

NBER WORKING PAPER SERIES

AN EMPIRICAL INVESTIGATION OF CONTINUOUS-TIME EQUITY RETURN MODELS

Torben G. Andersen  
Luca Benzoni  
Jesper Lund

Working Paper 8510  
<http://www.nber.org/papers/w8510>

NATIONAL BUREAU OF ECONOMIC RESEARCH  
1050 Massachusetts Avenue  
Cambridge, MA 02138  
October 2001

We are grateful to Alexandre Baptista, Menachem Brenner, Sanjiv Das, Bjørn Eraker, Ron Gallant, Rick Green, Jack Kareken, Marti Subrahmanyam, George Tauchen, Harold Zhang, an anonymous referee and seminar participants at Boston College, Brown University, University of Chicago, CIRANO, Harvard University, University of Maryland, Michigan State University, University of Michigan, University of Minnesota, the Newton Institute at Cambridge (UK), University of North Carolina, Northwestern University (Finance and Statistics Departments), NYU, the Center for NonLinear Methods in Economics at Svinkløv, Denmark, the Econometric Society Winter Meeting 1999, and the Option Pricing Conference in Montreal, March 2000, for helpful comments and suggestions. Also, we would like to thank the MSI center at the University of Minnesota for providing computing resources. Of course, all errors remain our sole responsibility. Previous versions of this paper were circulated under the title "Estimating Jump-Diffusions for Equity Returns." The views expressed herein are those of the authors and not necessarily those of the National Bureau of Economic Research.

© 2001 by Torben G. Andersen, Luca Benzoni and Jesper Lund. All rights reserved. Short sections of text, not to exceed two paragraphs, may be quoted without explicit permission provided that full credit, including © notice, is given to the source.

An Empirical Investigation of Continuous-Time Equity Return Models  
Torben G. Andersen, Luca Benzoni and Jesper Lund  
NBER Working Paper No. 8510  
October 2001  
JEL No. G12, C50, G13

### **ABSTRACT**

This paper extends the class of stochastic volatility diffusions for asset returns to encompass Poisson jumps of time-varying intensity. We find that any reasonably descriptive continuous-time model for equity-index returns must allow for discrete jumps as well as stochastic volatility with a pronounced negative relationship between return and volatility innovations. We also find that the dominant empirical characteristics of the return process appear to be priced by the option market. Our analysis indicates a general correspondence between the evidence extracted from daily equity-index returns and the stylized features of the corresponding options market prices.

Torben G. Andersen  
Department of Finance  
Kellogg School of Management  
Northwestern University  
2001 Sheridan Road  
Evanston, IL 60208  
and NBER  
t-andersen@nwu.edu

Luca Benzoni  
Department of Finance  
Carlson School of Management  
University of Minnesota  
321 19<sup>th</sup> Avenue South  
Minneapolis, MN 55455  
lbenzoni@umn.edu

Jesper Lund  
Department of Finance  
The Aarhus School of Business  
Fuglesangs Alle 4  
DK-8210 Aarhus V  
Denmark  
jl1204@worldonline.dk

MUCH ASSET AND DERIVATIVE PRICING THEORY is based on diffusion models for primary securities. However, prescriptions for practical applications derived from these models typically produce disappointing results. A possible explanation could be that analytic formulas for pricing and hedging are available for only a limited set of continuous-time representations for asset returns and risk-free discount rates. It has become increasingly evident that such “classical” models fail to account adequately for the underlying dynamic evolution of asset prices and interest rates. Not surprisingly, the inadequacy of these specifications also shows up in bond and derivatives pricing, where the standard representations falter systematically. For example, the Black-Scholes pricing formula, although widely used by practitioners, is well known to produce pronounced and persistent biases in the pricing of options. Deviations of actual prices from Black-Scholes benchmarks result in the extensively documented “volatility smiles” and “smirks” reported in, e.g., Rubinstein (1994).

The above observations suggest that practical financial decision making based on the continuous-time setting will be satisfactory only if it builds upon reasonable specifications of the underlying asset price processes. Specifically, bond and derivatives prices are very sensitive to volatility dynamics. Likewise, the extent of skewness and the presence of outliers in the underlying return distribution are critical inputs to hedging and risk management decisions as well as for option pricing. It has long been asserted that jumps or stochastic volatility may account for such return characteristics. Unfortunately, these features are generally explored separately and not within a unified framework. Until recently, a major obstacle was the lack of feasible techniques for estimating and drawing inference on general continuous-time models using discrete observations. Over the last few years, new methods for diffusion estimation have emerged. Nonetheless, the inference tools are often specific to a limited number of models. Moreover, the related empirical findings have so far been inconclusive or contradictory and most studies fail to produce a satisfactory fit to the underlying asset return dynamics.

From a different vantage point, a number of studies extract information about the parameters of the underlying returns process from derivatives prices and contrast the findings to the time series behavior of the return series; see, e.g., Bates (1996a, 2000), Bakshi, Cao and Chen (1997), and Chernov and Ghysels (2000). The results are striking: there is a strong disparity between the characteristics of the return dynamics implied by the derivative prices and those inferred from the actual time series of underlying returns. The lack of consensus about the proper specification of continuous-time models for asset returns and the inability to link the estimated representations coherently to corresponding features of associated financial instruments remain a cause for concern since they suggest that a critical element may be missing from the model specification.

The objective of this paper is to identify a class of jump-diffusion models that are successful in approximating the S&P500 return dynamics and should therefore constitute an adequate basis for continuous-time asset pricing applications. We explore alternative representations for the daily equity-index return dynamics within a general jump-diffusion setting. We consider specifications both inside and outside the popular class of affine models that generally provide tractable pricing

and estimation procedures; see, for example, Duffie, Pan and Singleton (2000). Our broad-based approach to model selection is critical in order to establish the merits of the different formulations. It further helps assess the severity of the constraints imposed by inference techniques that are tailored on the family of affine models, which are widely used in the literature. Along the way, we identify the features of the return dynamics that account for the inadequate performance of the classical models. Finally, we explore the relationship between our estimated specification and the associated derivatives prices. First, we contrast our estimates for the model parameters that are unaffected by the adjustments for volatility and jump risks with those extracted from options in previous empirical work. Second, we provide a qualitative comparison of the pricing implications of our model estimated solely from equity returns and the stylized evidence from actual options data.

The need for a general, yet efficient framework for inference leads us to adopt a variant of the simulated method of moments (SMM) technique of Duffie and Singleton (1993). Moment conditions are obtained from an implementation of the efficient method of moments (EMM) procedure of Gallant and Tauchen (1996) and performance is judged through the associated specification tests and model diagnostics. We apply this approach to daily observations from the S&P500 index. Besides being a broad indicator of the equity market, this index is the asset underlying the SPX option, an important and highly liquid contract in the derivatives market. Both data sets have been studied extensively, so we have natural reference points for our analysis. Finally, the daily sampling frequency allows us to capture high-frequency fluctuations in the returns process that are critical for identification of jump components, while avoiding modeling the intra-day return dynamics which are confounded by market microstructure effects and trading frictions.

Our results indicate that *both* stochastic volatility and discrete jump components are critical ingredients of the data generating mechanism. Also, a pronounced negative correlation between return and volatility innovations appears necessary to capture the skewness in S&P500 returns. A relatively low-frequency jump component accounts for the fat tails of the returns distribution. We estimate that jumps occur on average 3-4 times a year. The discontinuities are relatively small, with most of the jumps lying within the  $\mp 3\%$  range. All variants of our model without a negative return-volatility relation or jumps are overwhelmingly rejected, while two stochastic volatility jump-diffusion (SVJD) specifications provide acceptable characterizations. Hence, we find that a combination of fairly standard and parsimonious representations of stochastic volatility and jumps accommodates the dominant features of the S&P500 equity-index returns, and offers an attractive alternative to, for example, the complex four-factor pure diffusion specification of Gallant and Tauchen (1997). We thus find empirical support for the affine jump-diffusion model of Bates (1996a) and Bakshi, Cao and Chen (1997), which provides a convenient setting for asset pricing applications.

Interestingly, our estimates for the model parameters that are unaffected by the adjustment for volatility and jump risks are generally similar to those obtained in previous work exploiting only equity options. In line with this observation, our model also produces option pricing implications

that correspond qualitatively to those obtained from actual derivatives data. For example, the jump component and, more prominently, the asymmetric return-volatility relationship induce a smirk in the typical implied volatility pattern which resembles that extracted from options data. Moreover, like in the actual data, this smirk becomes less pronounced as maturity increases. Finally, relatively small premia for the uncertainty associated with volatility and jumps are sufficient to replicate most of the salient features of the term structure of implied volatility. Hence, a large number of characteristics of the stock price process, which seem to be implied or priced by the derivatives contracts, are independently identified as highly significant components of the underlying dynamics uncovered in our empirical analysis of the S&P500 returns. Consequently, our results indicate a general correspondence between the dominant features of the equity-index returns and option prices.

We deliberately avoid exploiting derivatives prices for estimation purposes. Although this is not fully efficient, there are important advantages associated with this approach. First, we are able to focus exclusively on the adequacy of the model under the “physical” measure. Features such as stochastic volatility, a negative correlation between return and volatility innovations, and jumps, are consequently not extracted from derivatives prices, and are therefore shown to be inherent characteristics of the underlying return dynamics. This analysis is an important benchmark since the option prices speak to the parameters of the return system under the “risk-neutral” measure. Hence, joint estimation requires distributional assumptions not only for the stock return dynamics, but also for the associated market premia for undiversifiable risks such as stochastic volatility and jumps. Rejection of the joint model may thus result from either the specification of the risk premia (the risk-neutral distribution), the stock return dynamics (the physical distribution), or both. This ambiguity is avoided when concentrating solely on the underlying asset return dynamics. Since the stock return dynamics is critical for a number of practical hedging, risk management, portfolio allocation and asset pricing decisions, it is advantageous to have a robust characterization that is immune to either misspecification or instability of the specified process for the risk premia. The relevance of this observation is underscored by the documentation of a structural break in option prices around the market correction in 1987. Prior to October 1987, the volatility implied by the equity-index option contracts displayed a largely symmetric smile pattern, but after October 1987 it turned into an asymmetric smirk, reflecting the higher prices for out-of-the-money put options (which are valuable for portfolio insurance strategies). Consequently, many studies utilizing option data end up having to focus on a limited sample from 1988 onwards. There is no evidence of a corresponding break in the underlying stock price dynamics. Thus, a second advantage of focusing exclusively on equity returns is that we may exploit a large daily sample originating with the initiation of the S&P500 index in 1953. A large sample allows for more accurate inference about the strongly persistent volatility process and its identification relative to the jump distribution. We are also able to gauge structural stability directly by considering subsample estimation. Third, drawing inference from joint return and derivatives data poses severe computational problems. To circumvent them, we would have to restrict our analysis to

models that deliver (near) closed-form solutions for the option price. This would in effect limit our applications to specifications within the affine class. But if explicit solutions for derivatives prices are not required, the EMM simulation approach readily accommodates alternative specifications for the diffusion and jump components, allowing for a broader study of candidate models including those outside of the tractable affine setting.

The remainder of the paper is structured as follows. In Section I we discuss the candidate continuous-time models for stock returns and provide a selective overview of the existing literature. Empirical results and the details of the EMM implementation are documented in Section II. In Section III we illustrate potential derivative pricing implications and compare our findings with the results reported in the empirical option pricing literature. Concluding remarks are in Section IV.

## I. Model Specification and Estimation Methodology

### A. Candidate Models

We focus on a class of continuous-time models that are sufficiently general to capture the salient features of equity-index returns, and also provide relatively straightforward comparisons to the representations appearing in the literature. We pay particular attention to specifications that facilitate derivative pricing, but without limiting ourselves to models that deliver (near) closed-form pricing formulas.

The following general representations turn out to be satisfactory for our application

$$\frac{dS_t}{S_t} = (\mu + c V_t - \lambda(t) \bar{\kappa}) dt + \sqrt{V_t} dW_{1,t} + \kappa(t) dq_t, \quad (1)$$

where the (log-)volatility process obeys a mean-reverting diffusion, given as

$$d \ln V_t = (\alpha - \beta \ln V_t) dt + \eta dW_{2,t}, \quad (2)$$

or

$$dV_t = (\alpha - \beta V_t) dt + \eta \sqrt{V_t} dW_{2,t}. \quad (3)$$

$W_1$  and  $W_2$  are standard Brownian motions with correlation  $\text{corr}(dW_{1,t}, dW_{2,t}) = \rho$ ,  $q$  is a Poisson process, uncorrelated with  $W_1$  and  $W_2$  and governed by the jump intensity  $\lambda(t)$ , i.e.,  $\text{Prob}(dq_t = 1) = \lambda(t) dt$ ,  $\lambda(t)$  is an affine function of the instantaneous variance,

$$\lambda(t) = \lambda_0 + \lambda_1 V_t. \quad (4)$$

$\kappa(t)$  denotes the magnitude of the jump in the return process if a jump occurs at time  $t$ . It is assumed to be log-normally distributed,

$$\ln(1 + \kappa(t)) \rightsquigarrow N(\ln(1 + \bar{\kappa}) - 0.5 \delta^2, \delta^2). \quad (5)$$

This class of models subsumes a number of important special cases that we consider in our empirical analysis. To establish a benchmark, we initially estimate a representation that is compatible with the Black-Scholes option pricing model. This is obtained by assuming that the drift and diffusion coefficients in (1) are constant and that there is no jump component, i.e.,  $\lambda(t) = 0$ . Thus, (1) reduces to

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t. \quad (6)$$

The option pricing formula associated with the Black-Scholes diffusion (6) is routinely used to price European options, although it is known to produce systematic biases. These are typically illustrated by the “smile” in implied volatilities extracted from a cross-section of options, sorted according to the degree of moneyness. That is, option prices consistently violate the basic assumption of a constant diffusion coefficient in the underlying stochastic differential equation for stock returns. This should not be surprising since high-frequency stock returns exhibit leptokurtosis, skewness and pronounced conditional heteroskedasticity, all characteristics ruled out by the Black-Scholes assumptions.

As a first extension of the representation (6), we estimate the Merton (1976) model, obtained by adding a jump component with constant intensity to the Black-Scholes formulation. This is also a special case of (1), obtained imposing the restriction  $\lambda(t) = \lambda$  and assuming volatility to be constant. Economically, jumps in stock returns are easily rationalized: the discrete arrival of new information induces an instantaneous revision of stock prices. Adding a jump component should improve the fit to the observed time-series of returns, since the jumps may help accommodate outliers as well as asymmetry in the return distribution. The presence of outliers is regulated by the magnitude and variability of the jump component, while the asymmetry is controlled by the average magnitude of the jump,  $\bar{\kappa}$ . The Merton jump-diffusion has been estimated from time series data on asset returns by a number of authors – among them, Press (1967), Jarrow and Rosenfeld (1984), Ball and Torous (1985), Akgiray and Booth (1986), and, more recently, Das and Uppal (1998) and Das (1999). The jump component has generally been found to be significant, and the specification does accommodate some of the observed skewness and leptokurtosis in the returns process. Nonetheless, the resulting representations are, as also documented below, seriously inadequate. They cannot account for the strong conditional heteroskedasticity of stock returns nor rationalize the substantial time-variation that is observed in the level and shape of the implied volatility smile.

These facts motivate estimation of alternative extensions of representation (6). As a first step, we rule out the jump component by setting  $\lambda(t) = 0$  in the diffusion (1), but allow volatility to be stochastic. This produces a pure two-factor diffusion,

$$\frac{dS_t}{S_t} = (\mu + c V_t) dt + \sqrt{V_t} dW_{1,t}, \quad (7)$$

where the variance process  $V$  follows either the log-variance specification (2) or the affine specification (3). Stochastic volatility induces excess kurtosis in the return process, governed largely by

the volatility diffusion parameters  $\alpha$ ,  $\beta$  and  $\eta$ . The asymmetry observed in the return process may be captured by a negative correlation between shocks to the variance and the return process, i.e.  $\text{corr}(dW_{1,t}, dW_{2,t}) = \rho < 0$ . The square-root variance specification in (3) is particularly attractive for option pricing applications since Heston (1993) provides a closed-form solution for the option price when the underlying return process obeys (7) and (3). On the other hand, the log-variance specification in (7) and (2) is more in line with standard discrete-time stochastic volatility models as well as the popular EGARCH representation for equity-index returns. This suggests that the log-variance model is a good starting point for our diffusion specification since it provides a basis for comparisons with the usual discrete-time results. The representation is, however, less convenient for derivatives pricing than the specification incorporating square-root variance, since numerical methods are required for the computation of option prices; see, e.g., Melino and Turnbull (1990) and Benzoni (1998). Also, note that we have included the volatility factor in the mean return (drift coefficient), and thus rule out arbitrage opportunities by ensuring that equities do not provide a fixed excess return over the risk-free rate when volatility approaches zero. However, given the mixed evidence on volatility-in-mean effects in the discrete-time oriented empirical literature, the associated coefficient is likely to be small.

