AN ENDPOINT $(1, \infty)$ BALIAN-LOW THEOREM

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Abstract. It is shown that a $(1, \infty)$ version of the Balian-Low Theorem holds. If $g \in L^2(\mathbb{R})$, $\Delta_1(g) < \infty$ and $\Delta_\infty(\widehat{g}) < \infty$, then the Gabor system $\mathcal{G}(g, 1, 1)$ is not a Riesz basis for $L^2(\mathbb{R})$. Here, $\Delta_1(g) = \int |t||g(t)|^2 dt$ and $\Delta_\infty(\widehat{g}) = \sup_{N>0} \int |\gamma|^N |\widehat{g}(\gamma)|^2 d\gamma$.

1. Introduction

Given a square integrable function $g \in L^2(\mathbb{R})$, and constants $a, b > 0$, the associated *Gabor system*, $\mathcal{G}(g, a, b) = \{g_{m,n}\}_{m,n\in\mathbb{Z}}$, is defined by

$$g_{m,n}(t) = e^{2\pi iamt} g(t - bn).$$

Gabor systems provide effective signal decompositions in a variety of settings ranging from eigenvalue problems to applications in communications engineering. Background on the theory and applications of Gabor systems can be found in [16], [12], [13], [3].

We shall use the Fourier transform defined by $\widehat{g}(\gamma) = \int g(t)e^{-2\pi i\gamma t} dt$, where the integral is over $\mathbb{R}$. We let $\mathbf{1}_S(t)$ denote the characteristic function of a set $S \subseteq \mathbb{R}$, and let S^c denote the complement of $S \subseteq \mathbb{R}$. Depending on the context, $|\cdot|$ will denote either the Lebesgue measure of a set, or the modulus of a function or complex number.

The Balian-Low Theorem is a classical manifestation of the uncertainty principle for Gabor systems.

Theorem 1.1 (Balian-Low). *Let $g \in L^2(\mathbb{R})$. If*

$$\int |t|^2 |g(t)|^2 dt < \infty \quad \text{and} \quad \int |\gamma|^2 |\widehat{g}(\gamma)|^2 d\gamma < \infty,$$

then $\mathcal{G}(g, 1, 1)$ is not an orthonormal basis for $L^2(\mathbb{R})$.

The Balian-Low Theorem has a long history and some of the original references include [1], [19], [2]. The theorem still holds if "orthonormal basis" is replaced by "Riesz basis". For this and other generalizations of the Balian-Low Theorem, we refer the reader to the survey articles [6], [9], as well as [4], [5], [7], [8], [10], [14], [17]. The issue of sharpness in the Balian-Low Theorem was investigated in [5], where the following was shown.

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Theorem 1.2. *If $\frac{1}{p}+\frac{1}{q}=1$, where $1<p,q<\infty$, and $d>2$, then there exists a function $g\in L^2(\mathbb{R})$ such that $\mathcal{G}(g,1,1)$ is an orthonormal basis for $L^2(\mathbb{R})$ and*

$$\int \frac{1+|t|^p}{\log^d(2+|t|)}|f(t)|^2dt<\infty \quad \textit{and} \quad \int \frac{1+|\gamma|^q}{\log^d(2+|\gamma|)}|\widehat{f}(\gamma)|^2d\gamma<\infty.$$

When $(p,q)=(2,2)$, this says that the Balian-Low Theorem no longer holds if the weights (t^2,γ^2) are weakened by appropriate logarithmic terms. In view of Theorem 1.2, it is also natural to ask if there exist versions of the Balian-Low Theorem for the general (p,q) case corresponding to the weights (t^p,γ^q). The best that is known is the following.

Theorem 1.3. *Suppose $\frac{1}{p}+\frac{1}{q}=1$ with $1<p<\infty$ and let $\epsilon>0$. If*

$$\int |t|^{(p+\epsilon)}|g(t)|^2dt<\infty \quad \textit{and} \quad \int |\gamma|^{(q+\epsilon)}|\widehat{g}(\gamma)|^2d\gamma<\infty$$

then $\mathcal{G}(g,1,1)$ is not an orthonormal basis for $L^2(\mathbb{R})$.

