. I.). — On the second cohomology group of a Kähler manifold of positive curvature, Proc. Amer. Math. Soc. 16 (19765), 119-122.

[3] BOMBIERI (E.) and GIUSTI (E.). — Harnack's inequality for elliptic differential equations on minimal surfaces, Inv. Math. 15 (1972), 24-46.

[4] CHEEGER (J.) and EBIN (D. G.). — Comparison Theorems in Riemannian Geometry, North-Holland Publishing Co., Amsterdam, 1975.

[5] CHEEGER (J.) and GROMOLL (D.). — The splitting theorem for manifolds of non-negative Ricci curvature, J. Diff. Geom., 6, (1971), 119-128.

[6] CHEEGER (J.) and GROMOLL (D.). — On the structure of complete manifolds of non-negative curvature, Ann. of Math. 96 (1972), 413-443.

[7] CHENG (S.-Y.) and YAU (S.-T.). — Differential equations on Riemannian manifolds and their geometric applications, Comm. Pure Appl. Math. 28 (1975), 333-354.

[8] CROKE (C.). — Some isoperimetric inequalities and consequences, Ann. Scient. Ec. Norm. Sup. 13 (1980).

[9] GREENE (R. E.) and WU (H.). — Analysis on non-compact Kähler manifolds, Proc. Symp. Pure Math., Vol. 30, Part II, Amer. Math. Soc. (1977).

[10] GREENE (R. E.) and WU (H.). — Function Theory on Manifolds which Possess a Pole, Lecture Notes in Math., Vol. 669, Springer-Verlag, Berlin-Heidelberg-New York, 1979.

[11] GOODMAN (J. E.). — Affine open subsets of algebraic varieties and ample divisors, Ann. of Math. 89 (1969), 160-183.

[12] HÖRMANDER (L.). — L^2-estimates and existence theorems for the $\bar{\partial}$-operator, Acta. Math. 113 (1965), 89-152.

[13] LELONG (P.). — Fonctions entières (n variables) et fonctions pluri-sousharmoniques d'ordre fini dans C^n, J. Anal. Math. 12 (1964), 365-407.

[14] LELONG (P.). — Sur la structure des courants positifs fermés, Séminaire Pierre Lelong (Analyse), Année 1975/76, Lectures Notes in Math., Vol. 578, pp. 136-156, Springer-Verlag, Berlin-Heidelberg-New York.

[15] MOK (N.). — Courbure bisectionnelle positive et variétés algébriques affines, Comptes Rendus, Acad. Sc. Paris, Vol. 296, Série A (1983), 473-476.

[16] MOK (N.). — Complete non-compact Kähler manifolds of positive curvature, Complex Analysis of Several Variables, Proc. of Sym. in Pure Math., Vol. 41, Amer. Math. Soc., 1984.

[17] MOK (N.), SIU (Y.-T.) and YAU (S.-T.). — The Poincaré-Lelong equation on complete Kähler manifolds, Comp. Math., Vol. 44, Fasc. 1-3 (1981), 183-218.

[18] MOSER (J.). — On Harnack's theorem for elliptic differential equations, Comm. Pure and Appl. Math., 14 (1961), 577-591.

[19] RAMANUJAM (C. P.). — A topological characterization of the affine plane as an algebraic variety, Ann. of Math. 94 (1971), 69-88.

[20] SERRE (J.-P.). — Faisceaux algébriques cohérents, Ann. of Math. 61 (1955), 197-278.

[21] SERRE (J.-P.). — Géométrie algébrique et géométrie analytique, Ann. Inst. Fourier 6 (1956), 1-42.

[22] SIU (Y.-T.). — Pseudoconvexity and the problem of Levi, Bull. Amer. Math. Soc. 84 (1978), 481-512.

[23] SIU (Y.-T.). — Analyticity of sets associated to Lelong numbers and the extension of closed positive currents, Invent. Math. 27 (1874), 53-156.

[24] SIU (Y.-T.) and YAU (S.-T.). — Complete Kähler manifolds with non-positive curvature of faster than quadratic decay, Ann. Math. 105 (1977), 225-264.

[25] SKODA (H.). — Application des tachniques L^2 à la théorie des idéaux d'une algèbre de fonctions holomorphes avec poids, Ann. Scient. Ec. Norm. Sup. 5 (1972), 548-580.

[26] STAMPACHIA (G.). – Equations elliptiques du second ordre à coefficients discontinus, Séminaire de Mathématiques Supérieures 16, 1966.

[27] THIE (P.). – The Lelong number in a point of a complex analytic set, Math. Ann. 172 (1967), 269-312.

[28] YAU (S.-T.). – Harmonic functions on complete Riemannian manifolds, Comm. Pure Appl. Math. 28 (1975), 201-228.

Added in proof

[29] BLANC (C.) and FIALA (F.). – Le type d'une surface et sa courbure totale, Comment. Math. Helv. 14 (1941-42), 230-233.

[30] GROMOLL (D.) and MEYER (W.). – On complete open manifolds of non-negative curvature, Ann. of Math. 90 (1969), 75-90.

[31] POOR (W.). – Some results on nonnegatively curved manifolds, J. Diff. Geom. 9 (1974), 583-600.

[32] WU (H.). – An elemetary method in the study of nonnegative curvature, Acta Math. 142 (1979), 57-78.