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An Engineering Approach to Model-Order Reduction and Its Application to Controller Design.

Alexius Ogbusua Kalu
Louisiana State University and Agricultural & Mechanical College

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AN ENGINEERING APPROACH TO MODEL-ORDER REDUCTION AND ITS APPLICATION TO CONTROLLER DESIGN

The Louisiana State University and Agricultural and Mechanical Col. PH.D. 1985

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AN ENGINEERING APPROACH TO MODEL-ORDER REDUCTION AND ITS APPLICATION TO CONTROLLER DESIGN

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in

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by
Alexius Ogbusua Kalu
B.S., The University of Texas, 1980
M.S., Louisiana Tech University, 1982
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LIST OF SYMBOLS

Symbols	Units
s = Complex Variable = $\sigma + j\omega$	
$j = \sqrt{-1}$	
ω = Cyclic Frequency	rad/sec
$H(s)$ = Transfer Function	
$M(s)$ = Transfer Function Matrix	
K_p = Position Error	
K_v = Velocity Error	
K_a = Acceleration Error	
K = Feedback Gain Matrix	
R^2 = Coefficient of Determination	
$E(s)$ = Sum of Errors	
Θ = Angle of Asymptotes	degrees
ϕ = Phase Angle	degrees

Notations

R^n = n-dimensional space

Im $\{.\}$ = Imaginary Part of .

Re $\{.\}$ = Real Part of .

[#] = Listed Reference #

F^* = The Complex Conjugate of F

F_{cal} = The F-Statistic (calculated)

$\overline{Y}$ = Average Value of Y

Notations (cont'd)	Units
$\hat{Y}$ = Estimated Value of Y	
A^T = Transpose of Matrix A	
degree {} = degree of	
A_p = State-Space Realization Matrix A for $H_p(s)$	
A_R = State-Space Realization Matrix A for $H_R(s)$	
$A_p(s)$ = Magnitude of $H_p(s)$	dB
$A_R(s)$ = Magnitude of $H_R(s)$	dB

Subscripts:

p = Plant quantity

R = Reduced-order Quantity

o = Initial Value

α = Level of Confidence for the F-Test.

Abbreviations:

CFE = Continued Fraction Expansion

MIMO = Multiple Input Multiple Output

SISO = Single Input Single Output

argH(jω) = Arguement of H(jω)

ABSTRACT

A new method of approximating a high-order system by a lower-order model in the frequency domain is developed. A reduced-order model of a transfer function obtained by the new method is compared with a reduced-order model obtained by another existing method to illustrate the power of the technique. Furthermore the approximant constructed by this new method is used to design a control system, which is applicable to the plant, to show its usefulness. The thesis also suggests some mathematical criteria for selecting the order of the approximant.

This technique is particularly attractive because of its simplicity and versatility. Furthermore, it is applicable to a large group of practical Single Input-Single Output systems' transfer functions.

CHAPTER I

INTRODUCTION AND LITERATURE REVIEW

1.1 General

A typical industrial system involves components from several engineering disciplines. For instance, just the reactor of a nuclear power generating plant involves chemical, electrical, electronic, hydraulic and mechanical components. Engineers and scientists are frequently confronted with the task of studying such complex physical systems. One of the first steps in studying any real world systems is the development of a mathematical model of the phenomenon being studied. In doing this engineers (and scientists) are guided by the thought that an oversimplified model will lead to conclusions which are invalid in the real world. Consequently, any study of a complex system involves complex mathematical description - high order differential equations or polynomials - in most cases.

The economic importance of the analysis and other studies of a plant to the industry cannot be over emphasized. It is the key to optimal performance, productivity and investment decisions, to mention a few. In some cases the mathematical models that represent the system are so complex that they defy solutions. In some other cases they may be solvable but so cumbersome that the economic advantage of the study could be outweighed by the cost of it in terms of human effort and time. To overcome this problem it is often desirable to

approximate a high-order dynamic system by a low-order model so that simplified studies can be carried out.

In this thesis, a procedure for obtaining an adequate low-order approximation to a high-order system is developed. Much emphasis is given to model adequacy while effort is made to make the procedure very computationally efficient. The procedure developed is applicable to both Single Input-Single Output (SISO) systems and Multiple Input-Multiple Output (MIMO) systems with some limitations.

The results obtained from the use of this new procedure are compared with those from other well-known methods to show its effectiveness and advantages. The usefulness of the method is also demonstrated by the use of the reduced-order model to design a controller which can adequately control the high-order plant.

There is a conflict between estimate 'fidelity', which is a measure of how much the estimate resembles the original plant, and estimate simplicity, which is determined by the order of the estimate. It therefore follows that for any given high-order system there is an optimal order of the reduced-order estimates. A method of obtaining such optimal order is presented in this thesis.

1.2 Order Reduction.

Model order reduction involves a trade off between model order and the degree to which the characteristics of the plant are neglected by the model. Because the relative importance of various plant characteristics is highly dependent upon the application, it is difficult to conceive a universal model reduction algorithm. Nevertheless

certain model order reduction algorithms can be considered more efficient than others. This judgement can be based on their relative simplicity and the amount of plant characteristics they can preserve. The main purpose of model order reduction is to obtain a simpler 'true' image of the plant so that studies about the plant can be simplified. Consequently, it is probably best for an order reduction algorithm to focus more on plant characteristics preservation while striving for simplicity. This philosophy calls for certain constraints to be imposed on the model parameters such that model retains all the important characteristics of the plant.

A high fidelity model may be achieved if the model and plant have (1) the same steady-state error constants; (2) high frequency responses which are asymptotically the same; and (3) transfer functions that match at some arbitrary frequency values. Any effort to achieve the above objectives forces the model parameters to have certain values which results in a unique estimate of the original plant. That is, mathematical relations between a set of unknowns which constitute the parameters of the low-order model and the parameters of the high-order plant are developed. These relations must be such that the two models satisfy the above outlined requirements. Again, it is not enough to have an adequate estimate but also a simple estimate. Therefore the set of unknown parameters must be of those parameters constituting a reduced model of predetermined order, alternatively, the method of determining the unknowns must be such that they could be chosen for any order of interest.

Also an order reduction algorithm should be approached from an engineering perspective rather than just as a mathematical exercise.

An engineering approach would be to obtain a reduced-order model by a process based on a set of simple design principles. Measures of relative stability (gain margin and phase margin) are often used as design criteria. A model order reduction process which insists, therefore, that the plant and the low-order model have the same gain margin and phase margin should be useful for engineering applications.

1.3 Review of Existing Literature

A common and quite legitimate complaint directed toward multivariate control literature is that the apparent strength of the theory is not accompanied by strong numerical tools. Practically every linear system text gives a discussion of minimal realization. The textbook algorithms are far from being satisfactory, however, serving mainly to illustrate the theory with textbook examples. Thus, simplification of dynamic systems with large order has received increased attention in recent years. Even so the problem has not been solved since the answer to the question "Do methods of system reduction exist which produce reduced systems suitable for control system design purposes?" is definitely not clear. Also, it is not clear how small the approximate model can be and still accurately represent the original plant.

Numerous methods for approximating high-order systems now exist. Comprehensive lists of references may be found in Genesio [1] (1976), Decester [4] (1976) and Marshall (1978). A good number of the existing methods are algebraic and are very computationally attractive. These include, for example, the Pade' class of methods like the continued fraction expansion [5] (CFE), time Moments [6] and Pade'

approximations [7]. It has been shown [8] that, under certain mild conditions, these methods yield the same Pade' approximants; the direct Pade' approximation being the more general one. However, being mostly approximations about a single frequency point (s = 0), the algebraic methods yield poor overall frequency response characteristics. Furthermore, some of the Pade' methods may produce an unstable reduced order model even though the original high order plant is stable. To deal with this problem another algebraic approach - the Routh approximation method [9], [10], has been introduced.