The above theorem follows by combining Theorem 4.4 of [11] and Theorem 1 in [15]. The $\epsilon>0$ can, of course, be removed in the case $(p,q)=(2,2)$, by the Balian-Low Theorem.

This note shows the existence of a Balian-Low Theorem in the case $(p,q)=(1,\infty)$, and thus extends Theorems 1.1 and 1.3. To define what this means, let $g\in L^2(\mathbb{R})$ and $1\le p<\infty$ and set

$$\Delta_p(g)=\int |t|^p|g(t)|^2dt \ \text{ and } \ \Delta_\infty(g)=\sup_{N>0}\int |t|^N|g(t)|^2dt.$$

With this notation, the classical Balian-Low Theorem says that if $\Delta_2(g)<\infty$ and $\Delta_2(\widehat{g})<\infty$ then $\mathcal{G}(g,1,1)$ is not an orthonormal basis for $L^2(\mathbb{R})$.

Our main result of this note is the following theorem.

Theorem 1.4. *Let $g\in L^2(\mathbb{R})$ and suppose that $\mathcal{G}(g,1,1)$ is a Riesz basis for $L^2(\mathbb{R})$. Then*

$$\Delta_1(g)=\infty \quad \textit{or} \quad \Delta_\infty(\widehat{g})=\infty.$$

This yields the following $(1,\infty)$ version of the classical Balian-Low Theorem.

Corollary 1.5. *Let $g\in L^2(\mathbb{R})$ and suppose*

$$\Delta_1(g)<\infty \quad \text{and} \quad \Delta_\infty(\widehat{g})<\infty.$$

Then $\mathcal{G}(g,1,1)$ is not an orthonormal basis for $L^2(\mathbb{R})$.

2. Background

A collection $\{e_n\}_{n\in\mathbb{Z}}\subseteq L^2(\mathbb{R})$ is a *frame* for $L^2(\mathbb{R})$ if there exist constants $0<A\le B<\infty$ such that

$$\forall f\in L^2(\mathbb{R}),\quad A||f||^2_{L^2(\mathbb{R})}\le\sum_{n\in\mathbb{Z}}|\langle f,e_n\rangle|^2\le B||f||^2_{L^2(\mathbb{R})}.$$

A and B are the *frame constants* associated to the frame. If $\{e_n\}_{n\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$, but is no longer a frame if any element is removed, then we say that $\{e_n\}_{n\in\mathbb{Z}}$ is a *Riesz basis* for $L^2(\mathbb{R})$. Riesz bases are also known as *exact frames* or *bounded unconditional bases*, e.g., see [3]. The Zak transform is an important tool for studying Riesz bases given by Gabor systems.

Given $g \in L^2(\mathbb{R})$, the *Zak transform* is formally defined by

$$\forall (t,\gamma) \in Q \equiv [0,1)^2, \quad Zg(t,\gamma) = \sum_{n\in\mathbb{Z}} g(t-n)e^{2\pi i n\gamma}.$$

This defines a unitary operator from $L^2(\mathbb{R})$ to $L^2(Q)$. Further background on the Zak transform, as well as the next theorem, can be found in [3], [16].

Theorem 2.1. *Let $g \in L^2(\mathbb{R})$. $\mathcal{G}(g,1,1)$ is a Riesz basis for $L^2(\mathbb{R})$ with frame constants $0 < A \leq B < \infty$ if and only if $A \leq |Zg(t,\gamma)|^2 \leq B$ for a.e. $(t,\gamma) \in Q$.*

A function $g \in L^2(\mathbb{R})$ is said to be in the *homogeneous Sobolev space* of order $s > 0$, denoted $\dot{H}^s(\mathbb{R})$, if $||g||^2_{\dot{H}^s(\mathbb{R})} \equiv \int |\gamma|^{2s}|\widehat{g}(\gamma)|^2 d\gamma < \infty$. Since the condition $\Delta_1(g) < \infty$ in Theorem 1.5 is equivalent to $\widehat{g} \in \dot{H}^{1/2}(\mathbb{R})$, we shall need some results on $\dot{H}^{1/2}(\mathbb{R})$. The following alternate characterization of $\dot{H}^{1/2}(\mathbb{R})$ will be useful, e.g., [18].