It is strictly an empirical issue whether the models (7) and (2) or (7) and (3) provide adequate descriptions of stock returns. Our empirical work leads us to expand both representations by including a jump component. This yields the most general specification, that of (1)-(5). The joint presence of jump and stochastic volatility factors provides additional flexibility in capturing the salient features of equity returns, including skewness and leptokurtosis. From an option pricing perspective, this extension has the advantage of delivering closed-form solutions if  $V$  obeys (3), see Bates (1996a, 2000), or numerical approximation schemes when  $V$  satisfies (2). Moreover, several studies have noted that the incorporation of a jump component is essential when pricing options that are close to maturity.<sup>1</sup> Indeed, if volatility follows a pure diffusion the implied continuous sample path may be incapable of generating a sufficiently volatile return distribution over short horizons to justify the observed prices of derivative instruments.

## *B. Estimation Methodology*

The main difficulty in conducting efficient inference for continuous-time model from discretely sampled data is that closed-form expressions for the discrete transition density generally are not available, especially in the presence of unobserved and serially correlated state variables. The latter is usually the case, for example, for stochastic volatility models. Although maximum likelihood estimation is, in principle, feasible via numerical methods – see, e.g., Lo (1988) – the computational demands are typically excessive if latent variables must be integrated out of the likelihood function.

In response to this challenge, a number of alternative consistent inference techniques for continuous-



time processes have been developed in recent years. Among the early contributions are the (semi-)nonparametric approaches of, e.g., Ait-Sahalia (1996) and Hansen and Scheinkman (1995); and, among the more recent, Conley et al. (1997), Stanton (1997), Jiang and Knight (1997), Johannes (1999), Bandi and Phillips (1998), Bandi and Nguyen (2000) and Poteshman (1998). Unfortunately, it is difficult to apply such methods in our setting because of the joint presence of stochastic volatility and jumps. At the same time, refinements of the Pedersen (1995) approach of treating the estimation problem as a missing values problem have appeared. Although the method may appear unable to accommodate latent factors, the development of Markov Chain Monte Carlo (MCMC) techniques has provided interesting progress; see Eraker (2001), Jones (1998), and Elerian, Chib and Shephard (2001).<sup>2</sup> Nonetheless, the approach is not ideal for our application since the implementation of the MCMC sampler must be tailored to the model of choice, and it is therefore hard to make comparisons across a broad range of different specifications. Yet another approach, developed in Duffie, Pan and Singleton (2000) and Liu (1997), has inspired work using empirical characteristic functions; see, e.g., Chacko and Viceira (1999), Jiang and Knight (1999), Singleton (2001) and Carrasco et al. (2001). However, this methodology is designed for the affine class and it is hard to adapt it to, e.g., the log-variance representation (2).

We turn instead to the EMM estimation procedure, a simulation-based method of moments technique. In principle, simulation approaches are feasible if it is possible to simulate the underlying diffusion paths arbitrarily well and obtain sufficient identifying information for parameter estimation via moment conditions. This typically produces inefficient inference, but careful moment selection can greatly alleviate this problem. The SMM procedure of Duffie and Singleton (1993) matches sample moments with simulated moments, i.e., moments computed using a long simulated series obtained from the assumed data generating mechanism. The EMM procedure of Gallant and Tauchen (1996) refines the SMM approach by providing a specific recipe for the generation of moment conditions. They are extracted from the expectation of the score of a discrete-time auxiliary semi-nonparametric model which closely approximates the distribution of the discretely sampled data. An attractive feature is that EMM achieves the same efficiency as maximum likelihood (ML) when the score of the auxiliary model (asymptotically) spans the score of the true model. Moreover, as for the generalized method of moments (GMM), the EMM criterion function may be used to construct a Chi-square statistic for an overall test of the over-identifying restrictions. Since this procedure is based on the identical moment conditions - the auxiliary model score vector - for all models under investigation, it allows for comparison of non-nested representations, like the log-variance and affine volatility processes in (2) and (3). Finally, the fit of individual scores may be used to gauge how well the model captures particular characteristics of the data. Although the EMM procedure has been used before to estimate stochastic volatility (SV) models, our extension to a fully specified SVJD setting, which produces efficient estimation of a model with both stochastic volatility and jumps, is to the best of our knowledge the first within the EMM literature.

In our application we use a sample of equity index returns. A number of authors has advocated instead the use of derivative prices, based on the conjecture that this data contains superior market-based information about the evolution of the data generating process. As a result, they obtain model parameters exclusively under the “risk-neutral” probability measure using, for example, the pricing techniques within general affine settings found in, e.g., Duffie, Pan and Singleton (2000) and Bakshi and Madan (2000). These approaches allow for inversion of option prices into (constant) model parameter and period-by-period implied volatility estimates, e.g., Bates (1996a, 2000), or even period-by-period implied parameters and period-by-period implied volatilities for affine stochastic volatility jump-diffusions, as explored by Bakshi, Cao and Chen (1997).<sup>3</sup> These studies invariably find the extracted time series of volatility to be inconsistent with the observed dynamics of the underlying equity returns, thus highlighting the difficulty of jointly rationalizing derivatives prices and the underlying asset price dynamics. These results can be ascribed to a misspecification of the return generating process, and thus the “physical” measure, as well as the factor risk premia, i.e., the “risk-neutral” measure. Our paper, along with a few other recent contributions reviewed in the following section, shed additional light on the sources of model misspecification.

### *C. Recent Empirical Findings*

Pan (1999) undertakes joint estimation of the return dynamics and the risk-neutral distribution underlying the derivatives prices using weekly data for 1989-1996 in an affine setting. Her constrained jump-diffusion appears to fit relatively well, although it fails to capture fully the volatility dynamics. What is more important for us, however, the use of weekly observations and a relatively small sample suggest that identification of the stochastic process governing the jump behavior is based almost exclusively on the derivatives prices. This may explain why our results indicate a much higher jump intensity in the return process than she reports. Jones (1999) exploits the VIX implied volatility index and daily S&P100 equity-index returns to estimate a constant elasticity of variance (CEV) extension of the square-root model using a Bayesian MCMC procedure. He finds this specification to do better than the square-root version and suggests that his extension may serve as a reasonable substitute for a jump component. As in prior work, however, the full option smirks cannot be rationalized by the estimated model, and the performance of the CEV specification relative to jump-diffusions is unclear. Benzoni (1998) estimates square-root and non-affine stochastic volatility models using S&P500 returns, and explores the pricing implications for the corresponding S&P500 index options. He finds that the two specifications have similar empirical properties and provide comparable fits for both equity returns and option prices. He concludes that volatility risk is priced by the market and documents different sources of misspecification in the models considered. Chernov and Ghysels (2000) also estimate the square-root stochastic volatility diffusion. In a noteworthy departure from the extant literature, their EMM based estimates from both stock-index returns and option prices

imply no significant asymmetry in the relation between return and volatility innovations. Their specifications, however, are overwhelmingly rejected by the associated goodness-of-fit tests. In recent work, Johannes, Kumar and Polson (1999) and Eraker, Johannes and Polson (1999) explore discretized versions of pure jump and square-root stochastic volatility jump-diffusions at the daily level with an emphasis on generalized jump representations. In particular, they consider models with jumps to volatility and perform estimation with the MCMC method using S&P500 index returns. Eraker (2000) extends this approach to a joint data set of S&P500 returns and option prices. More general jump-diffusion specifications are also explored in current EMM-based work by Chernov et al. (1999, 2000).<sup>4</sup>

## II. Empirical Results

In the following sections we report on our EMM implementation. In Section A we outline the semi-nonparametric (SNP) estimation of the conditional return density by quasi-maximum likelihood. In Section B we discuss the EMM estimation results and interpret the specification tests used to gauge the performance of the different models along various dimensions.

### A. A SNP Model for the S&P500 Returns

The key to a successful application of the EMM procedure is the choice of an auxiliary model that closely approximates the conditional distribution of the return process. Loosely speaking, Gallant and Long (1997) have shown that if the score function of the auxiliary model asymptotically spans the score of the true model, then EMM is (asymptotically) efficient. Also, they have demonstrated that, within the class of discrete-time auxiliary models, SNP densities are good candidates for this task.

SNP models are based on the notion that a polynomial expansion can be used as a nonparametric estimator of a density function; see Gallant and Nychka (1987). In addition, SNP densities allow for a leading parametric term that may be used to capture the dominant systematic features of the (discrete-time) return dynamics, thus providing a parsimonious representation of the conditional density for the observed series. An ARMA term, potentially extended by a volatility-in-mean effect, is a natural candidate for the conditional mean, and the ARCH specification generally provides a reasonable characterization of the pronounced conditional heteroskedasticity of stock returns. Hence, we search over ARMA-ARCH type models for an initial (leading) term in the SNP-representation. Our specification analysis suggests an EGARCH representation for the conditional variance process.<sup>5</sup> Besides superior in-sample performance as measured by standard information criteria statistics, the EGARCH form has other attractive properties. As originally argued by Nelson (1991), it readily accommodates an asymmetric response of the conditional volatility process to return innovations and

it renders non-negativity restrictions for the volatility parameters unnecessary.

To provide a basis for comparison with prior studies, we initially fit the pure ARMA-EGARCH model by quasi-maximum likelihood (QML), thus excluding the nonparametric expansion of the conditional density that is unique to the SNP approach. Our main results are based on daily returns for the S&P500 equity index<sup>6</sup> from 01/02/1953 to 12/31/1996, a sample of 11,076 observations. We also make use of a smaller sample of 4,298 daily observations from 01/03/1980 to 12/31/1996 to gauge the temporal stability of our findings. Summary statistics are provided in Table I. The unit root hypothesis is convincingly rejected in favor of stationarity for the return series – a condition required for any method of moments approach predicated on stability of the return generating mechanism. The price and return series are depicted in Figure 1.

The QML estimates of the pure ARMA-EGARCH (not reported) are indicative of strong temporal persistence in the conditional variance process: the parameters governing the persistence are within, but close to, the boundaries of the covariance stationary region. Also, the correlation between the return innovations and the conditional variance is negative and highly significant, as observed in many prior studies. A relatively high-order AR term in the mean equation is necessary to capture the autocorrelation structure in the S&P500 returns, although this pattern is accommodated nicely by a single MA(1) term. Such pronounced short-run return predictability is somewhat difficult to reconcile with market efficiency and is likely spurious since it is consistent with non-synchronous trading in the stocks of the underlying index; see, e.g., Lo and MacKinlay (1990). Finally, the short-run return autocorrelation is quantitatively less important than the pronounced volatility fluctuations for most applications, and the inference on the volatility process is largely unaffected by the short-run mean dynamics – a result confirmed by the findings reported below. For these reasons, we prefilter the data using a simple MA(1) model for the S&P500 daily returns and rescale the residuals to match the sample mean and variance in the original data set. This residual series is then treated as the observed return process.<sup>7</sup>

Quasi-maximum likelihood estimation is performed on the fully specified semi-nonparametric (SNP) auxiliary model,

$$f_K(r_t|x_t; \xi) = \left( \nu + (1 - \nu) \times \frac{[P_K(z_t, x_t)]^2}{\int_{\mathbb{R}} [P_K(z_t, x_t)]^2 \phi(u) du} \right) \frac{\phi(z_t)}{\sqrt{h_t}},$$

where  $\nu$  is a small constant (fixed at 0.01),<sup>8</sup>  $\phi(\cdot)$  denotes the standard normal density,  $x_t$  is a vector containing a set of lagged observations, and

$$\begin{aligned} z_t &= \frac{r_t - \mu_t}{\sqrt{h_t}}, \\ \mu_t &= \phi_0 + c h_t + \sum_{i=1}^s \phi_i r_{t-i} + \sum_{i=1}^u \delta_i \varepsilon_{t-i}, \\ \ln h_t &= \omega + \sum_{i=1}^p \beta_i \ln h_{t-i} + (1 + \alpha_1 L + \dots + \alpha_q L^q) [\theta_1 z_t + \theta_2 (b(z_t) - \sqrt{2/\pi})], \end{aligned}$$

$$b(z) = |z| \text{ for } |z| \geq \pi/2K, \quad b(z) = (\pi/2 - \cos(Kz))/K \text{ for } |z| < \pi/2K,$$

$$P_K(z, x) = \sum_{i=0}^{K_z} a_i(x) z^i = \sum_{i=0}^{K_z} \left( \sum_{|j|=0}^{K_x} a_{ij} x^j \right) z^i, \quad a_{00} = 1,$$

where  $j$  is a multi-index vector,  $x^j \equiv (x_1^{j_1}, \dots, x_M^{j_M})$  and  $|j| \equiv \sum_{m=1}^M j_m$ . As in Andersen and Lund (1997),  $b(z)$  is a smooth (twice-differentiable) function that closely approximates the absolute value operator in the EGARCH variance equation, with  $K = 100$ .

With this specification, the main task of the nonparametric polynomial expansion in the conditional density is to capture any excess kurtosis in the return process and, to a lesser extent, any asymmetry which has not already been accommodated by the EGARCH leading term. In practice, the nonparametric specification is implemented via an orthogonal Hermite polynomial representation. We also allow for heterogeneity in the polynomial expansion ( $K_x > 0$ ), but these terms are insignificant, indicating that the EGARCH leading term provides an adequate characterization of the serial dependence in the conditional density.

Within this class of SNP models, we rely on the Bayesian (BIC) and Hannan-Quinn (H-Q) information criteria for model selection, as the commonly used Akaike criterion (AIC) tends to overparameterize the models. (Actual values of the statistics are not reported.) This selection strategy points towards an ARMA(0,0)-EGARCH(1,1)-Kz(8)-Kx(0). It is theoretically desirable to include a volatility-in-mean effect, but, since the term is insignificant and induces additional estimation uncertainty for the remaining drift coefficients, we report EMM results both excluding and including this effect in the auxiliary model. Table II reports the value of the parameter estimates and corresponding standard errors. Ljung-Box tests for the autocorrelation of the residuals (not reported) confirm that the selected specification successfully removes the systematic first- and second-order dependencies in the data.

## B. EMM Estimation

The expectation of the auxiliary model score function provides the moment conditions for simulated method of moment estimation of the continuous-time SVJD.

Let  $\{r_t(\psi), x_t(\psi)\}_{t=1}^{\mathcal{T}(N)}$  denote a sample simulated from the SVJD using the parameter vector  $\psi = (\mu \ \alpha \ \beta \ \eta \ \rho \ \lambda_0 \ \lambda_1 \ \bar{\kappa} \ \delta)$ . The EMM estimator of  $\psi$  is then defined by

$$\hat{\psi}_N = \arg \min_{\psi} m_{\mathcal{T}(N)}(\psi, \hat{\xi})' W_N m_{\mathcal{T}(N)}(\psi, \hat{\xi}),$$

where  $m_{\mathcal{T}(N)}(\psi, \hat{\xi})$  is the expectation of the score function, evaluated by Monte Carlo integration at the quasi-maximum likelihood estimate of the auxiliary model parameter  $\hat{\xi}$ , i.e.,

$$m_{\mathcal{T}(N)}(\psi, \hat{\xi}) = \frac{1}{\mathcal{T}(N)} \sum_{t=1}^{\mathcal{T}(N)} \frac{\partial \ln f_K(r_t(\psi)|x_t(\psi); \hat{\xi})}{\partial \xi},$$

and the weighting matrix  $W_N$  is a consistent estimate of the inverse asymptotic covariance matrix of the auxiliary score function. Following Gallant and Tauchen (1996), we estimate the covariance matrix of the auxiliary score from the outer product of the gradient. In simulating the return sequence  $\{r_t(\psi), x_t(\psi)\}_{t=1}^{\mathcal{T}(N)}$ , two antithetic samples of  $75,000 \times 10 + 5,000$  returns are generated from the continuous-time model at time intervals of  $1/10$  of a day<sup>9</sup>. The first 5,000 observations are discarded to eliminate the effect of the initial conditions. Lastly, a sequence of  $\mathcal{T}(N) = 75,000$  daily returns is obtained by summing the elements of the simulated sample in groups of 10.

### B.1. Black-Scholes

To obtain an initial benchmark, we estimate the Black and Scholes specification (6). The results for the auxiliary model without volatility-in-mean effect are given in Table III, and for the auxiliary model including the effect in Table IV. Note that, from here on,  $dt = 1$  corresponds to one trading day and parameter estimates are expressed in percentage form on a daily basis.