Routh approximation methods have their own weaknesses too. Ashoor and Singh [11] have shown that for some systems the Routh approximant may be much superior to the Pade' approximants whereas Shamash [7] has shown that for certain other systems, the Routh approximant [9] may be much inferior. Thus none of the above classes of algebraic approximations can be applied with any certainty. Much other work on order reduction revolves around these two classes of approximation - being only extensions, modifications, or alternative versions of either of them.

There is still another group of order-reductions methods based on error minimization. This includes the work by Eitelberg [11], Obinata and Inoka [12], [13]. The proponents of these methods claim that their greatest advantage over other approximation methods lies in the fact that the equation error (residual) depends linearly on the unknown quantities. But Eitelberg points out that there may be disadvantages in the application of some aspects of the works of Obinata and Inoka [12], [13]. Almost all the existing methods based on error minimization have been disapproved by some scholars for one reason or

the other-indicating that this class of reduction methods are still not applicable with any certainty.

Most of the above methods are applicable to SISO Systems only, and so only a few of them are adaptable to nonlinear systems. However, it does not seem a formidable task to extend any method intended for SISO Systems for application to MIMO Systems. Furthermore, some authors have developed approximation methods which are exclusively applicable to Multivariable Systems. A popular technique [15], [16], [17] is the Eigenvalue Preservation Method. One of the early complaints against this method is its failure to preserve the d.c. steady state gain. Hicken and Sinha [18], have shown that the method of aggregation implicitly implies eigenvalue preservation.

Other methods of order reduction include Moments Matching Techniques [19], [20], [21] and orthogonal projection (geometrical) techniques [22]. Other more general methods include the Singular pertubation technique [23] and the uniform approximation technique [24].

Most of the above system reduction methods are of a mathematical nature and hardly address engineering problems. Obtaining a good approximation of the system response to particular inputs should not be the only goal of a reduction technique. More important is the application of the reduced order model to engineering design. The first efforts in this direction were made by Aoki [25], Sannuti and Kokotovic [23], and Milanese and Negro [27]. It was noticed by Lambo and Rao [28] that Davison's model [15] was a special case of aggregation, so that Aoki's analysis could be applied. Disappointingly, similar results could not be obtained using the moment matching

methods and the above eigenvalue preservation methods. This casts doubt on their reliability. Although Aoki's method is computationally efficient it cannot be used to design static compensator's for the original plant. An effort by Hickis and Sinha [18] to bridge this gap results in a formidable computational task.

Methods of reduction based on simple design principles have been suggested by Marshall [31] recently. He included in his set of constraints a requirement that the model and plant have the same gain margin and phase margin. But as attractive as Marshall's reduction techniques seem, they do not differ very much from many of the others in that the philosophy is to preserve some of the dominant modes. This requires that the poles of the open loop transfer function be known. This may involve a tedious mathematical task for system of very high order. Furthermore, no set or rules (or algorithms) is laid out for any class of transfer functions. In some cases part of the procedure is the design of a compensator which will make the estimate meet the prescribed requirements. In some other cases Marshall [31] starts the reduction process from the response curve, such as polar plot, of the original system. It may be observed that not only do his methods require a lot of information about the plant responses but they also lack generality.

As alluded to earlier, many of the existing methods have been criticized by different scholars. There is so much comment on order reduction techniques and rebutals in the literature that in some cases the arguments become subjective and the choice is left to the reader. Trying to settle the problems involved in the eigenvalue preservation

method, for instance, resulted in what Towil [29] described as a "running battle" between Davidson and Chidambara.

There is just as much effort to improve the techniques as there are criticisms about them. Most critics follow their comments with a suggestion for improvement. A brief but closer look of some of the more general methods in the literature follows.

Reduction methods based on retaining only the eigenvalues of the original systems close to the origin are referred to as the dominant eigenvalues while neglecting these eigenvalues farthest from the origin are common. The philosophy is that by retaining the dominant eigenvalues of the original systems the dominant time constants of the original system which govern the system behavior will be retained in the reduced model. This implies that the overall behavior of the approximate system will be very similar to the original system. The proponents of these methods argue that the above assumption is true because the contribution of the eigenvalues far away from the origin to the system response are important only at the beginning of the response, whereas the dominant eigenvalues have significant contribution throughout the whole of the response and, consequently determines the type of system response.

The main disadvantages of this approach are: it requires the determination of the poles of the original system which may pose computational problems for very high order systems or for systems with widely separated poles. Also these methods cannot be applied to systems where there are no dominant poles or where the dominant poles

are difficult to identify, example [39], a system with poles at -1, -1±j 27, -1.5. Another complaint against the dominant mode concept is that it does not preserve the d.c. steady state gain.[!]

Another class of model reduction technique in the frequency domain is the Pade' approximation method mentioned before. In these methods the Taylor series expansion about $s = 0$ for the original and reduced models are matched up to the maximum number of terms. These methods are particularly useful, the authors claim, where the original system has no clearly dominant poles or where the dominant modes are difficult to identify and thus the methods based on the dominant eigenvalue concept discussed in the previous paragraph cannot be employed.

A very serious problem with this approach is that it may result in an unstable reduced model even when the original system is stable, in which case the reduced-order model is worthless. Furthermore, since the Pade' approximations are approximations about a single frequency point ($s \to 0$), they may yield poor response characteristics at several other frequencies.

To deal with the stability problem in Pade' methods, a number of methods based on the idea of selecting the denominator of the reduced-order model, which will ensure stability, a priori have been suggested. The numerator coefficients are then used to match terms in the Taylor series thus producing the so-called partial Pade' approximation. Unfortunately, [44] the partial Pade' approximation

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[!] Marshall's method [31] does not suffer from this problem.

results in a worse overall frequency response characteristics than the full Pade' approximation does.

The problem of poor overall frequency response characteristics that results from the algebraic (Pade' class) methods has been challenged by Bistriz and Langholz [45]. Instead of matching the Taylor series expansion about a single frequency point ($s \to 0$) or about two frequency points ($s \to 0$, $s \to \infty$) as in the modified Pade' approach, Bistriz and Langholz manipulate two Chebyshev polynomial series, one representing the original high-order system and the other representing the approximating low-order model. They prefer to regard their approach as a generalization of the classical Pade' approximations, with the Chebyshev polynomial series expansion being over a desired frequency interval instead of a power series about a single frequency point. But in their work [45] only low-pass amplitude approximations were considered. Also the stability question was not specifically addressed. Furthermore, this approach lacks the simplicity of the Pade' methods.

The next group of reduction methods in frequency domain - the Routh approximation methods - were introduced basically to circumvent the stability problem resulting from the Pade' group of methods. The general philosophy is to seek an approximant which has equal sign changes (in its Routh array) to the original system's Routh's array. No other constraints are imposed on the reduced-order model, thus though stable estimates are produced from stable high-order plants using this approach, the estimates in many cases are much inferior to those obtained via the other methods. Furthermore, it has been

found that [38] there exists a high degree of nonuniqueness in the Routh methods.

The above three general approaches represent most the efforts to reduce the order of a system in frequency-domain available in the literature. The other methods mentioned are reduction techniques in time-domain. These methods seem to be more of mathematical exercises instead of solutions to engineering problems. A common defect of most of the available techniques is their lack of simplicity. There is still a need for a technique which has engineering implications and is conceptually simple.

CHAPTER II

MODEL-ORDER REDUCTION CRITERIA

2.1 General

If an order-reduction technique is to be applicable to control system design or/and analysis problems it should meet certain criteria other than the loose requirement, namely, that it produces a low-order estimate which has approximately the same response characteristics with the original high-order system. These criteria consist of the constraints the technique imposes on the reduced-order model forcing it to preserve certain characteristics of the original system which are vital for certain engineering applications.

The constraints imposed on the reduced-order model by the reduction technique introduced in this thesis are: -

(1) That the approximant have the same gain margin and phase margin with the plant.

(2) The order difference, d-n, be preserved, where d is the order of the denominator polynomial of the transfer function and n the order of the numerator polynomial.

(3) That the reduced-order model have the same response characteristics with the plant as $s \to 0$.