Theorem 2.2. *If $f \in \dot{H}^{1/2}(\mathbb{R})$ then*

$$||f||^2_{\dot{H}^{1/2}(\mathbb{R})} = \frac{1}{4\pi^2}\int\int \frac{|f(x)-f(y)|^2}{|x-y|^2}dxdy.$$

Given $f \in L^2(\mathbb{R})$, the *symmetric-decreasing rearrangement* f^* of f is defined by

$$f^*(t) = \int_0^\infty \mathbf{1}_{S_x}(t)dx,$$

where $S_x = (-s_x/2, s_x/2)$ and $s_x = |\{t : |f(t)| > x\}|$. An important property of a symmetric-decreasing rearrangement is that it decreases the $\dot{H}^{1/2}(\mathbb{R})$ norm of functions, [18].

Theorem 2.3. *If $f \in \dot{H}^{1/2}(\mathbb{R})$ then*

$$||f||_{\dot{H}^{1/2}(\mathbb{R})} \geq ||f^*||_{\dot{H}^{1/2}(\mathbb{R})}.$$

This has the following useful corollary, [18].

Corollary 2.4. *If $S \subset \mathbb{R}$ is a measurable set of positive and finite measure then $||\mathbf{1}_S||_{\dot{H}^{1/2}(\mathbb{R})} = \infty$.*

3. Proof of the $(1,\infty)$ Balian-Low Theorem

The proof of Theorem 1.4 requires the following preliminary technical theorem.

Theorem 3.1. *Let f be a non-negative measurable function supported in the interval $[-1,1]$ and suppose that there exist constants $0 < A \leq B < \infty$ such that*

$$A \leq |f(x) \pm f(x-1)| \leq B, \quad a.e.\ x \in [-1,1]. \tag{3.1}$$

Then $||f||_{\dot{H}^{1/2}(\mathbb{R})} = \infty$.

Proof. We begin by defining the measurable sets

$$S = \{x \in [0,1] : f(x-1) \leq f(x)\},$$
$$T = S^c \cap [0,1] = \{y \in [0,1] : f(y) < f(y-1)\},$$

and note that (3.1) implies

$$A \leq f(x) - f(x-1), \qquad a.e.\ \ x \in S, \tag{3.2}$$

$$A \leq f(y-1) - f(y), \qquad a.e.\ \ y \in T. \tag{3.3}$$

We break up the proof into two cases depending on whether or not S is a proper non-trivial subset of $[0,1]$.

Case I. We shall first consider the case where

$$0 < |S| < 1, \tag{3.4}$$

and hence that $0 < |T| < 1$.

Define the following capacity type integral over the product set $S \times T$.

$$I = \int_S \int_T \frac{1}{|x-y|^2} dy dx. \tag{3.5}$$

Conditions (3.2) and (3.3) allow one to bound I in terms of the $\dot{H}^{\frac{1}{2}}(\mathbb{R})$ norm of f as follows.

$$\begin{aligned} I \ &\leq \ \frac{1}{4A^2} \int_S \int_T \frac{\left|f(x) - f(x-1) + f(y-1) - f(y)\right|^2}{|x-y|^2} \, dy \, dx \\ &\leq \frac{1}{2A^2} \left(\int_S \int_T \frac{\left|f(x) - f(y)\right|^2}{|x-y|^2} \, dy \, dx \ + \ \int_S \int_T \frac{\left|f(y-1) - f(x-1)\right|^2}{|x-y|^2} \, dy \, dx \right) \\ &\leq \ \frac{1}{A^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\left|f(x) - f(y)\right|^2}{|x-y|^2} \, dy \, dx \\ &= \frac{4\pi^2}{A^2} \ ||f||^2_{\dot{H}^{1/2}} \ . \end{aligned}$$

It therefore suffices to show that $I = \infty$.