As is evident from both tables, the model is overwhelmingly rejected at any reasonable confidence level, based on the Chi-square test for over-identifying restrictions. Parameter estimates are therefore largely uninterpretable, the reason being that the EMM procedure confronts the model with auxiliary scores moments and not sample return moments, as maximum likelihood reduces to in this case. The former are vastly more informative about the dynamic features of the return data than are the sample moments. However, without the ability to accommodate any of the dominant SNP moments, the parameter estimates are determined by features that have little to do with their natural interpretation in the underlying (misspecified) model. For example, the  $\sigma$  estimate is approximately 0.6 and significantly different from the return standard deviation of 0.83. Since the return standard deviation for the auxiliary model matches that of the data (we checked that it does by simulating a long return sample from the SNP density), the  $\sigma$  estimate of 0.6 is explained by the poor fit of the Black-Scholes specification, rather than a problem with the SNP model.

The significance of individual score t-ratios associated with corresponding SNP parameters, reported in Table V, are suggestive of the source of model misspecification, even though, given the above observation, the statistics for the Black-Scholes model must be interpreted with caution. The moment associated with the asymmetry parameter in the EGARCH variance equation is highly significant, indicating that the model fails to accommodate the observed asymmetry in the return process. Also, the moments associated with the even terms in the polynomial approximation are non-zero, and this suggests that the excess kurtosis of the S&P500 returns exceeds what can be rationalized by the model. This is consistent with the findings in, e.g., Gallant, Hsieh and Tauchen (1997). As expected, the Black-Scholes representation does not provide an acceptable characterization of the time-series properties of daily stock-index returns.

### B.2. Black-Scholes with Jumps

We first extend the Black-Scholes model by incorporating a jump component with constant intensity:  $\lambda(t) = \lambda_0$ . Initial experimentation reveals that the  $\bar{\kappa}$  parameter is insignificant, but also somewhat poorly identified by the auxiliary score moments, and we consequently impose the restriction  $\bar{\kappa} = 0$ . The results obtained from this constrained specification are also reported in Tables III and IV for the (slightly) different versions of the auxiliary score moments. Of particular interest is the jump intensity parameter  $\lambda_0$ . The coefficient is significant and implies an average of about 14 jumps per year. But even though incorporating a jump component improves the fit considerably – the Chi-Square statistic drops from 127.41 to 90.82 – the model is nevertheless overwhelmingly rejected.

The score t-ratio diagnostics in the BSJ column of Table V are indicative of numerous problems. The fat tails of the return innovations are somewhat better accommodated, but the model still fails in this dimension. Further, the conditional variance process appears seriously misspecified. In particular, the symmetric ( $\bar{\kappa} = 0$ ) jump-diffusion does not capture the asymmetry manifest in the  $\theta_1$  coefficient. Relaxing the  $\bar{\kappa} = 0$  constraint did not lead to a marked improvement in the fit of the score component corresponding to  $\theta_1$ , suggesting that an asymmetric jump component cannot capture the skewness in the S&P500 returns.

### B.3. Stochastic Volatility: Log-Variance Model

Next, we investigate the stochastic volatility diffusion (7) and (2); again the results are reported in Tables III and IV. The estimation is first performed with  $\rho = 0$ , but this constraint is subsequently removed allowing the model to possibly accommodate the asymmetries in the stock returns and variance. The resulting estimates of  $\rho$  are strongly negative and highly significant. Moreover, with an unconstrained  $\rho$  the overall fit is substantially improved. The model is, however, rejected. This result is consistent with the findings in empirical studies of corresponding discrete-time stochastic volatility models, Gallant, Hsieh and Tauchen (1997), Liu and Zhang (1997), van der Sluis (1997) and others.

Turning to the remaining parameters, we note first that the estimate of the (daily) drift term  $\mu$  (in Table 3, 0.0314) implies an annual return of 7.91%, which is in line with the sample mean of 7.59% for 1953-1996. Second, the significantly positive  $\beta$  estimate ensures that the (log-)variance process is stationary and controls the persistence of shocks to the process. The solution to (2) takes the form [see, e.g., Andersen and Lund (1997)]

$$\ln \sigma_s^2 = \exp\{-\beta(s-t)\} \ln \sigma_t^2 + (\alpha/\beta)(1 - \exp\{-\beta(s-t)\}) + \eta \int_t^s \exp\{-\beta(s-v)\} dW_{2,v}, \quad (8)$$

which is a discrete-time representation of the variance process governed by the diffusion parameters. For  $s-t=1$ ,  $\exp\{-\beta\} = 0.9865$  without the volatility-in-mean effect (Table 3) and 0.9841 with that effect (Table 4). Those estimates, which imply a strong daily (log-)volatility persistence, are consistent with estimates reported in the discrete-time literature. The score t-ratio diagnostics are again reported

in Table V. The symmetric ( $\rho = 0$ ) stochastic volatility representation fares somewhat better than the Black-Scholes model, but also fails to accommodate the skewness and kurtosis in the returns. Allowing for an asymmetric volatility response improves the fit dramatically. The corresponding score parameter  $\theta_1$  is no longer significant, which suggests that the asymmetry is induced by a so-called “leverage” or “volatility feedback” effect. However, the score moments associated with the even terms of the nonparametric expansion are again non-zero, indicating that the pure stochastic volatility diffusion is incompatible with the degree of kurtosis observed in the data.

#### B.4. Stochastic Volatility: Square-Root Model

Here we consider the alternative stochastic volatility diffusion (7) and (3), inspired by the square-root, or Cox, Ingersoll and Ross (1985) type representation. The model is also estimated with and without the restriction  $\rho = 0$ . Although the square-root model appears to do marginally worse than the log-variance version, the findings reported in Tables III and IV imply similar characteristics. For example, the square-root model successfully accommodates the asymmetry in the S&P500 returns, but fails to induce a sufficient degree of kurtosis in the return series, as confirmed by the score t-ratios (not reported).<sup>10</sup> In summary, allowing for an asymmetric stochastic volatility factor greatly enhances performance, and yet does not provide an adequate description of the S&P500 returns. We consequently turn to another generalization.

#### B.5. Stochastic Volatility with Jumps

The models in this section incorporate both a stochastic volatility and a jump component. We focus initially on the model (1) and (2) with a log-variance specification and constant jump intensity:  $\lambda(t) = \lambda_0$ . As for the simple jump-diffusion, the  $\bar{\kappa}$  parameter is insignificant and imprecisely estimated. This suggests that the asymmetry is more appropriately captured through a negative correlation between the return innovations and the diffusion variance (i.e.,  $\rho < 0$ ). Consequently, the restriction  $\bar{\kappa} = 0$  is imposed in the following estimation results.

The SVJD provides a substantial improvement over the earlier results, as is evident from the results in Tables III and IV. The Chi-square test statistics for overall goodness-of-fit decrease to 13.34 without the volatility-in-mean effect and 13.13 with that effect. The associated p-values are 6.4% and 6.9%, so the model is not rejected at a 5% significance level. Furthermore, the stochastic volatility parameters are virtually unaffected by the introduction of the jump component. Thus, the earlier interpretation of these parameters remains valid, except that it applies only to the diffusion component of the return variance. We must include a genuine return jump component in the characterization of the overall volatility process.

The estimates of the jump parameters are of independent interest. The average number of jumps per day,  $\lambda_0$ , is 0.0137, which implies 3 to 4 jumps per year. The variability of the jump magnitude is



characterized by the  $\delta$  coefficient. Our estimate for  $\delta$  implies a standard deviation of 1.5%, so most jumps should fall within the  $\mp 3\%$  range.<sup>11</sup>

The score t-ratio diagnostics, reported in the  $SVJ_1$  column of Table V, reveal that almost all score moments are insignificant at the 5% level. These results do not point to any particular inadequacy of the SVJD, but rationalizing exceptional episodes such as the sharp market drop in October 1987 remains difficult. It may be that adding another jump component or an alternative distribution for the jump magnitude  $\kappa(t)$  that allows for larger (negative) jumps would help. However, given the very few instances of such jumps, overfitting is a very real possibility, and we did not consider either of those extensions. Overall, allowing for Poisson jumps as well as stochastic volatility appears to capture the main characteristics of the return series quite well.

We also estimate the SVJD (1) and (3) with a square-root volatility specification and constant jump intensity which then becomes the Bates (1996a) model. The results are included in Tables III and IV. The model performs marginally worse than the SVJD (1) and (2) with a log-variance specification – for the square-root representation, the test associated with the over-identifying restrictions attains a p-value of 3.73% (4.96%) – but, realistically, we cannot differentiate between the two models. Again, the t-ratio diagnostics do not identify any particular source of misspecification: all score moments are insignificant at the 5% level.

We come now to the most general model considered, where the jump intensity is a function of the volatility level:  $\lambda(t) = \lambda_0 + \lambda_1 V_t$ . This generalizes not only the constant intensity representations, but also the model of Pan (1999) for which  $\lambda(t) = \lambda_1 V_t$ . Our results are summarized in Tables III and IV. Across the different specifications, the estimates of  $\lambda_1$  are all positive, suggesting that the jump probability may depend on the instantaneous volatility. On the other hand, the point estimates of  $\lambda_1$  are also imprecise and insignificant in all cases.

The point estimates of  $\lambda_0$  are also insignificant. This finding, somewhat surprising, could be an artifact of the approximation used to compute the (Wald) standard errors.<sup>12</sup> To investigate that possibility, we also constructed confidence intervals by inverting a critical region for the criterion function, as proposed by Gallant and Tauchen (1997). The constructed intervals are in general good approximations to those computed from the Wald standard errors, except for  $\lambda_0$ . For the SVJD with log-variance, the 95% confidence interval for  $\lambda_0$  becomes [0.008, 0.022]. Consequently, when confidence intervals are constructed by inverting the criterion function, the affine term  $\lambda_0$  is statistically significant and, overall, results are similar to those for the corresponding model with constant jump intensity. Identical conclusions hold for the square-root specification.

Consistent with the above observations, the p-values associated with the overall goodness-of-fit test are marginally lower than in the models with constant jump intensity. Hence, the linear part of the jump intensity specification seems to have little explanatory value for the distribution of daily equity-index returns. Of course, there is a possibility that the parameter  $\lambda_1$  simply cannot be estimated precisely from our return data. Moreover, even if it is insignificant under the “physical”

measure, the identical effect may still be important under the risk-neutral distribution and thus for valuation of derivatives. Specifically, a large risk premium associated with the jump intensity and the use of option prices for estimation may explain in part the distinctly different conclusions obtained by Pan (1999) for both the average jump intensity and the (conditional) dependence of the jump intensity on concurrent (diffusion) volatility.

As estimated, both SVJD's indicate a weak volatility-in-mean effect. The  $c$  coefficients are small, but positive. Hence, there is some evidence of a positive association between equity volatility and expected return, which is consistent with a volatility risk premium. However, since the estimated premium is small and statistically insignificant, the empirical importance of the effect remains questionable.<sup>13</sup> Note also that, except for a compensating decrease in the drift term constant, none of the parameter estimates change significantly with its inclusion. To conclude, the volatility-in-mean effect has a limited impact on the quality of our characterization of the equity-index return process and will likely be of minimal concern for practical option pricing applications.

### *B.6. Estimation over a Shorter Sample*

EMM results for our continuous-time models based on the relatively small daily sample 1980-1996, and an SNP auxiliary model with a leading EGARCH(2,1) term, are given in Table 6. The auxiliary model is further characterized in Table 7.

The small sample findings are generally consistent with those obtained from the full sample. Consequently, there is no need to qualify our conclusions regarding the inadequacy of the specifications without stochastic volatility or jump components. Furthermore, the SVJD's appear to fit the return distribution better over the small than over the full sample, although the specification tests with the small sample are less powerful. As was found using the full sample, allowing the jump intensity to depend on volatility does not improve the quality of the fit, and the small sample estimate of  $\lambda_1$  remains insignificant.

Between the two sets of estimates, one main difference is in the strength of the asymmetric return-volatility relationship. For the small sample, it is also highly significant but appears somewhat weaker, with estimates of  $\rho$  around -0.40. Not surprisingly, given the dramatic market corrections observed over this sample, the jump intensity is now estimated marginally higher with a jump probability per day of around 1.9% or about 5 jumps per year. Moreover, the estimated average jump size increases as reflected in a larger  $\delta$  coefficient, implying a standard deviation of 2.15%, i.e. jumps will typically fall within the  $\mp 4.3\%$  range. Finally, the volatility persistence measure given by  $\exp\{-\beta\}$  drops marginally to approximately 0.980 at the daily level.

In summary, there are no indications of a substantial structural change in the return generating mechanism during the years 1953-1996. What the relatively smaller sample yields, less precise inference aside, is a marginal increase in the level and variability of volatility, as reflected in more

frequent and larger jumps and a quicker mean reversion in the volatility diffusion process. Overall, we deem the longer sample more useful in providing identifying information regarding the jumps in the equity-index return process and the asymmetric relation between returns and volatility. But on the record, the EMM procedure seems fully capable of extracting meaningful inference from the shorter sample as well. We conclude that the empirical findings discussed in the previous sections are not an artifact of our choice of sample period.

### III. Implications for Option Pricing

A common finding of the empirical derivatives pricing literature is that the return dynamics implied by option prices are incompatible with the time series properties of the underlying asset prices. To the contrary, in this section we establish a general correspondence between the dominant characteristics of the equity return process and options prices. More specifically, we show that most parameter estimates obtained from the daily S&P500 returns under the “physical” probability measure are similar to those extracted in previous studies from option prices under the “risk-neutral” distribution. We also show that the jump component and the asymmetric return-volatility relationship identified from the equity return series are qualitatively consistent with the dynamics implied by derivative prices. For example, our EMM point estimates generate a pronounced volatility “smirk” effect for short-maturity contracts, which, just as in the actual data, becomes less pronounced as maturity increases. Finally, we illustrate how small and sensible risk premia for jump and volatility components are able to reconcile the typical shape of the term structure of implied volatilities in the option prices generated by our model to that observed in market prices. Hence, a large number of characteristics of the stock return process which seem to be implied or priced by associated derivative contracts are independently identified in our empirical analysis as highly significant components of the underlying dynamics of the S&P500 returns.

The computations below rely on the jump-diffusion with square-root volatility defined through equations (1) and (3).<sup>14</sup> The parameters  $\alpha$ ,  $\beta$ ,  $\eta$ ,  $\rho$ ,  $\lambda_0$ ,  $\lambda_1$  and  $\delta$  under the “physical” measure are fixed at the EMM estimates in Table IV. In the presence of jumps and stochastic volatility, appropriate risk adjustment must be incorporated into derivative prices. As suggested by Bates (2000), this can be done in a representative agent economy by rewriting the model (1) and (3) in “risk-neutral” form:

$$\frac{dS_t}{S_t} = (r - d - \lambda^*(t) \bar{\kappa}^*) dt + \sqrt{V_t} dW_{1,t}^* + \kappa^*(t) dq_t^*, \quad (9)$$

$$dV_t = (\alpha - \beta V_t - \xi V_t) dt + \eta \sqrt{V_t} dW_{2,t}^*, \quad (10)$$

where  $r$  and  $d$  are respectively the instantaneous risk-free interest rate and the dividend yield for the underlying stock,  $q^*$  is a Poisson process with parameter  $\lambda^*$ ,  $W_1^*$  and  $W_2^*$  are Standard Brownian Motions under the risk adjusted measure with correlation  $\text{corr}(dW_{1,t}^*, dW_{2,t}^*) = \rho$ , and  $\kappa^*$  is the jump

in the return when the Poisson event occurs, with expected value  $\bar{\kappa}^*$  and variance  $\text{var}(\kappa^*)$ . With this specification, a (semi-)closed form solution – reproduced in Appendix B – is available for computing option prices.

### A. Stochastic Volatility and Jumps

The effects of stochastic volatility and jumps on the pricing of options are shown in Figure 2. Each panel displays Black-Scholes implied volatilities extracted from put option prices computed from our SV diffusion, both with and without a jump component. In all panels, the independent variable is “moneyness,” defined as the ratio of the strike price  $K$  to the underlying price  $S$  minus unity,  $K/S - 1$ . All option prices are computed for a value of the underlying equity-index price  $S$  of \$800. The interest rate  $r$  and dividend yield  $d$  equal 5.1% and 2%, respectively, and the risk premia on volatility and jump risk are fixed at zero. Maturities go from a week to six months.