(4) That the reduced-order model have the same response characteristics with the plant as $s \to \infty$.

It shall be shown in this chapter that these constraints have serious engineering implications and are thus essential requirements. It will

also be seen that an approximant which satisfies all these requirements will have an overall frequency response characteristics similar to those of the plant.

Some of the commonly used control system design techniques in the frequency domain are the Nyquist and the Root Locus approach. Since the reduction procedure introduced here is in the frequency domain the above approaches shall be used as the bases for justifying the constraints imposed on the approximant by the new reduction technique.

2.2 Implications of the Relative Stability Constraints

The Nyquist stability criterion is a very valuable tool for determining the degree of stability, or instability of a feedback control system. This criterion is stated algebraically as [2]

$$N = Z - P \tag{2-1}$$

Where N is the number of clockwise encirclements of the $-1 + j0$ point by the Nyquist locus, P is the number of poles of the open loop transfer function H(s) having positive real parts, and Z is the number of roots of the characteristic equation $1 + H(s) = 0$ having positive real parts. For a stable system

$$Z = 0 \tag{2-2}$$

and thus the criterion of Equation (2-1) becomes

$$N = -P \tag{2-3}$$

Consider the Nyquist diagram of figure 2-1. The number of encirclements, N, of the point $-1 + j0$ can be determined by knowing the points at which the Nyquist locus crosses the negative real axis. Also the direction of the locus can be determined by knowing the point

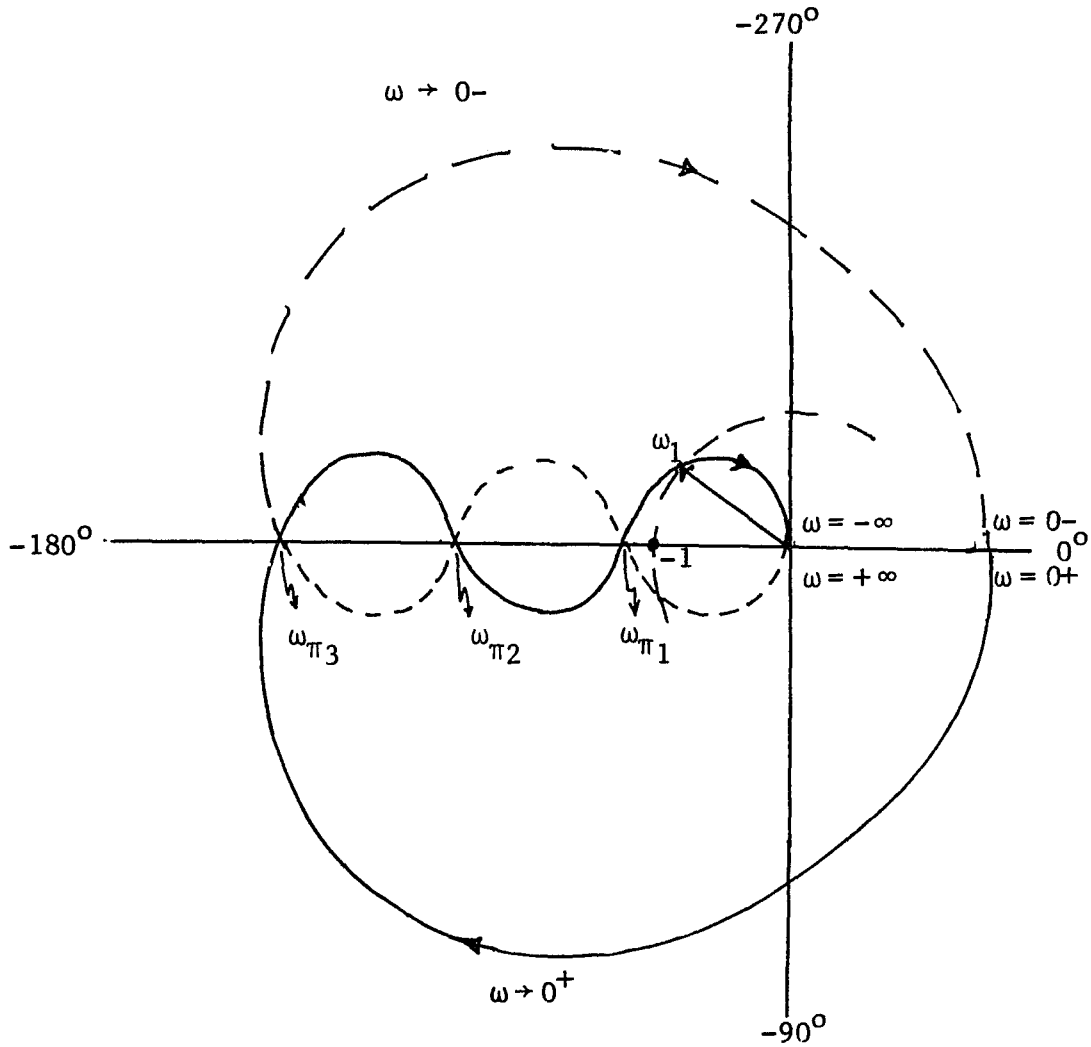

Figure 2-1

Illustrating the Relationship Between ω_π, ω_1 and the Nyquist Criterion.

at which the locus intercepts a circle of unit radius (with center at the origin) which passes through the point -1 + j0. This gives the angle H(s), makes with the negative real axis when its magnitude is unity.

Consider, then, two different open loop transfer functions $H_p(s)$ and $H_R(s)$ whose Nyquist diagrams cross the real axis at the same points and intercept with the unit radius circle at the same point, as shown in Figure 2-2. Their Nyquist diagrams must then encircle the -1 + j0 point equal number of times and will also have the same direction. The frequencies at which the Nyquist locus crosses the negative real axis are called the phase crossover frequency ω_π and the frequency at which the magnitude of H(s) is unity is the gain crossover frequency ω_1. Thus any two transfer functions whose gain margin and phase margin are equal will have identical stability provided they have equal number of poles P with positive real parts. The application of this to model order reduction can be summarized in the following theorem.

THEOREM I

Given a transfer function $H_p(s)$ which has P poles in the right half plane (RHP); if a reduced-order transfer function $H_R(s)$ can be constructed such that

(1) $H_R(s)$ has P poles in the RHP

(2) $H_p(s)$ and $H_R(s)$ have same ω_π and

(3) $H_p(s)$ and $H_R(s)$ have the same ω_1 then any K that yields

stable $\dfrac{KH_R(s)}{1 + KH_R(s)}$ will yield a stable $\dfrac{KH_p(s)}{1 + KH_p(s)}$.