Since by the Lebesgue differentiation theorem almost every point of T is a point of density, it follows from (3.4) that we may chose $a \in (0,1)$ such that a is point of density of T which satisfies either

$$0 < |S \cap [0,a]| < a \tag{3.6}$$

or

$$0 < |S \cap [a,1]| < 1-a. \tag{3.7}$$

Without loss of generality, we assume (3.6). If (3.7) holds then our arguments proceed analogously; for example in the first subcase below we would symmetrize about $x = 1$ instead of $x = 0$.

To estimate I, we shall proceed separately depending on whether $\int_0^a \frac{\mathbf{1}_S(x)}{|x-a|} dx$ is finite or infinite.

Subcase i. Suppose $\int_0^a \frac{\mathbf{1}_S(x)}{|x-a|} dx < \infty$. It will be convenient to work with the following set

$$\widetilde{S} = (S \cup (-S)) \cap [-a,a].$$

By (3.6) we have $|\widetilde{S}| = 2|S\cap[0,a]| \neq 0$. It follows from Corollary 2.4 and the definition of $\widetilde{S}$ that

$$\begin{aligned}\widetilde{I} &\equiv \int_{-a}^{a}\int_{-\infty}^{\infty} \frac{\mathbf{1}_{\widetilde{S}}(x)\mathbf{1}_{(\widetilde{S})^c}(y)}{|x-y|^2}dydx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\mathbf{1}_{\widetilde{S}}(x)\mathbf{1}_{(\widetilde{S})^c}(y)}{|x-y|^2}dydx\\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{|\mathbf{1}_{\widetilde{S}}(x)-\mathbf{1}_{\widetilde{S}}(y)|^2}{|x-y|^2}dxdy\\ &= 4\pi^2\ ||\mathbf{1}_{\widetilde{S}}||^2_{\dot{H}^{1/2}(\mathbb{R})} = \infty.\end{aligned}$$

The symmetric definition of $\widetilde{S}$ implies that

$$\widetilde{I} = 2\int_0^a\int_{-\infty}^{\infty} \frac{\mathbf{1}_{\widetilde{S}}(x)\mathbf{1}_{(\widetilde{S})^c}(y)}{|x-y|^2}dydx = 2(I_1+I_2+I_3), \tag{3.8}$$

where

$$\begin{aligned}I_1 &\equiv \int_0^a\int_{-\infty}^{-a} \frac{\mathbf{1}_{\widetilde{S}}(x)\mathbf{1}_{(\widetilde{S})^c}(y)}{|x-y|^2}dydx \le \int_0^a\int_{-\infty}^{-a}\frac{1}{|y|^2}dydx < \infty,\\ I_2 &\equiv \int_0^a\int_a^{\infty} \frac{\mathbf{1}_{\widetilde{S}}(x)\mathbf{1}_{(\widetilde{S})^c}(y)}{|x-y|^2}dydx \le \int_0^a \frac{\mathbf{1}_S(x)}{|x-a|}dx < \infty,\\ I_3 &\equiv \int_0^a\int_{-a}^{a} \frac{\mathbf{1}_{\widetilde{S}}(x)\mathbf{1}_{(\widetilde{S})^c}(y)}{|x-y|^2}dydx.\end{aligned}$$

A simple calculation for I_3 shows that

$$I_3 = \int_0^a\int_0^a \frac{\mathbf{1}_{\widetilde{S}}(x)\mathbf{1}_{(\widetilde{S})^c}(y)}{|x-y|^2}dydx + \int_0^a\int_0^a \frac{\mathbf{1}_{\widetilde{S}}(x)\mathbf{1}_{(\widetilde{S})^c}(y)}{|x+y|^2}dydx \le 2I, \tag{3.9}$$

where the inequality for the second term in the middle of (3.9) follows from the fact that $|x-y|\le|x+y|$ in the square $[0,a]\times[0,a]$.