The left column panels are drawn for an instantaneous volatility level corresponding to an annual return volatility of 11.5%, which is consistent with the estimates for the long-run mean of volatility reported in previous sections. The pronounced skew in implied volatility patterns, induced by the negative relationship between return innovations and volatility, is evident across all maturities. The jump component adds an upward tilt to the pattern at the right end for shorter maturities. This is indicative of a return distribution with fatter tails. Interestingly, with our specification the smile is induced by jumps rather than stochastic volatility, which traditionally has been identified as causing smiles. When the jump intensity is constant (dashed line), the relatively small number of jumps identified in our EMM estimation does not affect the long-term return distribution much: jumps simply add to the unconditional long-term mean of the volatility process. Allowing the jump intensity to depend on volatility (dotted line) does not alter this conclusion: the two plots are virtually indistinguishable, except for minor differences at longer maturities. This should not be surprising, given the small and insignificant  $\lambda_1$  estimate. Thus, in sum, the presence of jumps manifests itself in the (asymmetric) smile pattern of implied volatility at shorter maturities, but the smile dissipates at longer horizons where we observe a pure skew. Since the long-run mean of volatility is near identical across our two models, implied volatilities are virtually the same at longer maturities.

In the right column panels of Figure 2 we illustrate the sensitivity of the option price to the level of (instantaneous) return volatility. The panels depict the Black-Scholes implied volatilities of put prices generated from the SVJD with constant jump intensity. The plots are constructed for instantaneous volatilities that correspond to annualized return volatility of 7%, 11.5%, and 15.5%, respectively. Obviously, an increase in the instantaneous return volatility increases implied volatilities, i.e. option prices. The effect is very strong at short maturities, but becomes less pronounced rather quickly as the time to expiration increases. Indeed, at longer maturities the mean-reverting component of the variance process pushes volatility back towards its long-run level, and the plots converge to that for

a long-run average of 11.5%. Also of interest, the implied volatility smile for the short maturity and low volatility case is nearly symmetric, while for the short maturity and high volatility scenario it is a pure negatively sloped smirk. This is because in the low instantaneous volatility case a large fraction of return volatility is attributable to the jump component, and for the high volatility scenario that is not so. In the latter case, the effect of the jump component not surprisingly weakens.

The evidence in Figure 2 indicates a qualitative correspondence between our estimation results, obtained from a sample of equity returns only, and the stylized characteristics of option prices. This analysis is reinforced by a comparison of our parameter estimates to those reported in studies in which models in “risk-neutral” form are estimated using derivative prices. One key parameter is the  $\rho$  asymmetry coefficient, which is independent of risk adjustments for volatility and jump uncertainty. Our point estimates fall in the -0.58 to -0.62 range for all model specification. These estimates are similar to those obtained from derivative prices. Bakshi, Cao and Chen (1997) report values between -0.57 and -0.64 for the corresponding specifications. Interestingly, they portray their estimates as inconsistent with the underlying equity return dynamics, and, in support of their claim, cite the typical estimate for the discrete-time EGARCH asymmetry coefficient,  $\theta_1$ , in the time-series literature, which is about -0.12; see, e.g., Nelson (1991). Our estimate of  $\theta_1$  is -0.17 (Table II), which, as our results indicate, is fully consistent with – indeed implies – a much more negative value for the corresponding asymmetry coefficient  $\rho$  in the continuous-time model. Another striking, albeit indirect, validation of this finding within the option pricing literature comes from Dumas, Fleming and Whaley (1998). They compute a correlation of -0.57 (page 2064) for the first-order differences of equity-index prices and Black-Scholes implied volatilities.

The importance of this strong negative return-volatility relation in the continuous-time representation of equity index returns is illustrated in Figure 3. The left column of Figure 3 conveys the impact of the  $\rho$  coefficient. The panels depict the Black-Scholes implied volatilities at different maturities, for both the asymmetric ( $\rho \neq 0$ , solid line) and symmetric ( $\rho = 0$ , dashed line) pure stochastic volatility model. Volatility risk premia are constrained to zero. It is evident that a negative  $\rho$  not only is critical for obtaining an adequate fit to the dynamics of equity returns, but it also induces an asymmetric volatility smile over both short and relatively longer maturities, consistent with the volatility “smirk” typically observed in equity-index option markets.

Another set of parameters,  $\alpha$  and  $\eta$ , are invariant to the risk adjustment in equations (9)-(10). Our estimates for the SVJD with constant jump intensity, annualized and expressed in decimal form, are  $\alpha = 0.0470$  and  $\eta = 0.1845$ . The corresponding (average) estimates based on option prices in Bakshi, Cao and Chen (1997) are 0.04 and 0.38 (Table III, p. 2018). The estimates of  $\alpha$  are essentially the same, so the only difference arises with the  $\eta$  coefficient, where the options data indicate a significantly higher volatility of volatility. There are several plausible explanations for this discrepancy. First, it may be evidence of model misspecification. An implausibly high value of  $\eta$  may be needed to accommodate the volatility smile/smirk observed in actual options data. This problem

may be alleviated by a model with two volatility factors or, possibly, by the presence of jumps in the volatility process. This latter hypothesis has also been conjectured by Pan (1999) and Bates (2000), and investigated by Eraker, Johannes and Polson (1999) and Eraker (2000). Second, it may be attributed to a misspecification of (time-varying) risk premia in option prices. Such misspecification may show up in a high “volatility of volatility” coefficient  $\eta$ , which helps accommodate the otherwise unexplained variability in the (risk neutral) volatility process. Third, there may be a small sample bias in the Bakshi, Cao and Chen (1997) coefficient estimate arising from the fact that they consider a short sample for which return volatility typically is well above the average for our longer sample. Fourth, it is likely that the volatility process estimated from daily (squared) returns data using our models is overly smooth, as the daily innovation variance in the volatility process is small – and thus hard to discern from daily data – relative to the variance of the daily return innovation. Hence, although the models generally perform well, they may fail to pick up high-frequency fluctuations in the volatility process. This is illustrated in Andersen, Bollerslev, Diebold and Labys (2001), who contrast volatility estimates obtained from daily data to more accurate high-frequency measures of volatility obtained from intraday time series. The implication is that estimation from daily return data will be able to identify the dominant characteristics of the volatility process, but systematically underestimate the extent of high-frequency movements in the volatility process. In contrast, option based volatility measures are better equipped to capture such movements.

In summary, the qualitative correspondence between the option pricing implications of our EMM estimates and the patterns in actual data is encouraging. This conclusion is supported by the close similarities observed for the key parameters that are invariant across the two probability measures. The only noteworthy exception is the  $\eta$  coefficient which, according to our estimates, is about one half the value reported in the option pricing literature. As indicated, this finding suggests a number of potential model misspecifications in both the objective and the risk-neutral probability measure representations. Since these conjectures are impossible to test efficiently using only daily return data, we leave them for future research.

## *B. Volatility and Jump Risk Premia*

It has been pointed out in numerous studies that Black-Scholes implied volatilities are systematically higher than realized (historical) volatilities, an observation which suggests that option prices embody premia for either volatility or jump risk, or both. In this section we illustrate the pricing of options for “moderate and reasonable” specifications of the risk premia. More specifically, we show that small risk adjustments to our parameters suffice to replicate many of the qualitative characteristics of the volatility “smirk.”

The panels on the right in Figure 3 illustrate how a variance risk premium changes option prices. Each panel contains plots of Black-Scholes implied volatilities computed from put prices generated

by the asymmetric ( $\rho \neq 0$ ) stochastic volatility model (no jumps). The assumption underlying the solid line plots is a zero volatility premium, and for the dashed line plots the underlying assumption is a  $\xi = -0.01$  risk adjustment for volatility uncertainty. The premium increases implied volatilities: a negative  $\xi$  increases the long-run mean of the (risk-neutral) volatility process, thus boosting option prices, and the effect is more pronounced at longer maturities.

In Figure 4 we illustrate how a jump risk premium affects option prices. The model used is one with constant jump intensity. Different plots reflect variation in the (average) jump size coefficient  $\bar{\kappa}^*$  (left column) and intensity  $\lambda_0^*$  (right column). Plotted in the left panels are Black-Scholes implied volatilities obtained from put prices generated by the SVJD with  $\bar{\kappa}^*$  equal to 0%, -1%, and -3%. Making the  $\bar{\kappa}^*$  coefficient more negative increases the skewness in the risk neutral return distribution and thus potentially rationalizes the even more pronounced “smirk” pattern. For  $\bar{\kappa}^* = 0$ , the annualized instantaneous return volatility is 11.5%. But because volatility changes with  $\bar{\kappa}^*$ , we adjust the (instantaneous) diffusion volatility at which we compute option prices to offset changes in  $\bar{\kappa}^*$ , and thus keep the level of instantaneous volatility constant over the values of  $\bar{\kappa}^*$ .<sup>15</sup> Given the mean reversion in the volatility process, significant differences in average volatility to maturity remain, but the adjustment renders the scenarios more comparable at the short end, where the impact of the jump component is most important. The negative  $\bar{\kappa}^*$  accentuates the smile asymmetry, suggesting that a negative premium for jump uncertainty may be helpful, or even necessary, in accommodating the volatility smirk observed in option prices, as argued by, e.g., Pan (1999). However, the impact of the jump specification is again only pronounced at shorter maturities. That is, the implied volatility plots flatten as time to expiration increases. At long maturities, the volatility smile approaches the same degree of asymmetry as for  $\bar{\kappa}^* = 0$ , even though, with a reversion of volatility to its higher long-run value, the implied volatility level is much higher. In summary, given our estimated asymmetry coefficient  $\rho$ , a small risk adjustment on  $\bar{\kappa}^*$  generates a deep volatility “smirk” in short-maturity options. This effect becomes less pronounced as maturity increases, as it does for actual S&P500 options data.

Figure 4, right column, relates to the identical model, but illustrates the effect of a risk adjustment involving  $\lambda_0^*$ . The unbroken line is for  $\lambda_0^* = 0.0202$ , or an average of 5 jumps per year. Increasing the average number of jumps to 10 per year increases implied volatilities (dotted line); and reducing the average number of jumps to 2 per year lowers the implied volatilities (dashed line). For short maturities, the increased jump intensity accentuates the upward tilt at the right end of the implied volatility pattern, making the smirk more of a symmetric smile. But the change is minimal for the longest maturities. Similar results (not reported) are obtained if the jump intensity is an affine function of volatility:  $\lambda^*(t) = \lambda_0^* + \lambda_1^* V_t$ . The impact of more jumps per year increases with the level of volatility and the magnitude of  $\lambda_0^*$  and  $\lambda_1^*$ . Higher values of the intensity parameters increase the probability of a jump, and a higher volatility level magnifies the effect. We can sum up as follows: changing the jump intensity impacts the qualitative characteristics of option prices mostly at the

shorter maturities, and a time-dependent jump intensity does not alter this result significantly.

### *C. The Term Structure of Implied Volatilities*

The term structure of implied volatilities in Figure 5 are for put prices obtained using the stochastic volatility model with constant intensity jumps. The full drawn line displays an “average” Black-Scholes implied volatility dynamics under the “physical” probability measure, as estimated in Table IV, and the dashed line incorporates instead the combined effect of jump and volatility risk premia. The panels on the left are constructed for instantaneous volatility fixed at 11.5%, while the panels on the right portray an average term structure across different volatility levels, as indicated below. To compare our term structures to those implicit in actual option data, we rely on the evidence in Table II (p. 2015) of Bakshi, Cao and Chen (1997) (BCC). Among their findings is that Black-Scholes implied volatilities for out-of-the-money puts are decreasing in term to maturity. That is consistent with the term structure in the upper left panel. Another finding of BCC is that the term structure for at-the-money puts is flat or sometimes slightly upward sloping. This is also in line with our model implications in the middle left panel. The term structure is flat under the “physical” probability measure, while very small values of the risk premia are enough to generate a moderate upward tilt, as observed in market prices. Finally, BCC report a downward sloping term structure for in-the-money puts. This is partially confirmed by the bottom panel of Figure 5. The downward sloping pattern is strong for maturities up to 2 months. For longer maturities, though, the term structure exhibits a slight upward tilt. Nevertheless, these results are readily reconciled. The key is to recognize that the BCC findings arise from an averaging of term structures across different (instantaneous) volatility levels. By comparison to our daily series, the sample used by BCC is small and characterized by alternating periods of average and well above average equity-index return volatility. Hence, their Table II describes the average Black-Scholes implied volatilities over a period with generally high return volatility. It is obviously not necessary for any individual term structure, even if it attains the average volatility over the sample, to replicate the shape of the term structure averaged across all realized volatility levels. Nonetheless, to assess the consequences of the BCC high volatility bias, we average our implied volatilities across medium- and high-volatility states (11.5%-15.5%). The averages are plotted in the right column of Figure 5. Averaging accentuates the downward trend in the term structure, as higher volatility states revert towards the lower long-run mean. The term structure in the bottom panel of the right column mimics that estimated using the BCC sample. Moreover, very small values of the volatility and jump risk premia are enough to make the term structure virtually flat for at-the-money contracts, as observed in market prices. Thus, our model, obtained under the “physical” measure, replicates most of the stylized facts for option prices. Volatility and jump risk premia provide additional flexibility, and only small risk adjustments are required to make model and market patterns of implied volatilities qualitatively very similar.



## IV. Conclusions

Much asset and derivative pricing theory is based on diffusion models for primary securities. Yet, there are very few estimates of satisfactory continuous-time models for equity returns. The objective of this paper is to identify a class of jump-diffusions that are successful in approximating the S&P500 return dynamics and therefore should constitute an adequate basis for continuous-time asset pricing applications. We extend the class of stochastic volatility diffusions by allowing for Poisson jumps of time-varying intensity in returns. We also explore alternative models both within and outside of the popular affine class. Estimation is performed by careful implementation of the EMM that provides powerful model diagnostics and specification tests. Finally, we explore the relationship between our estimated models and option prices. We contrast those of our parameter estimates which are invariant to adjustments for volatility and jump risk to those reported in the option literature, and provide a qualitative comparison of the pricing implications of our estimated system and the stylized evidence from actual option data.

We find that every variant of our stochastic volatility diffusions without jumps fails to jointly accommodate the prominent characteristics of the daily S&P500 returns. Further, every specification that does not incorporate a strong negative correlation between return innovations and diffusion volatility fails as well. In contrast, two versions of our SVJD's that incorporate discrete jumps and stochastic volatility, with return innovations and diffusion volatility strongly and negatively correlated, accommodate the main features of the daily S&P500 returns. This is true not only of the models estimated using the entire sample of daily return observations, but also as estimated using subsamples. The models therefore appear to be structurally stable. Finally, we find that those parameter estimates which are invariant to adjustments for volatility and jump risk generally are similar to those reported in the option literature, and document that "small" risk premia suffice to produce pronounced patterns in Black-Scholes option implied volatilities that are qualitatively consistent with the stylized evidence from derivatives markets. Thus, the main characteristics of the stock price process implied by options data are independently identified as highly significant components of the underlying S&P500 returns dynamics.

One potential extension of our work is to obtain direct estimates for the underlying volatility process. That could be done, as Gallant and Tauchen (1998) suggest, by means of "reprojection" within the EMM setting. Obtaining such estimates will facilitate forecasting of future return distributions, with obvious implications for portfolio choice and derivatives pricing. Further, providing a reasonable fit to the long memory characteristics of the volatility process – excluded by us from our diffusion specifications – appears to be another interesting extension. Finally, there is more experimentation with alternative jump specifications to be done, in light of the extreme and infrequent outliers that have been observed and not yet fully rationalized. On this dimension, the recent work of Eraker, Johannes and Polson (1999) and Chernov et al. (1999) provides an interesting starting point.

## Appendix A: Numerical Implementation of EMM

In this Appendix we provide more details on the algorithm used to simulate returns from the SVJD model.

First, log-returns are used to fit the auxiliary model, hence Itô's Lemma is applied to the continuous-time model to obtain a characterization for the log-return process. Expressing returns in decimal form, this yields:

$$d \ln(S) = (\mu - \lambda(t)\bar{\kappa} - 0.5 V_t)dt + \sqrt{V_t}dW_{1,t} + \ln(1 + \kappa(t))dq_t \quad (11)$$

where  $\ln(1 + \kappa(t)) \rightsquigarrow N(\ln(1 + \bar{\kappa}) - 0.5 \delta^2, \delta^2)$  and the (log-)variance process  $V$  obeys either (2) or (3). Log-returns in percentage form satisfy an expression similar to (11), obtained by multiplying (11) by a factor of 100.

The Euler scheme - see, e.g., Kloeden and Platen (1992) - is then applied to generate a sample  $\{r_t(\psi), x_t(\psi)\}_{t=1}^{T(N)}$  from the continuous-time model for log-returns. Simulation from the stochastic volatility model is not problematic, hence we refer to Andersen and Lund (1997) for more details and focus exclusively on the jump component.