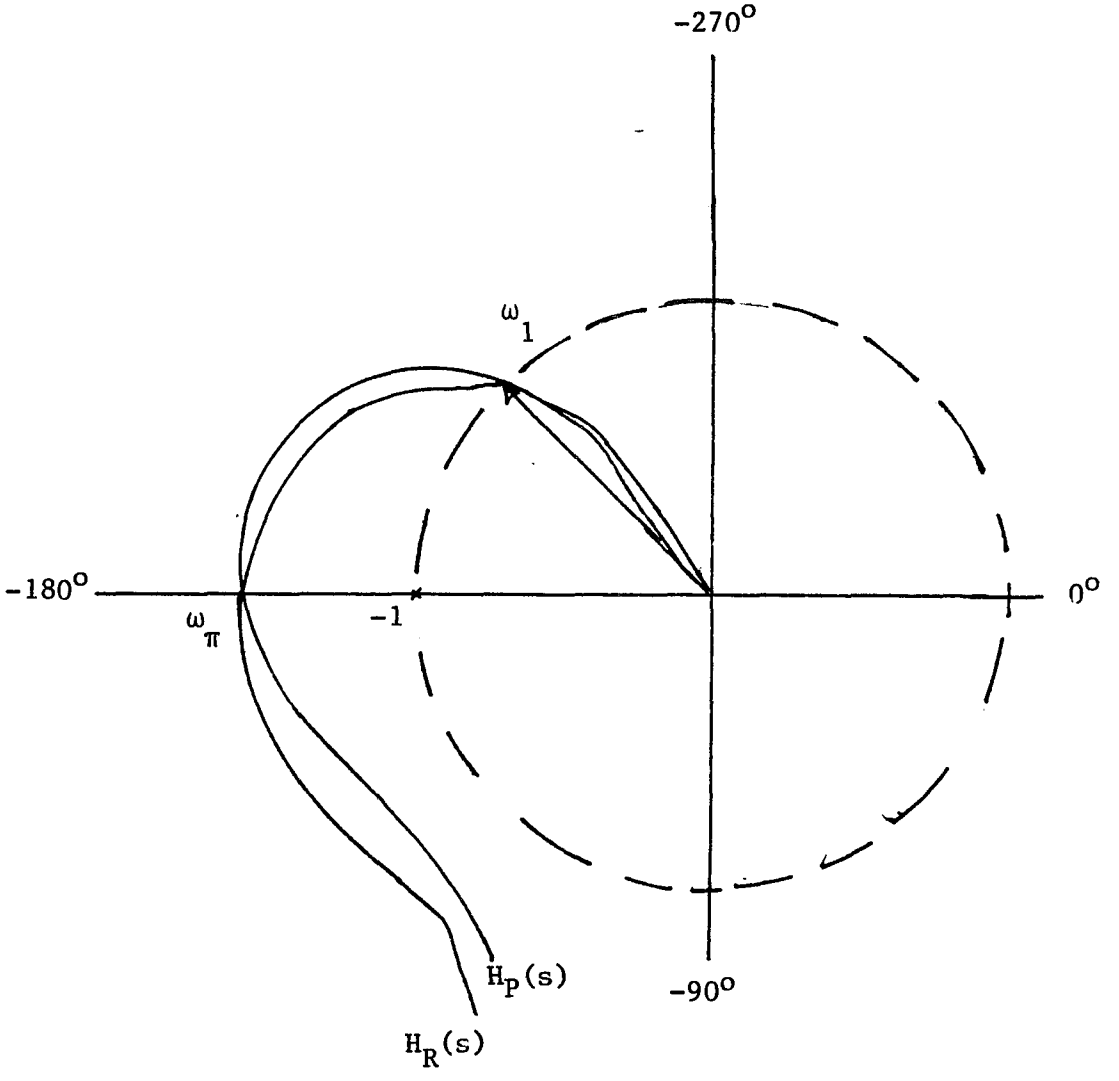

Figure 2-2

Two Systems with Common Phase Crossover
Frequency and Gain Crossover Frequency

similarly any e^{ϕ} that yields stable $\dfrac{e^{\phi}H_R(s)}{1 + e^{\phi}H_R(s)}$ will yield a stable

$$\frac{e^{\phi}H_p(s)}{1 + e^{\phi}H_p(s)}$$

Theorem I shows that the preservation of gain margin and phase margin will result in an estimate which can be useful in design of controllers, specifically simple gain compensator and simple phase compensator. More importantly, the preceeding discussion shows that the preservation of gain margin and phase margin will guarantee that $H_R(s)$ will be stable if $H_p(s)$ is stable provided $H_R(s)$ has no poles on the RHP. If this is the case then one would expect $H_R(s)$ to be useful in other applications particularly if certain other characteristics of $H_p(s)$ are preserved in $H_R(s)$. Thus preservation of the relative degree of stability is a necessary condition.

2.3 Implications of the Order-Difference Preservation

The purpose of this section is to show that constructing the estimate $H_R(s)$ such that

$$d_R - n_R = d - n \tag{2-4}$$

contributes to the estimate having similar response characteristics with the plant $H_p(s)$; where n_R and d_R are the order of the numerator and denominator of $H_R(s)$, respectively, and n and d are the order of the numerator and denominator of $H_p(s)$, respectively.

The general strictly proper transfer function

$$H_p(s) = \frac{a_n s^n + a_{n-1} s^{n-1} + \ldots + a_o}{b_d s^d + f_{d-1} s^{d-1} + \ldots + b_m s^m} \; ; \; d > n \qquad (2\text{-}5)$$

may be written in factored form as

$$H_p(s) = \frac{K(1+sT_a)(1+sT_b)\ldots(1+sT_n)}{s^m(1+sT_1)(1+sT_2)\ldots(1+sT_{d-m})} \; , \qquad (2\text{-}6)$$

where m denotes the system type. Substituting $s = j\omega$, then

$$\lim_{\omega \to 0} H_p(j\omega) = \begin{cases} K\angle 0^o & ; \; m = 0 \\ \infty\angle -m90^o & ; \; m \neq 0 \end{cases} \qquad (2\text{-}7)$$

$$\lim_{\omega \to \infty} H_p(j\omega) = 0 \; \angle -(d-n)90^o \quad \text{for all m.} \qquad (2\text{-}8).$$

Thus the angle at which the polar plot (Nyquist locus) approaches the axis as $\omega \to \infty$ is a function of the difference d-n. The following conclusions can be drawn.

Theorem II.

Given a transfer function $H_p(s)$ with numerator polynomial of order n and denominator polynomial of order d then a reduced-order model $H_R(s)$ constructed such that the difference between the orders of its denominator and numerator polynomials is d-n will have a Nyquist path which approaches the axis at the same angle as $H_p(s)$, as $\omega \to \infty$.

Next it will be shown that the shape of the root locus, another important design technique, depends in part on the difference d-n. For any point on the s-plane to lie on the root locus of $H_p(s)$, that value of s must satisfy the following [2].

Magnitude Condition:

$$\left| H_p(s) \right| = 1, \text{ for all values of K} \qquad (2\text{-}9)$$

Angle Condition:

$$\arg H_p(s) = \begin{cases} (1+2h)180^0; & K > 0 \\ h360^0; & K < 0 \end{cases} \tag{2-10}$$

where K is the loop sensitivity and

$$h = 0, \pm 1, \pm 2, \ldots \pm(d-n-1). \tag{2-11}$$

Now the characteristic equation of the system is given by

$$B(s) = 1 + H_p(s) = 0 \tag{2-12}$$

Therefore

$$H_p(s) = \frac{K(s-Z_1)(s-Z_2)\ldots(s-Z_n)}{(s-P_1)(s-P_2)\ldots(s-P_d)} = -1 \tag{2-13}$$

The asymptotes that the loci approach as s approaches infinity can be determined by evaluating the limit of $H_p(s)$ as $s \to \infty$.

$$\lim_{s \to \infty} H_p(s) = \lim_{s \to \infty} \left\{ \frac{K \sum_{i=1}^{n} (s-Z_i)}{\sum_{j=1}^{d} (s-P_j)} \right\} = \frac{K}{s^{d-n}} = -1 \tag{2-14}$$

Therefore the angle condition of Equation (2-10) becomes

$$\angle -K = \angle s^{d-n} = (1+2h)180^0 \tag{2-15}$$

or

$$(d-n)\angle s = (1+2h)180^0 \tag{2-16}$$

That is, the angle of the asymptotes the locus approach as s approaches infinity are given by

$$\angle s = \theta = \frac{(1+2h)180^0}{d-n} \tag{2-17}$$

and they are d-n asymptotes. This condition leads to the following theorem.

Theorem III.

Given a transfer function $H_p(s)$ with numerator polynomial of order n and denominator polynomial of order d then a reduced-order model $H_R(s)$ constructed such that the difference between the orders of its denominator and numerator polynomials is d-n will have the same number of asymptotes on its root locus as does $H_p(s)$. These asymptotes also have the same angles as those of $H_p(s)$.

2.4 Implications of the Zero Frequency Response Constraint

Static accuracy is an important characteristic of a feedback control system. The designer always strives to design the system to minimize error for a certain class of inputs. It therefore seems important that the accuracy of the approximant be similar to that of the plant if the approximant is to be used for studies about the plant.