It follows from (3.8) and (3.9) that

$$\infty = \widetilde{I} \le 2I_1 + 2I_2 + 4I.$$

Since I_1 and I_2 are finite, we have $I=\infty$, as desired.

Subcase ii. Suppose $\int_0^a \frac{\mathbf{1}_S(x)}{|x-a|}dx = \infty$. Define

$$I_D = \int_0^a\int_0^a \frac{\mathbf{1}_S(x)\mathbf{1}_T(y)}{|x-y|^2}\mathbf{1}_D(x,y)dydx \le I,$$

where $D=\{(x,y)\in\mathbb{R}^2 : x<y\}$. To compute a lower bound for I_D first note that since a is a point of density of T, there exists a sufficiently large constant $0<C<\infty$ such that

$$|a-x| \le C\ |T\cap[x,a]|, \quad a.e.\ x\in[0,1].$$

Therefore for $a.e.$ $x\in[0,a)$

$$\begin{aligned}\frac{1}{|a-x|} &\le \frac{C|T\cap[x,a]|}{|a-x|^2} = C\ |\ T\cap[x,a]\ |\ \cdot \min_{y\in[x,a]}\left\{\frac{1}{|x-y|^2}\right\}\\ &\le C\int_x^a \frac{\mathbf{1}_T(y)}{|x-y|^2}dy.\end{aligned}$$

This implies that

$$\infty = \int_0^a \frac{\mathbf{1}_S(x)}{|a-x|}dx \leq C \int_0^a \int_x^a \frac{\mathbf{1}_S(x)\mathbf{1}_T(y)}{|x-y|^2}dydx = CI_D,$$

and it follows that $I_D = \infty$, and hence $I = \infty$, as desired.

Case II. We conclude by addressing the cases where $|S| = 0$ or $|S| = 1$. Without loss of generality we only consider $|S| = 1$, and hence assume that $S = [0,1]$ up to a set of measure zero. It follows from (3.2) and the positivity of f that

$$A \leq f(x), \quad a.e.\ x \in [0,1].$$

This, together with the fact that f is supported in $[-1,1]$, implies that

$$\begin{aligned} \infty &= \int_1^\infty \int_0^1 \frac{1}{|x-y|^2}dxdy \\ &\leq \frac{1}{A^2}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|f(x)-f(y)|^2}{|x-y|^2}dxdy \\ &= \frac{4\pi^2}{A^2}||f||^2_{\dot{H}^{1/2}(\mathbb{R})}, \end{aligned}$$

as desired. This completes the proof.

□

We are now ready to prove Theorem 1.4.

Proof of Theorem 1.4. We proceed by contradiction. Assume that $g \in L^2(\mathbb{R})$, that $\mathcal{G}(g,1,1)$ is a Riesz basis for $L^2(\mathbb{R})$ with frame constants $0 < A \leq B < \infty$, and that $\Delta_1(g) < \infty$ and $\Delta_\infty(\widehat{g}) < C < \infty$, for some constant C.

By Theorem 2.1,

$$\sqrt{A} \leq |Zg(x,w)| \leq \sqrt{B} \quad \text{a.e. on} \quad [0,1)^2.$$

Since $Z\widehat{g}(x,w) = e^{2\pi ixw}Zg(-w,x)$ we have

$$\sqrt{A} \leq |Z\widehat{g}(x,w)| \leq \sqrt{B} \quad \text{a.e. on} \quad [0,1)^2.$$