Poisson jumps are first approximated with a Binomial distribution, i.e., we replace  $dq_t$  with a random variable  $Y$  such that  $\text{Prob}\{Y = 1\} = \lambda(t) dt$  and  $\text{Prob}\{Y = 0\} = (1 - \lambda(t) dt)$ . For this purpose, we generate a random variable  $U \text{ Uniform}(0,1)$  and we smooth the discontinuity of  $Y$  over an interval centered around  $1 - \lambda(t)$ :

$$Y = \begin{cases} 0 & \text{if } 0 \leq U < 1 - \lambda(t) dt - h/2, \\ g(X) & \text{if } 1 - \lambda(t) dt - h/2 \leq U < 1 - \lambda(t) dt + h/2, \\ 1 & \text{if } 1 - \lambda(t) dt + h/2 \leq U \leq 1, \end{cases}$$

where  $X = U - (1 - \lambda(t) dt - h/2)$  and  $g(X) = -2/h^3 X^3 + 3/h^2 X^2$  for  $0 \leq X \leq h$ . Notice that  $g$  is a  $C^\infty$  function, and that it becomes steeper as the interval length  $h$  goes to zero. In our application we fine-tune  $h$  by choosing the smallest possible size for the interpolation interval that eliminates the numerical problems in the EMM criterion function. This yields an accurate approximation to the jumps in the simulated return sequence.

Convergence conditions for the Euler approximations in a jump-diffusion setting are discussed in, e.g., Kloeden and Platen (1989) and Protter and Talay (1997). These conditions are not explicitly verified for our specific approximation algorithm. As is often the case with these high-level assumptions, it is very hard to do. Nevertheless, it does not appear to constitute a problem for our application as extensive simulations verify that the moments of the simulated process converge.

As a final remark, at any iteration of the minimization each jump  $\kappa(t)$  is generated, in the event  $dq_t = 1$ , using the identical seed. Also, we obtain variance reduction through the use of antithetic variates in the simulation; see, e.g., Geweke (1996) for a discussion of this technique.

## Appendix B: Option prices in the presence of stochastic volatility and jumps

Given the risk-adjusted model (9)-(10) a closed-form formula is available for computing option prices. As shown in, e.g., Bates (2000), it is given by

$$f(S_t, V_t, \tau; K) = e^{-d\tau} S_t P_1 + e^{-r\tau} K P_2,$$

where, for  $j = 1, 2$  :

$$\begin{aligned} P_j &= \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \frac{\text{imag} \left( F_j(i\Phi) e^{-i\Phi x} \right)}{\Phi} d\Phi, \\ F_j(\Phi; V, \tau) &= \exp \{ A_j(\tau; \Phi) + B_j(\tau; \Phi) V + \lambda_0^* \tau C_j(\Phi) \}, \\ A_j(\tau; \Phi) &= (r - d) \Phi \tau - \frac{\alpha \tau}{\eta^2} (\rho \eta \Phi - \beta_j - \gamma_j) - \\ &\quad \frac{2\alpha}{\eta^2} \ln \left( 1 + \frac{1}{2} (\rho \eta \Phi - \beta_j - \gamma_j) \frac{1 - e^{\gamma_j \tau}}{\gamma_j} \right), \\ B_j(\tau; \Phi) &= -2 \frac{1/2 (\Phi^2 + (3 - 2j)\Phi) + \lambda_1^* C_j(\Phi)}{\rho \eta \Phi - \beta_j + \gamma_j \frac{1 + e^{\gamma_j \tau}}{1 - e^{\gamma_j \tau}}}, \\ C_j(\Phi) &= (1 + \bar{\kappa}^*)^{(2-j)} \left( (1 + \bar{\kappa}^*)^\Phi e^{1/2 \delta^2 (\Phi^2 + (3-2j)\Phi)} - 1 \right) - \bar{\kappa}^* \Phi, \\ \gamma_j &= \sqrt{(\rho \eta \Phi - \beta_j)^2 - 2\eta^2 (1/2 (\Phi^2 + (3 - 2j)\Phi) + \lambda_1^* C_j(\Phi))}, \\ \beta_j &= \beta + \xi + \rho \eta (j - 2), \quad x = \ln(K/S_t), \end{aligned}$$

and  $r$ ,  $d$  are respectively the instantaneous risk-free interest rate and the dividend yield for the underlying stock.

## REFERENCES

- Aït-Sahalia, Yacine, 1996, Nonparametric Pricing of Interest Rate Derivative Securities, *Econometrica* 64, 527-560.
- Akgiray, Vedat, and Geoffrey Booth, 1986, Stock Price Processes with Discontinuous Time Paths: An Empirical Examination, *The Financial Review* 21, 163-184.
- Andersen, Torben G., Tim Bollerslev, Francis X. Diebold, and Paul Labys, 2001, Modeling and Forecasting Realized Volatility, Working Paper, Northwestern University.
- Andersen, Torben G., Hyung-Jin Chung, and Bent E. Sørensen, 1999, Efficient Method of Moments Estimation of a Stochastic Volatility Model: A Monte Carlo Study, *Journal of Econometrics* 91, 61-87.
- Andersen, Torben G., and Jesper Lund, 1996, Stochastic Volatility and Mean Drift in The Short Term Interest Rate Diffusion: Sources of Steepness, Level and Curvature in The Yield Curve, Working Paper #214, Northwestern University.
- Andersen, Torben G., and Jesper Lund, 1997, Estimating Continuous-Time Stochastic Volatility Models of the Short Term Interest Rate, *Journal of Econometrics* 77, 343-377.
- Backus, David K., and Allan W. Gregory, 1993, Theoretical Relations between Risk Premiums and Conditional Variances, *Journal of Business and Economic Statistics* 11, 177-185.
- Backus David K., Silverio Foresi, Kai Li, and Liuren Wu, 1997, Accounting for Biases in Black-Scholes, Working Paper, New York University.
- Bakshi, Gurdip, Charles Cao, and Zhiwu Chen, 1997, Empirical Performance of alternative Option Pricing Models, *Journal of Finance* 52, 2003-2049.
- Bakshi, Gurdip, Charles Cao, and Zhiwu Chen, 2000, Pricing and Hedging Long-Term Options, *Journal of Econometrics* 94, 277-318.
- Bakshi, Gurdip, and Nikunj Kapadia, 2001, Delta-Hedged Gains and the Negative Market Volatility Risk Premium, Working Paper, University of Maryland.
- Bakshi, Gurdip, and Dilip Madan, 2000, Spanning and Derivative-Security Valuation, *Journal of Financial Economics* 55, 205-238.
- Ball, Clifford A., and Walter N. Torous, 1985, On Jumps in Common Stock Prices and Their Impact on Call Option Pricing *Journal of Finance* 40, 155-173.

- Bandi, Federico M., and Thong Nguyen, 2000, On the Functional Estimation of Jump-diffusion Processes, Working Paper, University of Chicago and Yale University.
- Bandi, Federico M., and Peter C. B. Phillips, 1998, Econometric Estimation of Diffusion Models, Working paper, Cowles Foundation, Yale University.
- Bates, David S., 1996a, Jumps and Stochastic Volatility: Exchange Rate Processes Implicit in Deutsche Mark Options, *The Review of Financial Studies* 9, 69-107.
- Bates, David S., 1996b, Testing Option Pricing Models, in G. S. Maddala and C. R. Rao, eds.: *Handbook of Statistics*, vol. 14, *Statistical Methods in Finance* (North-Holland, Amsterdam).
- Bates, David S., 2000, Post-'87 Crash Fears in S&P 500 Futures Options, *Journal of Econometrics* 94, 181-238.
- Benzoni, Luca, 1998, Pricing Options under Stochastic Volatility: an Econometric Analysis, Working Paper, University of Minnesota.
- Brenner, Menachem, and Young H. Eom, 1997, No-Arbitrage Option Pricing: New Evidence on the Validity of the Martingale Property, Working Paper, New York University.
- Carrasco, Marine, Mikhail Chernov, Eric Ghysels, and Jean-Pierre Florens, 2001, Estimating Diffusions With a Continuum of Moment Conditions, Working Paper, University of North Carolina.
- Chacko, George, and Luis M. Viceira, 1999, Spectral GMM Estimation of Continuous-Time Processes, Working Paper, Harvard University.
- Chernov, Mikhail, and Eric Ghysels, 2000, A Study towards a Unified Approach to the Joint Estimation of Objective and Risk Neutral Measures for the Purpose of Options Valuation, *The Journal of Financial Economics* 56, 407-458.
- Chernov, Mikhail, A. Ronald Gallant, Eric Ghysels, and George Tauchen, 1999, A New Class of Stochastic Volatility Models with Jumps: Theory and Estimation, Working Paper, Penn State University, University of North Carolina, Duke University.
- Chernov, Mikhail, A. Ronald Gallant, Eric Ghysels, and George Tauchen, 2000, Alternative Models for Stock Price Dynamics, Working Paper, Columbia University, University of North Carolina, Duke University.
- Conley, Timothy G., Lars P. Hansen, Erzo G. J. Luttmer, and José A. Scheinkman, 1997, Short-Term Interest Rates as Subordinated Diffusions, *The Review of Financial Studies* 10, 525-577.

- Cox, John C., Jonathan E. Ingersoll, Jr., and Stephen A. Ross, 1985, A Theory of the Term Structure of Interest Rates, *Econometrica* 53, 385-407.
- Das, Sanjiv R., 1999, The Surprise Element: Interest Rates as Jump-Diffusions, Working Paper, Harvard Business School.
- Das, Sanjiv R., and Silverio Foresi, 1996, Exact Solutions for Bond and Option Prices with Systematic Jump Risk, *Review of Derivatives Research* 1, 7-24.
- Das, Sanjiv R., and Rangarajan K. Sundaram, 1999, Of Smiles and Smirks: A Term-Structure Perspective, *Journal of Financial and Quantitative Analysis*, 34, 211-239.
- Das, Sanjiv R., and Raman Uppal, 1998, International Portfolio Choice with Systemic Risk, Working Paper, Harvard Business School and MIT.
- Duffie, Darrell, and Kenneth J. Singleton, 1993, Simulated Moments Estimation of Markov Models of Asset Prices, *Econometrica* 61, 929-952.
- Duffie, Darrell, Jun Pan, and Kenneth J. Singleton, 2000, Transform Analysis and Asset Pricing for Affine Jump Diffusions, *Econometrica* 68, 1343-1376.
- Dumas, Bernard, Fleming, Jeff, and Robert E. Whaley, 1998, Implied Volatility Functions: Empirical Tests, *Journal of Finance* 53, 2059-2106.
- Elerian, Ola, Siddhartha Chib, and Neil Shephard, 2001, Likelihood Inference for Discretely Observed Non-linear Diffusions, forthcoming *Econometrica*.
- Eraker, Bjørn, 2000, Do Stock Prices and Volatility Jump? Reconciling Evidence from Spot and Option Prices, Working Paper, University of Chicago.
- Eraker, Bjørn, 2001, MCMC Analysis of Diffusion Models with Application to Finance, *Journal of Business and Economic Statistics*, 19, 177-191.
- Eraker, Bjørn, Michael S. Johannes, and Nicholas G. Polson, 1999, The Impact of Jumps on Volatility and Returns, Working Paper, University of Chicago.
- Gallant, A. Ronald, David A. Hsieh, and George Tauchen, 1997, Estimation of Stochastic Volatility Models with Diagnostics, *Journal of Econometrics* 81, 159-192.
- Gallant, A. Ronald, Chien-Te Hsu, and George Tauchen, 1999, Using Daily Range Data to Calibrate Volatility Diffusions and Extract the Forward Integrated Variance, *The Review of Economics and Statistics*, 81, 617-631.

- Gallant, A. Ronald, and Jonathan R. Long, 1997, Estimating Stochastic Differential Equations Efficiently by Minimum Chi-Square, *Biometrika* 84, 125-142.
- Gallant, A. Ronald, and Douglas W. Nychka, 1987, Semi-Nonparametric Maximum Likelihood Estimation, *Econometrica* 55, 363-390.
- Gallant, A. Ronald, Peter E. Rossi, and George Tauchen, 1992, Stock Prices and Volumes, *The Review of Financial Studies* 5, 199-242.
- Gallant, A. Ronald, and George Tauchen, 1996, Which Moments to Match?, *Econometric Theory* 12, 657-681.
- Gallant, A. Ronald, and George Tauchen, 1997, Estimation of Continuous-Time Models for Stock Returns and Interest Rates, *Macroeconomic Dynamics* 1, 135-168.
- Gallant, A. Ronald, and George Tauchen, 1998, Reprojecting Partially Observed Systems With Application to Interest Rate Diffusions, *Journal of the American Statistical Association* 93, 10-24.
- Geweke, John, 1996, Monte Carlo Simulation and Numerical Integration, in Amman, Hans M., David A. Kendrick and John Rust, eds.: *Handbooks in Economics*, vol. 13, *Handbook of Computational Economics*, vol. 1, (North-Holland, Amsterdam).
- Ghysels, Eric, Andrew C. Harvey, and Eric Renault, 1998, Stochastic Volatility, in G. S. Maddala and C. R. Rao, eds.: *Handbook of Statistics*, vol. 14, *Statistical Methods in Finance* (North-Holland, Amsterdam).
- Glosten, Lawrence R., Ravi Jagannathan, and David Runkle, 1993, On the Relation Between the Expected Value and the Volatility of the Nominal Excess Return on Stocks, *Journal of Finance* 48, 1779-1801.
- Hansen, Lars P. and José A. Scheinkman, 1995, Back to the Future: Generating Moment Implication for Continuous-Time Markov Processes, *Econometrica* 63, 767-804.
- Heston, Steven L., 1993, A Closed Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options, *The Review of Financial Studies* 6, 327-343.
- Ho, Mun S., William R. M. Perraudin, and Bent E. Sørensen, 1996, A Continuous-Time Arbitrage Pricing Model With Stochastic Volatility and Jumps, *Journal of Business and Economic Statistics* 14, 31-43.
- Jacquier, Eric, Nicholas G. Polson, and Peter E. Rossi, 1994, Bayesian Analysis of Stochastic Volatility Models, *Journal of Business and Economic Statistics* 12, 371-389.

- Jarrow, Robert, and Eric Rosenfeld, 1984, Jump Risks and the Intertemporal Capital Asset Pricing Model, *Journal of Business* 57, 337-351.
- Jarrow, Robert, and Andrew Rudd, 1982, Approximate Option Valuation for Arbitrary Stochastic Processes, *Journal of Financial Economics* 10, 347-369.
- Jiang, George J., and John L. Knight, 1997, A Nonparametric Approach to the Estimation of Diffusion Processes, with an Application to a Short-Term Interest Rate Model, *Econometric Theory* 13, 615-645.
- Jiang, George J., and John L. Knight, 1999, Efficient Estimation of the Continuous Time Stochastic Volatility Model Via the Empirical Characteristic Function, Working Paper, York University.
- Jiang, George J., and Pieter J. van der Sluis, 1999, Index Option Pricing Models with Stochastic Volatility and Stochastic Interest Rates, *European Finance Review* 3, 273-310.
- Johannes, Michael, 1999, A Non-Parametric Approach to Jumps in Interest Rates, Working Paper, Columbia University.
- Johannes, Michael, Rohit Kumar, and Nicholas G. Polson, 1999, State Dependent Jump Models: How do US Equity Indices Jump?, Working Paper, University of Chicago.
- Jones, Christopher S., 1998, Bayesian Estimation of Continuous-Time Finance Models, Working Paper, Rochester University.
- Jones, Christopher S., 1999, The Dynamics of Stochastic Volatility: Evidence from Underlying and Option Markets, Working Paper, Rochester University.
- Jorion, Philippe, 1988, On Jump Processes in the Foreign Exchange and Stock Markets, *The Review of Financial Studies* 1, 427-445.
- Kim, Sangjoon, Neil Shephard, and Siddhartha Chib, 1998, Stochastic Volatility: Likelihood Inference and Comparison with ARCH Models, *Review of Economic Studies* 65, 361-393.
- Kloeden, Peter E., and Eckhard Platen, 1989, A Survey of Numerical Methods for Stochastic Differential Equations, *Stochastic Hydrology and Hydraulics* 3, 155-178.
- Kloeden, Peter E., and Eckhard Platen, 1992, Numerical Solutions of Stochastic Differential Equations, Springer-Verlag.
- Liu, Jun, 1997, Generalized Method of Moments Estimation of Affine Diffusion Processes, Working Paper, Graduate School of Business, Stanford University.