The relation between the resulting system error, E(s), for a given input R(s) is given by [2]

$$\frac{E(s)}{R(s)} = \frac{1}{1 + H_p(s)} \tag{2-18}$$

The steady state error is given by

$$e_{ss} = \lim_{t \to \infty} e(t) = \lim_{s \to 0} \left\{ \frac{sR(s)}{1 + H_p(s)} \right\} \tag{2-19}$$

The usual inputs of interests are

Position input: $R(s) = 1/s$

Velocity (or ramp) input: $R(s) = 1/s^2$

Acceleration input: $R(s) = 1/s^3$

and sometimes higher order derivative inputs $R(s) = 1/s^i$

where i is a positive integer. The general form of the steady-state error can be expressed as

$$e_{ss} = \lim_{s \to 0} \left\{ \frac{s(1/s^i)}{1 + H_p(s)} \right\} \tag{2-20}$$

or

$$e_{ss} = \frac{1}{1 + \lim_{s \to 0} \left\{ s^{i-1} H_p(s) \right\}} = \frac{1}{\lim_{s \to 0} \left\{ s^{i-1} H_p(s) \right\}} \tag{2-21}$$

The quantity $\lim_{s \to 0} s^{i-1} H_p(s)$ is called the error constant denoted by K_i, thus

$$K_i = \lim_{s \to 0} \left\{ s^{i-1} H_p(s) \right\} \tag{2-22}$$

and the steady-state error is given by

$$e_{ss} = \frac{1}{K_i} \tag{2-23}$$

Where the subscript i indicates the type of input, for instance, for a velocity input ($R(s) = 1/s^2$) K_i is denoted by K_v, and for an input $R(s) = 1/s^4$ K_i is denoted by K_4.

From Equation (2-22) it can be seen that for transfer functions whose values are equal at s=0, the constant K_i will be the same. The implication of this to order-reduction is as follows.

Theorem IV

If an estimate, $H_R(s)$, of a high-order transfer function $H_p(s)$ is constructed so that $H_R(0) = H_p(0)$ then $H_R(s)$ and $H_p(s)$ will have the same error constants.

2.5 The Need for the Preservation of Response Characteristics as $s \to \infty$.

As previously indicated plotting the root locus of a system is greatly facilitated if one can determine the asymptotes approached

by the various branches as s takes on large values. This implies determining the value of $H_p(s)$ as s approaches infinity. To use $H_R(s)$ for studies about $H_p(s)$ one would like it to have similar root locus and one of the necessary conditions for this is $H_R(\infty) = H_p(\infty)$.

Similar reasoning holds when one looks at the Nyquist plot of a system. One of the assumptions of the Nyguist criterion [2] is that $\lim_{s \to \infty} H(s) \to 0$. Thus to evaluate the stability conditions of $H_R(s)$, which will be used for designs compatible with $H_p(s)$, one would require among other things that $\lim_{s \to \infty} H_R(s) = \lim_{s \to \infty} H_p(s)$.

However, it should be noticed from Theorem II that the preservation of d-n implicitly results in the preservation of response at $s = \infty$. Thus a reduction process may not insist on the latter if it can guarantee the former.

2.6 Versatility of Procedure

Lastly, the model order reduction technique should be versatile. One of the requirements of Theorem I, for instance, is that $H_p(s)$ and $H_R(s)$ have equal number of poles with positive real parts. Suppose a reduction algorithm produces an estimate that does not meet this requirement, then there should be part of the whole process that forces $H_R(s)$ to meet this requirement so that Theorem I will be applicable.

Furthermore, the reduction process should be able to allow the engineer to preserve any part of the plant's response characteristics as his intended application may dictate. For instance, it might be more important to match the low frequency responses of $H_p(s)$ and $H_R(s)$ if the reduced-order model is to be used for low pass filter designs.

CHAPTER III

DEVELOPMENT OF THE NEW ALGORITHM FOR MODEL ORDER REDUCTION

3.1 General

The high-order system here referred to as the plant is represented by its transfer function $H_p(s)$ if it is a SISO system or by its transfer function matrix $M_p(s)$ for a MIMO system. The basic problem is to obtain another transfer function $H_R(s)$ (or another transfer function matrix $M_R(s)$) of lower order such that controllers designed for $H_R(s)$ (or $M_R(s)$) will adequately control $H_p(s)$ (or $M_p(s)$) respectively. The philosophy is to seek a reduction procedure which is both conceptually simple and computationally efficient, and will result in a reduced-order model which has identical performance characteristics with the plant.

Many methods of reduction require a prior knowledge of the poles of the plant. This involves the factorization of a high-degree polynomial which is a formidable task. To use the reduction method introduced here one does not have to perform this task since knowledge of the plant's poles is not required. Also much of the existing literature in model order reduction may be difficult to understand becuase of the level of mathematics employed. The reduction method developed in this thesis does not require any more knowledge of mathematics than elementary algebra and thus can claim simplicity.

The reduction process in this work is done in the frequency domain. The comparisons of plant and estimate performance

characteristics are also done in the frequency domain. The constraints, therefore, imposed on the estimate are that it has the same frequency specifications and similar overall frequency response. Some of the most commonly used design criteria in the frequency domain are the gain margin and the phase margin. Consequently, a mathematical process which will lead to an estimate whose gain margin and phase margin are the same with those of the original plant shall constitute a part of the reduction procedure. Another essential part of the reduction procedure consists of the mathematical process which will force the resultant model to have the same steady state response and high frequency response with the plant.

There should be little or no doubt (as was shown in Chapter II) that if a reduction method can preserve the above mentioned plant performance characteristics it will yield an adequate estimate of the plant for design purposes. As mentioned in Chapter II the procedure should be versatile enough to allow the exact matching of plant and estimate response at any frequency of interest. This will enable one to emphasize those plant characteristics which are most significant according as the application for which the model is intended. In this chapter is developed a reduction procedure which guarantees exact matching of plant and estimate steady-state response, asymtotic matching of high frequency responses, and the same gain margin and phase margin. Furthermore the procedure is capable of exact matching of plant and estimate response at any frequency. This is useful in finetuning the reduced-order model to have an overall frequency response similar to that of the plant.

3.2 Preserving the D.C. Steady-State and High Frequency Response Characteristics

Consider a strictly proper transfer function $H_p(s) = N(s)/D(s)$ where, $H(s) \in R^n$ and $D(s) \in R^d$; $d > n$.

$$H_p(s) = \frac{1}{D(s)/N(s)} = \frac{1}{P_1(s) + H_1(s)} \tag{3-1}$$

Where $P_1(s)$ is the quotient polynomial of $1/H_p(s)$ and $H_1(s)$ is a strictly proper transfer function given by

$$H_1(s) = \frac{R_1(s)}{N(s)} \tag{3-2}$$

Where $R_1(s)$ is the remainder polynomial of $1/H_p(s)$. Since $H_1(s)$ is inherently strictly proper it vanishes as $s \to \infty$. This phenomenum can be summarized in the form of a theorem, viz:

Theorem V

Given any $H_p(s)$ which is strictly proper, $H_p(s) \to \frac{1}{P_1(s)}$ as $s \to \infty$.

Where $P_1(s)$ is the quotient polynomial of $1/H_p(s)$.

The stipulation of Theorem V assures us that retaining the quotient of $1/H_p(s)$ in the reduced-order model $H_R(s)$ will result in the asymptotic matching of $H_p(s)$ and $H_R(s)$ at high frequency responses. Thus, as far as satisfying the high frequency response requirement is concerned $H_1(s)$ is arbitrary and can be chosen for convenience with respect to satisfying other constraints in the reduction process.

The transfer function $H_1(s)$ (Equation (3-2)) can be expressed as the sum of its d.c. steady-state component and its frequency dependent component, thus:

$$H_1(s) = H_1(o) + H_2(s) \tag{3-3}$$

Where,

$$H_2(s) = H_1(s) - H_1(o). \tag{3-4}$$

Obviously $H_2(s)$ is not strictly proper. It is of the form

$$H_2(s) = sH_3(s) \tag{3-5}$$

where $H_3(s)$ is a strictly proper transfer function with no poles at the origin.