Next, the assumption $\int |\gamma|^N |\widehat{g}(\gamma)|^2 d\gamma < C$ for all $N > 0$ implies that

$$\text{supp } \ \widehat{g} \subseteq [-1,1].$$

Thus, for $(x,w) \in [0,1)^2$, we have

$$Z\widehat{g}(x,w) = \sum_{n\in\mathbb{Z}} \widehat{g}(x-n)e^{2\pi inw} = \widehat{g}(x) + \widehat{g}(x-1)e^{2\pi iw},$$

so that we have

$$\sqrt{A} \leq |\widehat{g}(x) + \widehat{g}(x-1)e^{2\pi iw}| \leq \sqrt{B} \quad \text{for a.e. } (x,w) \in [0,1)^2. \tag{3.10}$$

In particular, it follows that

$$\sqrt{A} \leq |\ |\widehat{g}(x)| \pm |\widehat{g}(x-1)|\ | \leq \sqrt{B}, \qquad \text{for } a.e.\ x\ \in [0,1].$$

It now follows from Theorem 3.1 that $|\widehat{g}| \notin \dot{H}^{1/2}(\mathbb{R})$, which implies that $\widehat{g} \notin \dot{H}^{1/2}(\mathbb{R})$. In other words, $\Delta_1(g) = ||\widehat{g}||^2_{\dot{H}^{1/2}(\mathbb{R})} = \infty$. This contradiction completes the proof.

□

Since orthonormal bases are Riesz bases with frame constants $A = B = 1$, Corollary 1.5 follows from Theorem 1.4.

4. Further Comments

1. Theorem 1.5 is sharp in the sense investigated in Theorem 1.2, see [5]. In fact, Theorem 1.5 no longer holds if one weakens the Δ_1 decay hypotheses by a certain logarithmic amount. For example, if $d > 1$ and $\widehat{g}(\gamma) = \mathbf{1}_{[0,1]}(\gamma)$ then $\mathcal{G}(g,1,1)$ is an orthonormal basis for $L^2(\mathbb{R})$, and

$$\int \frac{|t|}{\log^d(|t|+2)}|g(t)|^2 dt < \infty \quad \text{and} \quad \sup_{N>0} \int |\gamma|^N |\widehat{g}(\gamma)|^2 d\gamma < \infty.$$

2. There are two noteworthy cases in which the proof of Theorem 1.4 can be significantly simplified. If one assumes that $\mathcal{G}(g,1,1)$ is an orthonormal basis for $L^2(\mathbb{R})$ then the frame constants satisfy $A = B = 1$ and it follows from (3.10) that $|\widehat{g}(x)| = \mathbf{1}_R(x)$ for some set $R \subset \mathbb{R}$ of positive and finite measure. Corollary 2.4 completes the proof in this case. Likewise, if $\mathcal{G}(g,1,1)$ is a Riesz basis for $L^2(\mathbb{R})$ whose frame bounds A and B are sufficiently close to one another, e.g., $\sqrt{B} < 3\sqrt{A}$, then a direct argument involving Theorem 2.2 and Theorem 2.3 completes the proof. The main difficulty in Theorem 1.4 and Theorem 3.1 arises when the frame constants A and B are far apart.

3. We conclude by noting that if one weakens the hypotheses in Theorem 1.4 to $\Delta_\infty(\widehat{g}) < \infty$ and $\Delta_{1+\epsilon}(g) < \infty$, for some $\epsilon > 0$, then the result is a simple consequence of the *Amalgam Balian-Low Theorem*. The Amalgam Balian-Low Theorem, e.g., [6], states that if $\mathcal{G}(g,1,1)$ is a Riesz basis for $L^2(\mathbb{R})$ then

$$g \notin W(C_0, l^1) \quad \text{and} \quad \widehat{g} \notin W(C_0, l^1),$$

where

$$W(C_0, l^1) = \{f : f \text{ is continuous and } \sum_{k\in\mathbb{Z}} ||f\mathbf{1}_{[k,k+1)}||_{L^\infty(\mathbb{R})} < \infty\}.$$

The assumptions $\Delta_{1+\epsilon}(g) < \infty$ and $\Delta_\infty(\widehat{g}) < \infty$ imply that $\widehat{g}$ is continuous and supported in $[-1,1]$, which, in turn, implies that $\widehat{g} \in W(C_0, l^1)$.

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