- Liu, Ming, and Harold H. Zhang, 1997, Specification Tests in the Efficient Method of Moments Framework with Application to the Stochastic Volatility models, Working Paper, The Chinese University of Hong Kong and Carnegie Mellon University.
- Lo, Andrew W., 1988, Maximum Likelihood Estimation of Generalized Itô Processes With Discretely Sampled Data, *Econometric Theory* 4, 231-247.
- Lo, Andrew W., and A. Craig MacKinlay, 1990, An Econometric Analysis of Nonsynchronous Trading, *Journal of Econometrics* 45, 181-212.
- Longstaff, Francis A., 1995, Option Pricing and the Martingale Restriction, *Review of Financial Studies* 8, 1091-1124.
- Melino Angelo, and Stuart M. Turnbull, 1990, Pricing Foreign Currency Options with Stochastic Volatility, *Journal of Econometrics* 45, 239-265.
- Merton, Robert C., 1976, Option Pricing when Underlying Stock Returns Are Discontinuous, *Journal of Financial Economics* 3, 125-144.
- Naik, Vasant, 1993, Option Valuation and Hedging Strategies with Jumps in the Volatility of Asset Returns, *Journal of Finance* 48, 1969-1984.
- Naik, Vasant, and Moon H. Lee, 1990, General Equilibrium Pricing of Options on the Market Portfolio with Discontinuous Returns, *The Review of Financial Studies* 3, 493-521.
- Nelson, Daniel B., 1991, Conditional Heteroskedasticity in Asset Returns: a New Approach, *Econometrica* 59, 347-370.
- Pan, Jun, 1999, The Jump-Risk in Options: Evidence from an Integrated Time-Series Study, forthcoming *Journal of Financial Economics*.
- Pastorello, Sergio, Eric Renault, and Nizar Touzi, 2000, Statistical Inference for Random Variance Option Pricing, *Journal of Business and Economic Statistics* 18, 358-367.
- Pedersen, Asger, 1995, A New Approach to Maximum Likelihood Estimation for Stochastic Differential Equations Based on Discrete Observations, *Scandinavian Journal of Statistics* 22, 55-71.
- Poteshman, Allen M., 1998, Estimating a General Stochastic Variance Model from Option Prices, Working Paper, University of Illinois at Urbana-Champaign.
- Press, S. James, 1967, A Compound Events Model for Security Prices, *Journal of Business* 40, 317-335.

- Protter, Philip, and Denis Talay, 1997, The Euler Scheme for Lévy Driven Stochastic Differential Equations, *Annals of Probability* 25, 393-423.
- Rubinstein, Mark, 1994, Implied Binomial Trees, *Journal of Finance* 49, 771-818.
- Singleton, Kenneth J., 2001, Estimation of Affine Asset Pricing Models Using the Empirical Characteristic Function, *Journal of Econometrics* 102, 111-141.
- Stanton Richard H., 1997, A Nonparametric Model of Term Structure Dynamics and the Market Price of Interest Rate Risk, *Journal of Finance* 52, 1973-2002.
- van der Sluis, Pieter J., 1997, Computationally Attractive Stability Tests for the Efficient Method of Moments, Working Paper, University of Amsterdam.

## Notes

<sup>1</sup>See, e.g., Das and Foresi (1996) and Bakshi, Cao and Chen (1997). Also, on the importance of jumps for option pricing and hedging see, among others, Ball and Torous (1985), Das and Sundaram (1999), Jorion (1988), Naik and Lee (1990) and Naik (1993).

<sup>2</sup>This work builds on the earlier contributions on discrete-time MCMC estimation by Jacquier, Polson and Rossi (1994) and Kim, Shephard and Chib (1998).

<sup>3</sup>See also the analysis in Jarrow and Rudd (1982), Longstaff (1995), Brenner and Eom (1997) and Backus et al. (1997) based on a semi-nonparametric approximation of the return density.

<sup>4</sup>Additional work on continuous- and discrete-time estimation of stochastic volatility models for asset returns based on method of moments procedures includes, among others, Andersen and Lund (1996, 1997), Gallant and Tauchen (1997, 1998), Gallant, Hsu and Tauchen (1999), Ho, Perraudin and Sørensen (1996), Jiang and van der Sluis (1999), Liu and Zhang (1997), Pastorello, Renault and Touzi (2000), and van der Sluis (1997).

<sup>5</sup>Gallant and Long (1997) show that certain non-Markovian score generators are valid auxiliary models, so that lower-order GARCH and EGARCH models may be used in lieu of less parsimonious pure ARCH representations. This is essential for good finite-sample performance, as is evident from the simulation evidence in Andersen, Chung and Sørensen (1999).

<sup>6</sup>Since the S&P500 index is not adjusted for dividends, it is more correct to state that we model the observed series of log-price differences. We adopt the term “return process” for ease of exposition.

<sup>7</sup>Gallant, Rossi and Tauchen (1992) also use dummy variables to capture day-of-the-week, week, month, and year effects in the S&P500 returns. Our approach falls somewhere between using their extensive prefiltering procedure and using the raw returns.

<sup>8</sup>The mixture in the conditional density  $f_K(r_t|x_t;\xi)$  is used to avoid instability during EMM estimation. For a given simulated trajectory,  $P_K(z, x)$  might equal zero, which would cause numerical problems when evaluating the score function (the practical importance of this point was noted by Qiang Dai).

<sup>9</sup>We also estimated the system using two antithetic simulated samples of  $150,000 \times 10 + 10,000$  returns, finding nearly identical results. Furthermore, when the data generating process contains a jump component, the simulation step involves an additional layer of approximation as our procedure for generating jumps render the EMM criterion function discontinuous in the parameter vector, and this creates problems for the numerical minimization of the EMM objective function. To avoid this problem, jumps are smoothed using a close continuously differentiable approximation, as described in Appendix A.

<sup>10</sup>To facilitate a comparison with the empirical option pricing literature, it may be useful to convert parameter estimates for the square-root model into decimal form on a yearly basis. Assuming 252 business days per year, they become:  $\mu = 0.0756$ ,  $\alpha = 0.0438$ ,  $\beta = 3.2508$ ,  $\eta = 0.1850$ ,  $\rho = -0.5878$ .

<sup>11</sup>t-ratios and confidence intervals are constructed in the usual way from the (Wald) estimate of

asymptotic standard errors. Gallant and Tauchen (1997) warn that this approach may be somewhat misleading if the EMM criterion function is highly nonlinear in the parameters as may occur near the boundary of the parameter space. To address this concern we also compute likelihood-ratio style confidence intervals via an inversion of the criterion function as illustrated in Gallant and Tauchen (1997). The constructed intervals are in general good approximations of those computed from the Wald standard errors. The only noteworthy exception concerns the  $\lambda_0$  coefficient in the case where the jump intensity depends on the volatility level. This case is discussed below.

<sup>12</sup>Alternatively, there is a multicollinearity type problem induced through a high degree of correlation between the estimates of  $\lambda_0$  and  $\lambda_1$ .

<sup>13</sup>There is, in fact, no compelling theoretical reason to believe that  $c$  must be positive over the entire support of the volatility process, as discussed by, e.g., Backus and Gregory (1993) and Glosten, Jagannathan and Runkle (1993). The violation of no-arbitrage conditions is limited to the case where stocks earn (risk-free) excess returns while the diffusion volatility is (near) zero. Volatility levels approaching zero are actually not observed over our sample.

<sup>14</sup>See also, among others, Bakshi, Cao and Chen (1997, 2000), Bates (1996a,b, 2000), Benzoni (1998), Chernov and Ghysels (2000), Das and Sundaram (1999), Duffie, Pan and Singleton (2000), Heston (1993), Jiang and van der Sluis (1999), Jones (1999), Poteshman (1998) and Pan (1999) for applications based on a similar model.

<sup>15</sup>With constant jump intensity, the variance of the jump component, evaluated under the risk-neutral measure, is given by  $V_J(\bar{\kappa}^*) \equiv \text{Var}_t(k(t) dq(t))/dt = (1 + \bar{\kappa}^*)^2(e^{\delta^2} - 1)\lambda_0^* + (\bar{\kappa}^*)^2\lambda_0^*$ . In our application, we adjust the return volatility by  $V_J(\bar{\kappa}^* = k) - V_J(\bar{\kappa}^* = 0)$ , where  $k = -1\%$  and  $k = -3\%$  respectively.

## Tables and Figures

**Table I**  
**S&P500 daily rate of return**

Summary statistics. Data on daily rates of return of the S&P500 index, 01/02/1953-12/31/1996 (N=11,076 observations) and 01/03/1980-12/31/1996 (N=4,298 observations). All figures expressed on a daily basis in percentage form.

	<i>1953-96</i>	<i>1980-96</i>
Mean	0.0301	0.0453
Std. Dev.	0.8324	0.9619
Skewness	-2.0353	-3.3390
Kurtosis	60.6019	83.4004

Autocorrelation of Returns:

	<i>1st</i>	<i>2nd</i>	<i>3rd</i>	<i>4th</i>	<i>5th</i>	<i>6th</i>
1953-96	0.1240	-0.0320	-0.0084	-0.0056	0.0222	-0.0131
1980-96	0.0535	-0.0322	-0.0324	-0.0424	0.0418	0.0089

Augmented Dickey Fuller test for the presence of unit roots. The test is based on the regression:

$$\Delta X_t = \mu + \delta t + \gamma X_{t-1} + \sum_{j=1}^{12} \psi_j \Delta X_{t-j} + \varepsilon_t.$$

		<i>S&amp;P500 daily prices</i>	<i>S&amp;P500 daily returns</i>
Augmented D. F.	53-96	2.85	-29.65
	80-96	-0.40	-18.97
5% critical value		-3.41	-3.41
1% critical value		-3.96	-3.96

**Table II**  
**SNP model estimates**

Data on daily rates of return of the S&P500 index, 01/02/1953-12/31/1996, filtered using an MA(1) model (N=11,076 observations). Parameter estimates are expressed in percentage form on a daily basis, and refer to the following model:

$$f_K(r_t|x_t;\xi) = \left( \nu + (1 - \nu) \times \frac{[P_K(z_t, x_t)]^2}{\int_{\mathbb{R}} [P_K(z_t, x_t)]^2 \phi(u) du} \right) \frac{\phi(z_t)}{\sqrt{h_t}}, \quad \nu = 0.01,$$

where  $\phi(\cdot)$  is the standard normal density,

$$\begin{aligned} z_t &= \frac{r_t - \mu_t}{\sqrt{h_t}}, \\ \mu_t &= \phi_0 + c h_t, \\ \ln h_t &= \omega + \sum_{i=1}^p \beta_i \ln h_{t-i} + (1 + \alpha_1 L + \dots + \alpha_q L^q) [\theta_1 z_t + \theta_2 (b(z_t) - \sqrt{2/\pi})], \\ b(z) &= |z| \text{ for } |z| \geq \pi/2K, \quad b(z) = (\pi/2 - \cos(Kz))/K \text{ for } |z| < \pi/2K, \quad K = 100, \\ P_K(z, x) &= \sum_{i=0}^{K_z} a_i(x) z^i = \sum_{i=0}^{K_z} \left( \sum_{|j|=0}^{K_x} a_{ij} x^j \right) z^i, \quad a_{00} = 1. \end{aligned}$$

Parameter	EGARCH leading term		EGARCH-M leading term	
	Estimate	(std. error)	Estimate	(std. error)
$\phi_0$	0.0331	(0.0142)	0.0546	(0.0394)
$c$			0.0315	(0.0331)
$\omega$	4.3769	(1.1249)	3.5526	(1.4211)
$\alpha$	-0.4391	(0.0635)	-0.4367	(0.0624)
$\beta_1$	0.9893	(0.0022)	0.9880	(0.0028)
$\theta_1$	-0.1581	(0.0195)	-0.1407	(0.0304)
$\theta_2$	0.2973	(0.0280)	0.3003	(0.0269)
$a_{10}$	-0.0102	(0.0109)	-0.0489	(0.0495)
$a_{20}$	-0.2499	(0.0291)	-0.2480	(0.0314)
$a_{30}$	-0.0208	(0.0069)	-0.0021	(0.0242)
$a_{40}$	0.1234	(0.0177)	0.1213	(0.0195)
$a_{50}$	-0.0065	(0.0077)	-0.0177	(0.0161)
$a_{60}$	-0.0516	(0.0089)	-0.0504	(0.0100)
$a_{70}$	0.0010	(0.0065)	0.0087	(0.0100)
$a_{80}$	0.0508	(0.0098)	0.0509	(0.0097)

**Table III**  
**EMM estimates of the Black and Scholes and the continuous-time, stochastic volatility models (1) and (2)**  
**and (1) and (3)**

Estimates are for to the sample period 01/02/1953-12/31/1996. Standard errors are reported in brackets. Parameter estimates are expressed in percentage form on a daily basis, and refer to the following model:

$$\begin{aligned}
 BS(J) : \quad \frac{dS_t}{S_t} &= (\mu - \lambda_0 \bar{\kappa}) dt + \sigma dW_t + \kappa(t) dq_t, \\
 SV_i(J) : \quad \frac{dS_t}{S_t} &= (\mu - \lambda(t) \bar{\kappa}) dt + \sqrt{V_t} dW_{1,t} + \kappa(t) dq_t, \quad i = 1, 2, \\
 SV_1 : \quad d \ln V_t &= (\alpha - \beta \ln V_t) dt + \eta dW_{2,t}, \\
 SV_2 : \quad dV_t &= (\alpha - \beta V_t) dt + \eta \sqrt{V_t} dW_{2,t}, \\
 \ln(1 + \kappa(t)) &\rightsquigarrow N(\ln(1 + \bar{\kappa}) - 0.5 \delta^2, \delta^2), \quad \bar{\kappa} = 0, \\
 \text{corr}(dW_{1,t}, dW_{2,t}) &= \rho, \quad \text{Prob}(dq_t = 1) = \lambda(t) dt, \quad \lambda(t) = \lambda_0 + \lambda_1 V_t.
 \end{aligned}$$

Parameter	BS	$BSJ$	$SV_1, \rho = 0$	$SV_1, \rho \neq 0$	$SV_1J, \rho \neq 0$	$SV_2, \rho = 0$	$SV_2, \rho \neq 0$	$SV_2J, \rho \neq 0$
$\mu$	0.0398 (0.0054)	0.0350 (0.0061)	0.0396 (0.0055)	0.0314 (0.0057)	0.0304 (0.0063)	0.0394 (0.0055)	0.0300 (0.0058)	0.0304 (0.0061)
$\sigma$	0.5933 (0.0255)	0.6243 (0.0294)						
$\alpha$			-0.0061 (0.0050)	-0.0114 (0.0022)	-0.0120 (0.0021)	0.0025 (0.0019)	0.0069 (0.0013)	0.0064 (0.0017)
$\beta$			0.0062 (0.0048)	0.0136 (0.0022)	0.0145 (0.0019)	0.0063 (0.0049)	0.0129 (0.0024)	0.0120 (0.0023)
$\eta$			0.0374 (0.0147)	0.1148 (0.0089)	0.1153 (0.0094)	0.0229 (0.0087)	0.0734 (0.0072)	0.0711 (0.0127)
$\rho$				-0.5755 (0.0432)	-0.6127 (0.0623)	-0.6125 (0.0632)	-0.5877 (0.0447)	-0.6220 (0.0542)
$\delta$				0.0151 (0.0008)	0.0150 (0.0008)	0.0150 (0.0008)	0.0134 (0.0007)	0.0134 (0.0010)
$\lambda_0$				0.0501 (0.0045)	0.0137 (0.0013)	0.0137 (0.0264)	0.0202 (0.0011)	0.0202 (0.0246)
$\lambda_1$				0.0010 (0.0197)	0.0010 (0.0197)	0.0010 (0.0197)	0.00002 (0.0650)	0.00002 (0.0650)
$\chi^2$ [ <i>d.f.</i> ]	127.35 [12]	89.60 [10]	121.11 [10]	30.92 [9]	13.34 [7]	121.15 [10]	31.94 [9]	14.90 [7]
( <i>P-Value</i> )	(< 10 <sup>-5</sup> )	(< 10 <sup>-5</sup> )	(< 10 <sup>-5</sup> )	(0.00031)	(0.06429)	(< 10 <sup>-5</sup> )	(0.00020)	(0.03736)

**Table IV**  
**EMM estimates of the Black and Scholes and the continuous-time, stochastic volatility models (1) and (2) and (1) and (3), with the volatility-in-mean effect in the drift term**

Estimates are for the sample period 01/02/1953-12/31/1996. Standard errors are reported in brackets. Parameter estimates are expressed in percentage form on a daily basis, and refer to the following model:

$$\begin{aligned}
 BS(J) : \quad \frac{dS_t}{S_t} &= (\mu - \lambda_0 \bar{\kappa}) dt + \sigma dW_t + \kappa(t) dq_t, \\
 SV_i(J) : \quad \frac{dS_t}{S_t} &= (\mu + cV_t - \lambda(t)\bar{\kappa}) dt + \sqrt{V_t} dW_{1,t} + \kappa(t) dq_t, \quad i = 1, 2, \\
 SV_1 : \quad d \ln V_t &= (\alpha - \beta \ln V_t) dt + \eta dW_{2,t}, \\
 SV_2 : \quad dV_t &= (\alpha - \beta V_t) dt + \eta \sqrt{V_t} dW_{2,t}, \\
 \ln(1 + \kappa(t)) &\rightsquigarrow N(\ln(1 + \bar{\kappa}) - 0.5\delta^2, \delta^2), \quad \bar{\kappa} = 0, \\
 \text{corr}(dW_{1,t}, dW_{2,t}) &= \rho, \quad \text{Prob}(dq_t = 1) = \lambda(t) dt, \quad \lambda(t) = \lambda_0 + \lambda_1 V_t.
 \end{aligned}$$