Equation (3-3) can then be rewritten as:

$$H_1(s) = H_1(o) + sH_3(s) \tag{3-6}$$

and Equation (3-1) which is an expression for the plant transfer function $H_p(s)$ becomes

$$H_p(s) = \frac{1}{P_1(s) + H_1(o) + sH_3(s)} \tag{3-7}$$

Defining another polynomial, $P_2(s)$, of the same order with $P_1(s)$:

$$P_2(s) = P_1(s) + H_1(o) \tag{3-8}$$

The plant transfer function $H_p(s)$ then becomes

$$H_p(s) = \frac{1}{P_2(s) + sH_3(s)} . \tag{3-9}$$

Since $H_3(s)$ does not have poles at zero then $sH_3(s)$ vanishes for s equal to zero and what is left of $H_p(s)$ for s equal to zero is

$$H_p(o) = \frac{1}{P_2(o)} . \tag{3-10}$$

The above derivation can be summarized in the following theorem:

Theorem VI

Given a strictly transfer function $H_p(s) = N(s)/D(s)$, $H_p(s) \to \frac{1}{P_1(s) + H_1(o)}$ as $s \to o$. Where $P_1(s)$ is the quotient polynomial of $\frac{1}{H_p(s)}$, $H_1(s) = R_1(s)/N(s)$, and $R_1(s)$ is the remainder polynomial of $1/H_p(s)$.

The above theorem assures us that retaining the sum of the quotient of $1/H_p(s)$ and the d.c. steady state component of $R_1(s)/N(s)$ in the estimate guarantees that the zero frequency responses of $H_p(s)$ and $H_R(s)$ are exactly matched. It should be noted with interest that the preservation of the d.c. steady-state response which is a consequence of Theorem VI requires that $P_1(s)$ be retained in the reduced model which is also a requirement for the asymptotic matching of the high frequency responses (as a result of Theorem V). The reduction process, thus, so far is decoupled in the sense that the satisfaction of one constraint does not destroy nor weaken an already satisfied requirement. Since both Theorems V and VI allow us the freedom to choose $H_3(s)$, the consequences of these theorems can be summarized as follows: any strictly proper transfer function $H_R(s) = \dfrac{1}{P_2(s) + sH_3(s)}$ is related to $H_p(s)$ by

$$H_R(o) = H_p(o)$$

$$H_R(\infty) = H_p(\infty)$$

and $H_p(s) \simeq H_R(s)$ for other values of s by appropriately choosing $H_3(s)$. Thus the application of Theorems V and VI ensure exact matching of d.c. steady-state responses and an asymptotic matching at high frequency responses which are typical points for evaluating and comparing system responses in the frequency domain. These conditions are also met by the Pade' methods [5, 6, 7].

3.3 Relative Stability Constraints

As was indicated earlier, if the reduced-order model is to be used for design purposes which will be applicable to the plant, then it

should have the same specifications with the plant. Commonly used design specifications in the frequency domains are the gain margin and the phase margin. The estimate can therefore be expected to be adequate for design purposes if it is forced to have the same relative stability with the plant, namely the same gain margin and phase margin.

In the previous section it was discovered that any transfer function $H_R(s)$ given by

$$H_R(s) = \frac{1}{P_2(s) + sH_3(s)} \tag{3-11}$$

will have the same d.c. steady-state response and high frequency response with the plant $H_p(s)$ for any $H_3(s)$ provided $H_3(s)$ has no poles at zero. Furthermore, it was concluded that, by appropriately choosing $H_3(s)$ $H_R(s) \simeq H_p(s)$. In this section the requirement that the plant $H_p(s)$ and the estimate $H_R(s)$ have the same gain margin and phase margin shall be used to determine an appropriate $H_3(s)$ which makes $H_R(s) \simeq H_p(s)$.

Equation (3-9) is of the form

$$H_p(s) = \frac{1}{P_2(s) + sN_3(s)}$$
$$D_3(s) \qquad \text{and}$$

can be rewritten as

$$H_p(s) = \frac{1}{P_2(s) + \{s/\{P_3(s) + H_4(s)\}\}} \tag{3-12}$$

where $P_3(s)$ is the quotient polynomial of $\frac{1}{H_3(s)}$, $H_4(s) = \frac{R_3(s)}{N_3(s)}$ and $R_3(s)$ is the remainder polynomial of $1/H_3(s) = D_3(s)/N_3(s)$. Thus $H_4(s)$ is strictly proper.

$$H_p(s) = \frac{1}{P_2(s) + \dfrac{s}{P_3(s) + H_4(s) + F(s) - F(s)}} . \qquad (3\text{-}13)$$

The function F(s) is chosen to be of the form*

$$F(s) = As + B \qquad (3\text{-}14)$$

Therefore

$$F(j\omega) = jA\omega + B \qquad (3\text{-}15)$$

and

$$F(-j\omega) = -jA\omega + B \qquad (3\text{-}16)$$

That is !,

$$F(j\omega) + F^*(j\omega) = 2B \qquad (3\text{-}17)$$

and

$$F(j\omega) - F^*(j\omega) = j2\omega A \qquad (3\text{-}18)$$

Equation (3-17) gives

$$B = \mathrm{Re}\{F(j\omega)\} \qquad (3\text{-}19)$$

and from Equation (3-18)

$$A = -\frac{\mathrm{Im}\{F(j\omega)\}}{\omega} \qquad (3\text{-}20)$$

consequently,

$$F(s) = \frac{\mathrm{Im}\{F(j\omega)\}s}{\omega} + \mathrm{Re}\{F(j\omega)\} \qquad (3\text{-}21)$$

If F(s) is chosen such that $F(j\omega_\pi) = H_4(j\omega_\pi)$ and $F(-j\omega_\pi) = H_4(-j\omega_\pi)$ then Equation (3-13) can be written as

---

* The function F(s) = As + B can be easily added to or subtracted from any polynomial or transfer function.

! $F^*(j\omega)$ is the complex conjugate of $F^*(j\omega)$ and is given $F^*(j\omega) = F^*(-j\omega)$.

$$H_p(s) = \frac{1}{P_2(s) + \dfrac{s}{P_3(s) + \dfrac{\mathrm{Im}\{H_4(j\omega_\pi)\}s + \mathrm{Re}\{H_4(j\omega_\pi)\} + (s^2 + \omega_\pi^2)H_5(s)}{\omega_\pi}}} \tag{3-22}$$

such that for $s = j\omega_\pi$,

$$H_p(j\omega_\pi) = \frac{1}{P_2(j\omega_\pi) + \dfrac{j}{P_3(j\omega_\pi) + H_4(j\omega_\pi)}} \tag{3-23}$$

where ω_π is the phase crossover frequency of the plant $H_p(s)$ and $H_5(s)$ is chosen to be

$$\omega_\pi H_5(s) = \frac{\omega_\pi H_4(s) - \mathrm{Im}\{H_4(j\omega_\pi)\}s - \omega_\pi \mathrm{Re}\{H_4(j\omega_\pi)\}}{(s^2 + \omega_\pi^2)} \tag{3-24}$$

Equation (3-23) is a demonstration of the fact that as long as the estimate $H_R(s)$ is of the form shown in Equation (3-22) then it must have the same characteristics with the plant $H_p(s)$ at $s = j\omega_\pi$. Moreover this characteristic is preserved for any value of $H_5(s)$. This suggests that one can manipulate $H_5(s)$ at will without destroying the already built-in characteristics in the model.

$H_5(s)$, a strictly proper transfer function can be written as

$$\frac{1}{H_5(s)} = D_5(s)/N_5(s) = P_5(s) + H_6(s) \tag{3-25}$$

where $P_5(s)$ is the quotient polynomial of $1/H_5(s)$ and $H_6(s) = R_5(s)/N_5(s)$; $R_5(s)$ is the remainder polynomial of $1/H_5(s)$. Defining another polynomial $P_4(s)$ of the same order with $P_3(s)$.

$$P_4(s) = P_3(s) + \frac{\mathrm{Im}\{H_4(j\omega_\pi)\}s}{\omega_\pi} + \mathrm{Re}\{H_4(j\omega_\pi)\}, \tag{3-26}$$

Equation (2-22) becomes

$$H_p(s) = \frac{1}{P_2(s) + \dfrac{s}{P_4(s) + \left\{\dfrac{(s^2 + \omega_\pi^2)}{P_5(s) + H_6(s)}\right\}}} . \qquad (3\text{-}27)$$

By the same argument for which Equation (3-22) was constructed it follows that

$$H_p(s) = \frac{1}{P_2(s) + \left\{\dfrac{s}{P_4(s) + \left\{\dfrac{(s^2 + \omega_\pi^2)}{P_5(s) + \dfrac{\mathrm{Im}\{H_6(j\omega_1)\}}{\omega_1} s + \mathrm{Re}\{H_6(j\omega_1)\} + (s^2 + \omega_1^2)H_a}\right\}}\right\}} \qquad (3\text{-}28)$$

Consequently, for $s = j\omega_1$ this equation becomes

$$H_p(j\omega_1) = \frac{1}{P_2(\omega_1) + \left\{\dfrac{j\omega_1}{P_4(j\omega_1) + \left\{\dfrac{(\omega_\pi^2 - \omega_1^2)}{P_5(j\omega_1) + H_6(j\omega_1)}\right\}}\right\}} \qquad (3\text{-}29)$$

Where ω_1 is the plant's gain crossover frequency. If $H_R(s)$ is of the form of Equation (3-28), therefore, it will have the same gain margin and phase margin with the $H_p(s)$, and will also have the same d.c. steady-state response and similar high frequency responses with $H_p(s)$. By appropriately choosing $H_a(s)$, $H_R(s) \simeq H_p(s)$.

3.4 Other Soft Constraints

The logic by which Equation (3-22) was constructed (from operating on $H_4(s)$) and Equation (3-28) was constructed (from operating on $H_6(s)$) can be used to match the responses of $H_R(s)$ and $H_p(s)$ at any

frequency by manipulating the appropriate transfer function. This reasoning is thus generalized in the following theorem:

Theorem VII

Given any transfer function $H_x(s)$ there exists a transfer function $H_a(s)$ such that $H_x(s) = \frac{\text{Im}\{H_x(j\bar{\omega})\}}{\bar{\omega}} s + \text{Re}\, H_x(j\bar{\omega}) + (s^2 + \bar{\omega}^2)H_a(s)$.

Some of the essential plant performance characteristics are preserved in the model of Equation (3-28). Viz: $H_R(o) = H_p(o)$; $H_R(\infty) = H_p(\infty)$; $H_R(\omega_\pi) = H_p(\omega_\pi)$; and $H_R(\omega_1) = H_p(\omega_1)$. These conditions which must be satisfied if $H_R(s)$ is to be used for design purposes which will be applicable to $H_p(s)$ may be referred to as 'hard' constraints. These conditions are also satisfied by Marshall's method [31]. But to improve the fidelity of the estimate more characteristics of the plant $H_p(s)$ should be included in the estimate $H_R(s)$. This process of fine-tuning may be referred to as 'soft' constraints.

This section discusses the use of Theorem VII for fine-tuning. In Equation (3-28), $H_a(s)$ is arbitrary with respect to satisfying any and all of the four constraints listed above. But for Equation (3-28) to be a true equality,

$$\omega_1 H_a(s) = \frac{\omega_1 H_6(s) - \text{Im}\{H_6(j\omega_1)\}s - \omega_1 \text{Re}\{H_6(j\omega_1)\}}{(s^2 + \omega_1^2)}$$

This suggests a reasonable way of choosing $H_a(s)$. However, if one is interested only in satisfying the hard constraints then $H_a(s) = 0$ or $H_a(s) = 1$ is an obvious choice.

The implication of Theorem VII is that the replacement of $H_a(s)$ by $H_a(j\bar{\omega})$ will result in exact matching of $H_p(s)$ and $H_R(s)$ at the frequency $\bar{\omega}$. And ofcourse to maintain the equality sign of Equation

(3-28) another transfer function $H_a'(s)$ must also be added, where $H_a'(s)$ must satisfy a relation similar to that of Equation (3-24). In this manner one can match the plant and estimate responses at any desired frequency and still have the opportunity to further fine-tune the estimate. Theorem VII can be used to construct the estimate for a specific application. By matching the response characteristics of $H_p(s)$ and $H_R(s)$ at desired frequencies.

3.5 The Algorithm

It will be observed that the preservation of any plant characteristics (or matching the responses of $H_p(s)$ and $H_R(s)$ at any frequency consists essentially of adding the value of the (arbitrary) remainder transfer function at the particular frequency to the already built-in quantities. At any stage a new $H_x(s)$ results which makes the equation a true equality. This $H_x(s)$ is then used to match responses at any other desired frequency.

Also, in the procedure, the transfer function $H_x(s)$ is always expressed as $H_x(s) = 1/[D_x(s)/N_x(s)]$. This is done not only to yield a new remainder for further response matching but also to preserve, in the last analysis, the order difference (d-n) of $H_p(s)$.

A summary of the steps used in obtaining the reduced-order model $H_R(s)$ is as follows.

Given $H_p(s) = N(s)/D(s)$, the purpose is to obtain another transfer transfer function $H_R(s)$ (of lower order than $H_p(s)$) such that

$$H_R(o) = H_P(o) ,$$

$$H_R(\infty) = H_P(\infty),$$

$$H_R(\omega_\pi) = H_P(\omega_\pi),$$

$$H_R(\omega_1) = H_P(\omega_1),$$

and $$H_P(s) \simeq H_P(s).$$

STEP 1: Find $P_1(s)$: the quotient polynomial of $D(s)/N(s)$

STEP 2: Find $H_1(o) = R(o)/N(o)$: $R(s)$ is the remainder polynomial of $D(s)/N(s)$.

STEP 3: Calculate $P_2(s) = P_1(s) + H_1(o)$

STEP 4: Calculate $H_2(s) = H_1(s) - H_1(o)$

STEP 5: Find $H_3(s) = H_2(s)/s = N_3(s)/D_3(s)$

STEP 6: Find $P_3(s)$; the quotient polynomial of $D_3(s)/N_3(s)$

NOTE: $H_4(s) = R_3(s)/N_3(s)$ where $R_3(s)$ is the remainder polynomial of $D_3(s)/N_3(s)$

STEP 7: Find $H_4(j\omega_\pi)$

STEP 8: Calculate

$$P_4(s) = P_3(s) + \{Im\{H_4(j\omega_\pi)\}s\}/\omega_\pi + Re\{H_6(j\omega_\pi)\}$$

STEP 9: Calculate $H_5(s)$ using Equation (3-24), see page 30.

STEP 10: Find $P_5(s)$ the quotient polynomial of $D_5(s)/N_5(s)$

NOTE: $H_6(s) = R_5(s)/N_5(s)$, where $R_5(s)$ is the remainder polynomial of $D_5(s)/N_5(s)$.

STEP 11: Find $H_6(j\omega_1)$

STEP 12: Calculate

$$P_6(s) = P_5(s) + \{Im\{H_6(j\omega_1)\}s\}/\omega_1 + Re\{H_6(j\omega_1)\}$$

STOP: Do you need to fine-tune or match $H_R(s)$ and $H_P(s)$ at any other frequency?

(a) NO: Then go to STEP 13.

(b) YES: Then go to STEP 15.

STEP 13: Calculate $H_R(s)$

$$H_R(s) = \frac{1}{P_2(s) + \left\{\frac{s}{P_4(s) + \left\{\frac{(s^2 + \omega_\pi^2)}{P_6(s) + (s^2 + \omega_1^2)H_a(s)}\right\}}\right\}}$$

NOTE: For (a) $H_a(s)$ is arbitrary.

STEP 14: If (a) STOP.

STEP 15: Calculate $H_a(s)$ using Equation (3-24), see page 30.