Parameter	BS	BSJ	SV <sub>1</sub> , ρ = 0	SV <sub>1</sub> , ρ ≠ 0	SV <sub>1</sub> J, ρ ≠ 0	SV <sub>2</sub> , ρ = 0	SV <sub>2</sub> , ρ ≠ 0	SV <sub>2</sub> J, ρ ≠ 0		
μ	0.0366 (0.0054)	0.0347 (0.0059)	0.0900 (0.0182)	0.0252 (0.0117)	0.0203 (0.0129)	0.0203 (0.0135)	0.0913 (0.0172)	0.0202 (0.0126)	0.0148 (0.0179)	0.0147 (0.0375)
c			-0.1446 (0.0551)	0.0149 (0.0255)	0.0258 (0.0267)	0.0258 (0.0280)	-0.1484 (0.0523)	0.0256 (0.0282)	0.0402 (0.0389)	0.0403 (0.0577)
σ	0.5999 (0.0238)	0.6069 (0.0260)								
α			-0.0076 (0.0037)	-0.0136 (0.0030)	-0.0125 (0.0026)	-0.0125 (0.0026)	0.0027 (0.0012)	0.0081 (0.0016)	0.0074 (0.0021)	0.0074 (0.0027)
β			0.0074 (0.0034)	0.0160 (0.0030)	0.0150 (0.0026)	0.0150 (0.0028)	0.0068 (0.0031)	0.0156 (0.0032)	0.0147 (0.0043)	0.0147 (0.0068)
η			0.0524 (0.0115)	0.1206 (0.0093)	0.1153 (0.0111)	0.1153 (0.0109)	0.0308 (0.0065)	0.0782 (0.0074)	0.0732 (0.0075)	0.0732 (0.0077)
ρ				-0.5778 (0.0433)	-0.6036 (0.0473)	-0.6035 (0.0495)	-0.5973 (0.0448)	-0.6196 (0.0665)	-0.6196 (0.0665)	-0.6196 (0.0855)
δ				0.0101 (0.0004)	0.0150 (0.0010)	0.0150 (0.0010)	0.0150 (0.0010)	0.0122 (0.0007)	0.0122 (0.0007)	0.0122 (0.0009)
λ <sub>0</sub>				0.0503 (0.0017)	0.0137 (0.0015)	0.0137 (0.0015)	0.0137 (0.0015)	0.0202 (0.0017)	0.0202 (0.0017)	0.0202 (0.0285)
λ <sub>1</sub>					0.00010 (0.0288)	0.00010 (0.0288)		0.00007 (0.0488)	0.00007 (0.0488)	0.00007 (0.0488)
χ <sup>2</sup> [d.f.]	132.01 [13]	88.10 [11]	116.98 [10]	31.53 [9]	13.13 [7]	13.10 [6]	116.21 [10]	30.74 [9]	14.09 [7]	14.07 [6]
(P-Value)	(< 10 <sup>-5</sup> )	(< 10 <sup>-5</sup> )	(< 10 <sup>-5</sup> )	(0.00024)	(0.06905)	(0.04142)	(< 10 <sup>-5</sup> )	(0.00033)	(0.04957)	(0.02882)





Table VI  
**EMM estimates of the Black and Scholes and the continuous-time, stochastic volatility models (1) and (2) and (1) and (3)**

Estimates are for the sample period 01/03/1980-12/31/1996. Standard errors are reported in brackets. Parameter estimates are expressed in percentage form on a daily basis, and refer to the following model:

$$\begin{aligned}
 BS(J) : \quad \frac{dS_t}{S_t} &= (\mu - \lambda_0 \bar{\kappa}) dt + \sigma dW_t + \kappa(t) dq_t, \\
 SV_i(J) : \quad \frac{dS_t}{S_t} &= (\mu - \lambda(t) \bar{\kappa}) dt + \sqrt{V_t} dW_{1,t} + \kappa(t) dq_t, \quad i = 1, 2, \\
 SV_1 : \quad d \ln V_t &= (\alpha - \beta \ln V_t) dt + \eta dW_{2,t}, \\
 SV_2 : \quad dV_t &= (\alpha - \beta V_t) dt + \eta \sqrt{V_t} dW_{2,t}, \\
 \ln(1 + \kappa(t)) &\rightsquigarrow N(\ln(1 + \bar{\kappa}) - 0.5\delta^2, \delta^2), \quad \bar{\kappa} = 0, \\
 \text{corr}(dW_{1,t}, dW_{2,t}) &= \rho, \quad \text{Prob}(dq_t = 1) = \lambda(t) dt, \quad \lambda(t) = \lambda_0 + \lambda_1 V_t.
 \end{aligned}$$

Parameter	BS	BSJ	SV <sub>1</sub> , ρ = 0	SV <sub>1</sub> , ρ ≠ 0	SV <sub>1</sub> J, ρ ≠ 0	SV <sub>2</sub> , ρ = 0	SV <sub>2</sub> , ρ ≠ 0	SV <sub>2</sub> J, ρ ≠ 0		
μ	0.0603 (0.0098)	0.0631 (0.0103)	0.0593 (0.0099)	0.0496 (0.0104)	0.0460 (0.0084)	0.0476 (0.0102)	0.0617 (0.0096)	0.0500 (0.0104)	0.0550 (0.0108)	0.0573 (0.0108)
σ	0.7176 (0.0272)	0.7166 (0.0281)								
α			-0.0200 (0.0133)	-0.0084 (0.0027)	-0.0102 (0.0014)	-0.0104 (0.0016)	0.0074 (0.0030)	0.0107 (0.0035)	0.0089 (0.0016)	0.0088 (0.0021)
β			0.0317 (0.0199)	0.0154 (0.0044)	0.0206 (0.0010)	0.0208 (0.0009)	0.0123 (0.0051)	0.0162 (0.0052)	0.0134 (0.0021)	0.0133 (0.0029)
η			0.1146 (0.0301)	0.1009 (0.0161)	0.1135 (0.0043)	0.1138 (0.0036)	0.0578 (0.0131)	0.0771 (0.0141)	0.0683 (0.0114)	0.0676 (0.0145)
ρ				-0.4000 (0.0709)	-0.3856 (0.0189)	-0.3844 (0.0229)		-0.3799 (0.0827)	-0.3234 (0.0267)	-0.3301 (0.0854)
δ		0.0135 (0.0003)			0.0217 (0.0005)	0.0217 (0.0006)			0.0195 (0.0006)	0.0194 (0.0005)
λ <sub>0</sub>		0.0591 (0.0005)			0.0192 (0.0011)	0.0192 (0.0010)			0.0195 (0.0025)	0.0193 (0.0115)
λ <sub>1</sub>					0.0008 (0.0016)	0.0008 (0.0016)			0.0004 (0.0031)	0.0004 (0.0331)
χ <sup>2</sup> [d.f.]	64.51 [13]	23.47 [11]	49.74 [11]	30.16 [10]	11.19 [8]	10.49 [7]	48.17 [11]	30.93 [10]	10.86 [8]	10.10 [7]
(P-Value)	(< 10 <sup>-5</sup> )	(0.0152)	(< 10 <sup>-5</sup> )	(0.0008)	(0.1911)	(0.1625)	(< 10 <sup>-5</sup> )	(0.0006)	(0.2095)	(0.1830)

**Table VII**  
**SNP model estimates and t-ratios of the average score components**

t-ratios correspond to the following models: Black and Scholes (with jumps):  $BS(J)$ . Stochastic volatility (with jumps), model (1) and (2):  $SV_1(J)$ . Stochastic volatility (with jumps), model (1) and (3):  $SV_2(J)$ . Estimates are for the sample period 01/03/1980-12/31/1996 (N=4,298 observations). The score vector components are relative to the parameters of the following auxiliary model:

$$f_K(r_t|x_t; \xi) = \left( \nu + (1 - \nu) \times \frac{[P_K(z_t, x_t)]^2}{\int_{\mathbb{R}} [P_K(z_t, x_t)]^2 \phi(u) du} \right) \frac{\phi(z_t)}{\sqrt{h_t}}, \quad \nu = 0.01, \quad \phi(\cdot) \text{ std. normal density,}$$

$$z_t = \frac{r_t - \mu_t}{\sqrt{h_t}},$$

$$\mu_t = \phi_0,$$

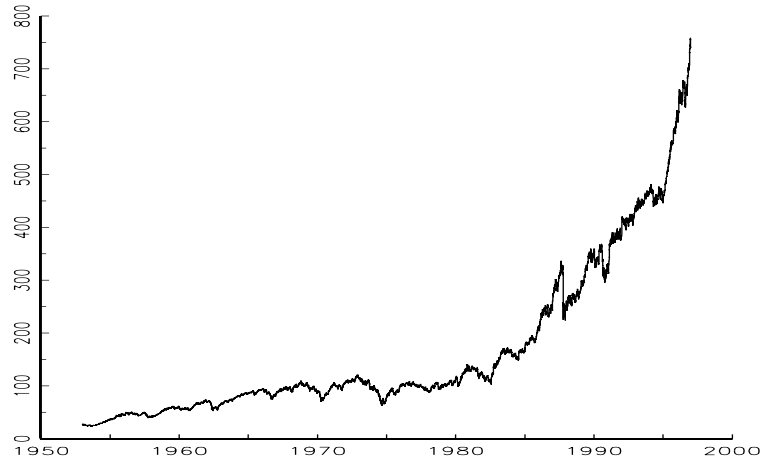
$$\ln h_t = \omega + \sum_{i=1}^p \beta_i \ln h_{t-i} + (1 + \alpha_1 L + \dots + \alpha_q L^q) [\theta_1 z_t + \theta_2 (b(z_t) - \sqrt{2/\pi})],$$

$$b(z) = |z| \text{ for } |z| \geq \pi/2K, \quad b(z) = (\pi/2 - \cos(Kz))/K \text{ for } |z| < \pi/2K, \quad K = 100,$$

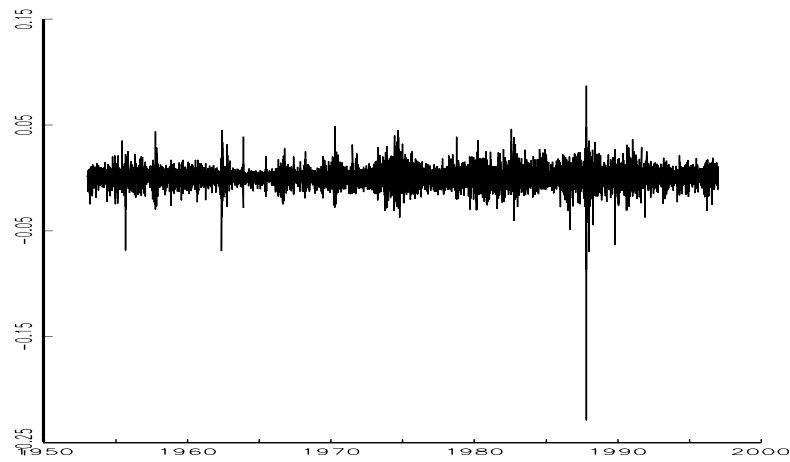
$$P_K(z, x) = \sum_{i=0}^{K_z} a_i(x) z^i = \sum_{i=0}^{K_x} \left( \sum_{|j|=0}^{K_x} a_{ij} x^j \right) z^i, \quad a_{00} = 1.$$

Parameter	Estimate	Std. Error	t-ratios									
			BS	BSJ	SV <sub>1</sub> , ρ = 0	SV <sub>1</sub> J	SV <sub>2</sub> , ρ = 0	SV <sub>2</sub> J				
φ <sub>0</sub>	-0.0398	0.0398	-1.3737	0.2456	-0.9776	-0.7946	-1.3666	-2.6811*	0.2495	-0.7181	-0.0908	0.9726
ω	2.6724	0.4565	-1.3856	0.8402	-1.0252	-0.7795	0.8203	0.9175	-1.0278	-0.9170	0.0620	0.1879
α	0.5712	0.1559	-0.0242	-1.2590	0.9601	1.9228	0.3604	0.1660	1.1443	1.8231	0.5458	0.4037
β <sub>1</sub>	0.0388	0.0199	0.8083	-1.2869	0.5728	1.1296	-0.6576	-0.7666	1.1316	1.0697	0.1150	-0.0592
β <sub>2</sub>	0.9213	0.0183	0.7378	-1.4187	0.4666	0.9534	-0.8842	-0.9736	1.0202	0.8913	-0.0108	-0.1597
θ <sub>1</sub>	-0.0916	0.0234	2.4965*	3.3108*	2.7965*	0.2105	1.2768	1.3482	2.6322*	0.5164	1.4941	1.1995
θ <sub>2</sub>	0.2011	0.0340	-0.1720	-1.9194	1.2242	0.6332	-1.7594	-1.9509	1.3473	0.6875	-0.5944	-0.6891
a <sub>10</sub>	0.0534	0.0249	1.5898	1.0289	1.1190	-1.3014	-1.4824	-1.6794	1.4046	-1.1897	-0.6531	-0.2666
a <sub>20</sub>	-0.2407	0.0257	-5.0370*	-1.1035	-3.0445*	-3.0283*	-0.8112	-3.4895*	-3.2545*	-1.8984	-1.8332	-1.8332
a <sub>30</sub>	-0.0297	0.0130	-0.1742	-0.0601	-0.0158	-0.0835	0.7839	0.7232	-0.1996	-0.0159	0.2459	0.1462
a <sub>40</sub>	0.0820	0.0119	1.3489	2.0423*	1.0267	0.5737	0.0632	0.1970	0.7397	0.5217	-0.1874	-0.2428
a <sub>50</sub>	0.0223	0.0103	-0.7371	-0.2057	-0.6979	-0.0787	0.2917	0.3376	-0.6146	-0.1458	0.0338	0.3228
a <sub>60</sub>	-0.0781	0.0068	4.4936*	0.6511	3.8508*	4.0672*	1.2939	1.2528	4.1496*	4.0909*	2.4054*	1.8978
a <sub>70</sub>	-0.0043	0.0123	-0.1250	0.7321	-0.3976	-0.8629	-1.1614	-1.2301	-0.3795	-0.8834	-0.7111	-0.3384
a <sub>80</sub>	0.0697	0.0123	-5.3750*	-2.0016*	-4.4820*	-4.7559*	-1.0532	-1.1688	-4.6979*	-4.7641*	-1.6401	-1.9388

\* Average score components significantly different from 0.

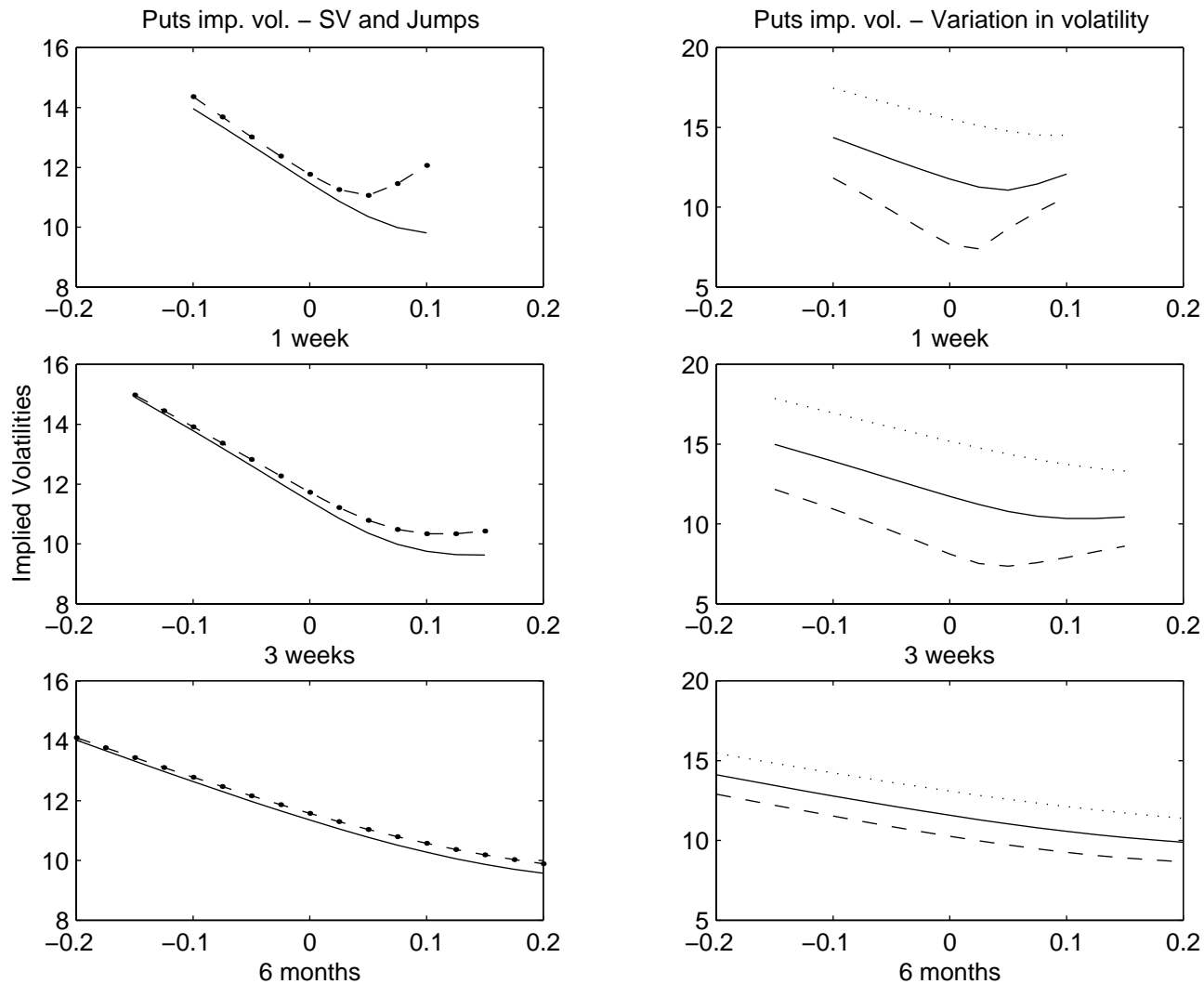


Panel 1: S&P500 price index.

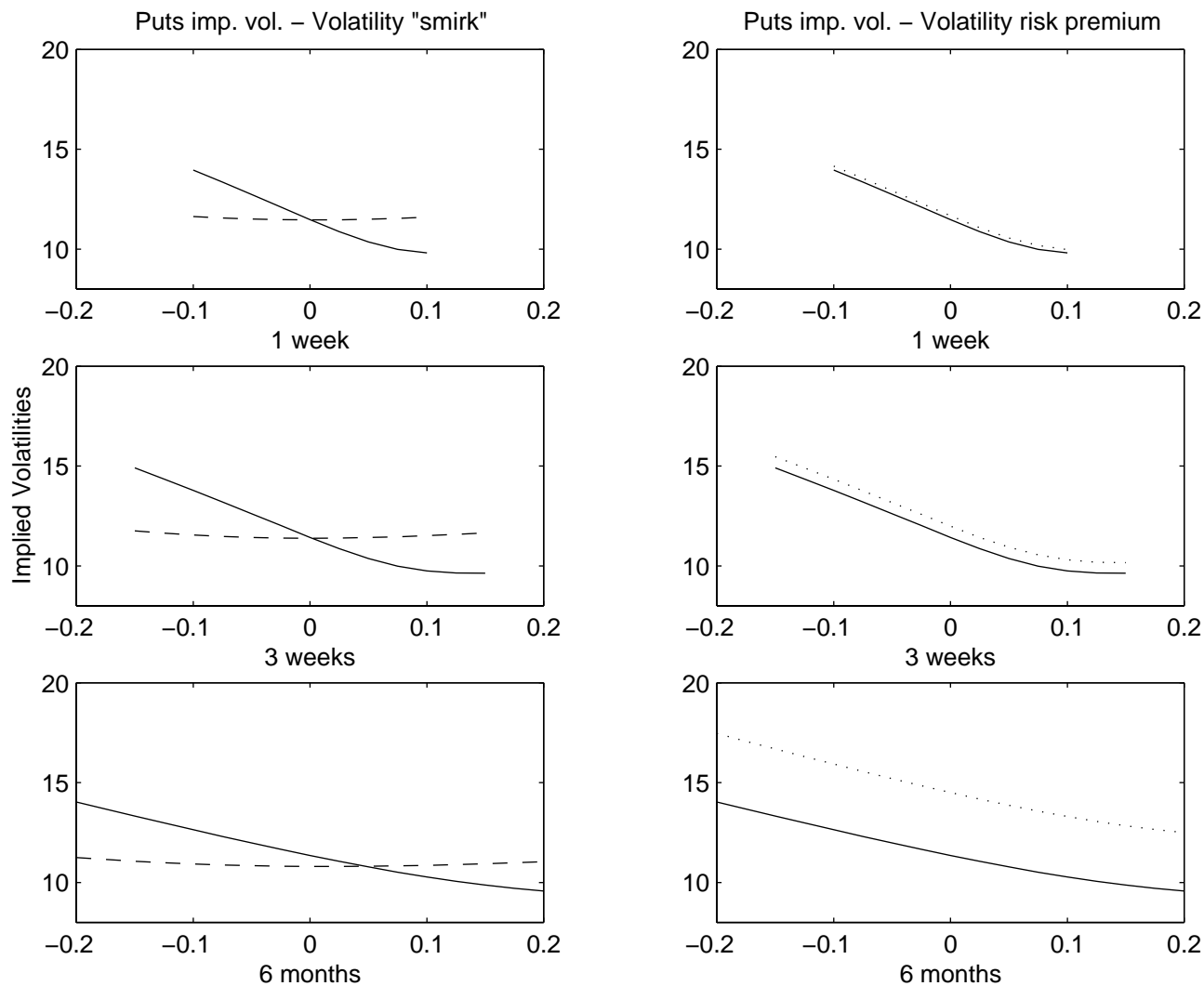


Panel 2: S&P500 daily rate of return.

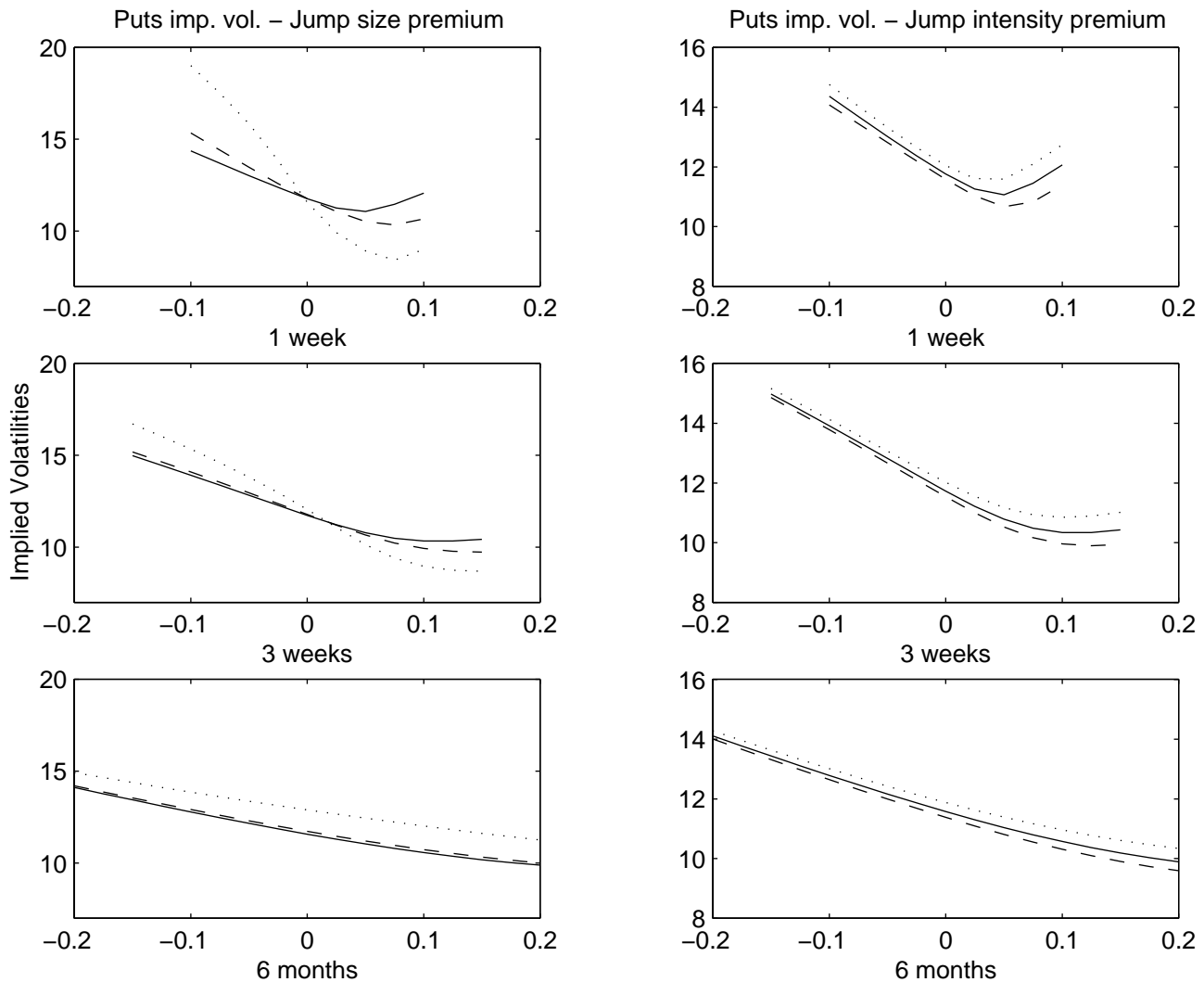
Figure 1. S&P500 prices and returns, 01/02/1953-12/31/1996.



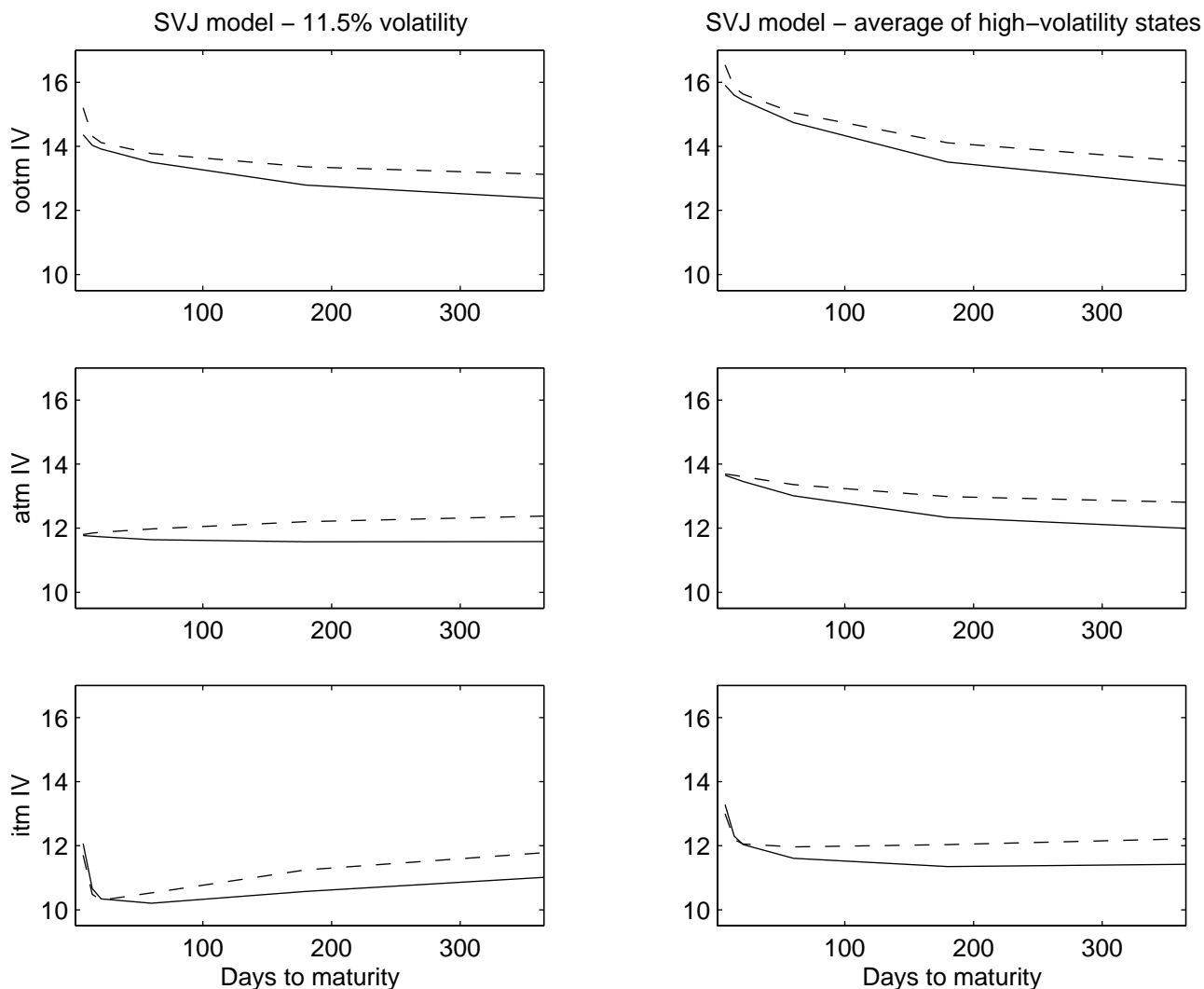
**Figure 2. Black-Scholes implied volatilities from option prices generated by the stochastic volatility model, square-root specification.** Model coefficients are equal to the EMM estimates in Table IV. Underlying stock price equals \$800. Volatility and jump risk premia are set equal to zero. Left column panels: Put prices are generated from the stochastic volatility model without (—) and with jumps: constant (- -) and affine (· · ·) jump intensity. The underlying returns volatility is 11.5% (equal to the long-run volatility mean). Right column panels: Put prices are generated from the stochastic volatility model with jumps, constant jump intensity. Instantaneous return volatility is set equal to 7% (- -), 11.5% (—), 15.5% (· · ·).



**Figure 3. Black-Scholes implied volatilities from option prices generated by the stochastic volatility model without jumps, square-root specification.** Model coefficients are equal to the EMM estimates in Table IV. Put prices are generated using a stock price of \$800 and a volatility of 11.5% (equal to the long-run volatility mean.) Left column panels: Different plots illustrate the effect of the asymmetry coefficient  $\rho$ :  $\rho = 0$  (- -) and  $\rho = -0.6$  (—). The volatility risk premium is set equal to zero. Right column panels: Different plots illustrate the effect of the volatility risk-adjustment:  $\xi = 0$  (—) and  $\xi = -0.0100$  (···). The  $\rho$  coefficient is set equal to -0.6.



**Figure 4. Black-Scholes implied volatilities from option prices generated by the stochastic volatility model with jumps, constant jump intensity, square-root specification.** Model coefficients are equal to the EMM estimates in Table IV. Put prices are generated using a stock price of \$800 and a volatility of 11.5% (equal to the long-run volatility mean.) Left column panels: Different plots illustrate the effect of the risk-adjustment on the jump size:  $\bar{\kappa}^* = 0$  (—),  $\bar{\kappa}^* = -0.01$  (- -),  $\bar{\kappa}^* = -0.03$  (···). Volatility and jump intensity risk premia are set equal to zero. Right column panels: Different plots illustrate the effect of the risk-adjustment on the jump intensity,  $\lambda_0^*$  coefficient:  $\lambda_0^* = 0.0202$  (—),  $\lambda_0^* = 0.0079$  (- -),  $\lambda_0^* = 0.0397$  (···). Volatility and jump size risk premia are set equal to zero.



**Figure 5. The term structure of Black-Scholes implied volatilities from option prices generated by the stochastic volatility model with jumps, constant jump intensity, square-root specification.** Model coefficients are equal to the EMM estimates in Table IV. Put prices are generated using a stock price of \$800. Out-of-the-money ( $K/S = .9$ ) puts in the top panels, at-the-money in the middle and in-the-money ( $K/S = 1.1$ ) bottom. Different plots illustrate the effect of jump and volatility risk adjustments:  $[\lambda_0^*, \bar{\kappa}^*, \xi] = [0.0202, 0, 0]$  (—) and  $[\lambda_0^*, \bar{\kappa}^*, \xi] = [0.0397, -0.005, -0.0015]$  (- -). Left column panels: Black-Scholes implied volatilities from put prices generated using a return volatility of 11.5% (equal to the long-run volatility mean.) Right column panels: Average of Black-Scholes implied volatilities computed across high-volatility states. (11.5% and 15.5%).