STEP 16: If (b) apply Theorem VII on $H_a(s)$ and continue.

An application of this algorithm is illustrated in the example of Section 6.2.

The implementation of this algorithm does not guarantee a stable reduced-order model from a stable plant. The method of making the reduced-order model stable is discussed in Section 5.1.

3.6 The MIMO Case

The above outlined procedure is useful for reducing a strictly proper transfer function. It is complete for a SISO system.

A MIMO system is represented by its transfer function matrix $M_p(s)$ whose elements are the constituent transfer functions: viz the transfer functions which describe the relations between the various outputs and the inputs. The number of rows of $M_p(s)$ corresponds to the number of system outputs while the number of columns corresponds to the number of control inputs. Precisely, $M_p(s)$ is of the form

$$M_p = \begin{bmatrix} H_{11}(s) & H_{12}(s) & \cdots & H_{1m}(s) \\ H_{21}(s) & H_{22}(s) & \cdots & H_{2m}(s) \\ \vdots & & & \vdots \\ H_{n1}(s) & H_{n1}(s) & \cdots & H_{nm}(s) \end{bmatrix} \qquad (3\text{-}30)$$

For an n - input, m - output system. $H_{ij}(s)$ is the transfer function which relates the input $U_i(s)$ to the output $Y_j(s)$.

The procedure of section 3-5 is used to reduce each of the elements of $M_p(s)$. But this might lead to a transfer function matrix $M_R(s)$ whose elements are of lower order than the corresponding elements of $M_p(s)$ but whose state-space realization matrix A_R is of a higher order than the state space realization matrix A_p of $H_p(s)$. This situation may arise if some of the transfer functions in $H_p(s)$ have common factors in their denominators.

This problem is not a serious one for a certain class of transfer function matrices because the state-space realization is not necessary for the determination of their stability conditions which need to be known (and sometimes adjusted) during synthesis. These transfer function matrices are those that are diagonal dominant. It has been shown [46] that using the concept of diagonal dominance of matrices in the field of complex numbers a MIMO system can be treated, to a certain extent, like a set of independent single-input/single-output subsystems or channels. An important consequence of this idea is summarized in Theorem 5.20 in Sinha's book [46].

Theorem:

"If a rational transfer function matrix $G(s) = g_{ij}(s)$, i,j 1, ..., m, is diagonal-dominant for every s on the contour D in the (complex

frequency) s-plane, then the number of encirclements of the origin by the Nyquist plot of det{G(s)} is the sum of the numbers of encirclements by the Nyquist plots of the diagonal elements of G(s)." Where m is the order of G(s). This theorem leads to the following Nyquist criterion for diagonal dominant MIMO systems [46]

$$\sum_{i=1}^{m} Nq_{ii} = -P \tag{3-31}$$

for asymptotically stable closed-loop system. Where P denotes the number of right-half plane poles of G(s) and Nq_{ii} the number of times the diagonal element q_{ii} encircles the (-1+jo) point.

Thus the application of the algorithm of section 3.5 to this type of transfer function matrix can lead to a useful reduced-order matrix provided the diagonal dominance is preserved.

CHAPTER IV

DETERMINATION OF OPTIMAL ORDER

4.1 General

Very little attention has been given to the problem of selecting the order of an estimate which will best represent the original system and still be sufficiently simple. Recently, Mahapatra [32] suggested a criterion for selecting the model order. His method however is applicable only to Davidson's [15] Model Simplification Technique. Since then his method has been modified by Rao et al [33] to improve its applicability to all cases of Davidson's Simplification Technique. In a different correspondence [34] Mahapatra introduced an alternative version of his order selection, another effort which was again augumented by those of Rao et al [35].

Developments of these criteria involve knowledge of the eigenvalues of the original system. Mahapatra suggests that the choice of the optimal order can be made by prespecifying the maximum allowable error. In this thesis, instead of having to know the eigenvalues of $H_R(s)$ a curve fitting approach is used to determine what order of $H_R(s)$ that will retain all the desired characteristics of $H_P(s)$. Also, instead of prespecifying the maximum allowable error, a hypothesis test is used to determine whether any further reduction in order is allowable without incurring an unacceptable level of error.

In comparing the relative merits and demerits of Pade' and Routh approximants, in a recent work [36], Ashoor and Singh discovered that

the mismatch in many of the k time moments of the estimate of order k with those of the original plant results in the model being a poor representation of the initial time response. Similarly, many mismatches of the Markov parameters indicates that the model will have a poor time response. Their work reveals that retaining k-terms (some time moments and some Markov parameters) in some instances resulted in good approximants and in very bad ones in other instances. From this, one draws the inference that to arbitrarily aim at matching k terms may not generally lead to a good approximation. What is obvious is that if many terms (multiple number of k terms) some of which are time moments and others Markov parameters, are matched a good approximation results. However, how many time moments and how many Markov parameters would correspond to the optimum choice for the model for a given transfer function is still to be investigated.

The curve fitting approach used in the establishment of a criterion for selecting a model order in this work is also based on the fact that a good model is one that matches the plant at more points than the other models. However, instead of matching the time moments which are the power series expansion of the transfer function about $s = 0$ and the Markov parameters which are the power series expansion of the transfer function about the point $s = \infty$, the steady-state responses of the actual transfer functions are compared at various frequencies.

One approach is to consider the estimates as assumed models of an unknown process. A test of goodness of fit is then used to established which estimate models it best. Since, however, the models are

already known to be in the form of ratios of polynomials the problem degenerates to that of determining how many terms (regressors) must be included in the model to best describe the process. This problem can therefore be addressed as a multiple regression problem. Consequently to resort to this approach one must first convince himself that a polynomial is a special case of a multiple regression. Once this is established all the assumptions, laws and tests applicable to multiple regression can be employed freely.

4.2 The Polynomial As a Multiple Regression Model

The standard multiple regression equation is of the form:

$$Y = a + bX + cZ + e \tag{4-1}$$

This equation is linear in the variables (X, Y, Z) as well as in the parameters (a, b, c). The variables X and Z are usually independent and Y is said to be regressed on X and Z better known as the regressors.

The process of modeling involves determining the estimates of the parameters, a, b and c. Examining the least squares method of determining $\hat{a}$, $\hat{b}$ and $\hat{c}$, will show that the estimating equations will be linear in the estimates $\hat{a}$, $\hat{b}$, and $\hat{c}$ provided $\hat{a}$, $\hat{b}$ and $\hat{c}$ appear in a linear fashion.

Consider a process that can be best described by a mathematical model of the form

$$Y = a + bQ + cQ^2 + e \tag{4-2}$$

Although this model is nonlinear in the variable Q, it is nevertheless linear in the parameters a, b,c; hence there should be no problem with ordinary least squares.

Equation (4-2) shows that Y is related to only one independent variable Q, but the fit involves regressing Y on two regressors, Q and Q^2. If the various powers of the independent variables of a polynomial are treated as regressors, the polynomial then represents a multiple regression model. When one variable is used to obtain several regressors, as in this instance, one may wonder if multicollinearity becomes a problem. This is not necessarily true as can be seen from the illustration below.

Let the regressors Q and Q^2 of Equation (4-2) be represented by X and Z respectively. Although Z_i and X_i are functionally dependent (i.e. one is the square of the other), they are not linearly dependent (i.e., one is not, say, three times the other). Geometrically, the points (X_i, Z_i) do lie on a curve [37], as shown in Figure 4-1; however, the important point is that they *do not* lie on a line. Thus the problem of complete multicolinearity is avoided. From a mathematical point of view, the physical or economic source of the X_i and Z_i values is irrelevant; just as long as X and Z are linearly independent. Thus, the laws of multiple regression apply to the polynomial of Equation (4-2) as long as the regressors are defined appropriately. Similarly, it can be shown that a transfer function (which is a ratio of polynomials) is a special case of multiple regression models if the regressors are appropriately defined.

4.3 Criterion for Goodness of Fit

Many criteria exist for evaluating goodness of fit. One of the most commonly used in multiple regression is the coefficient of

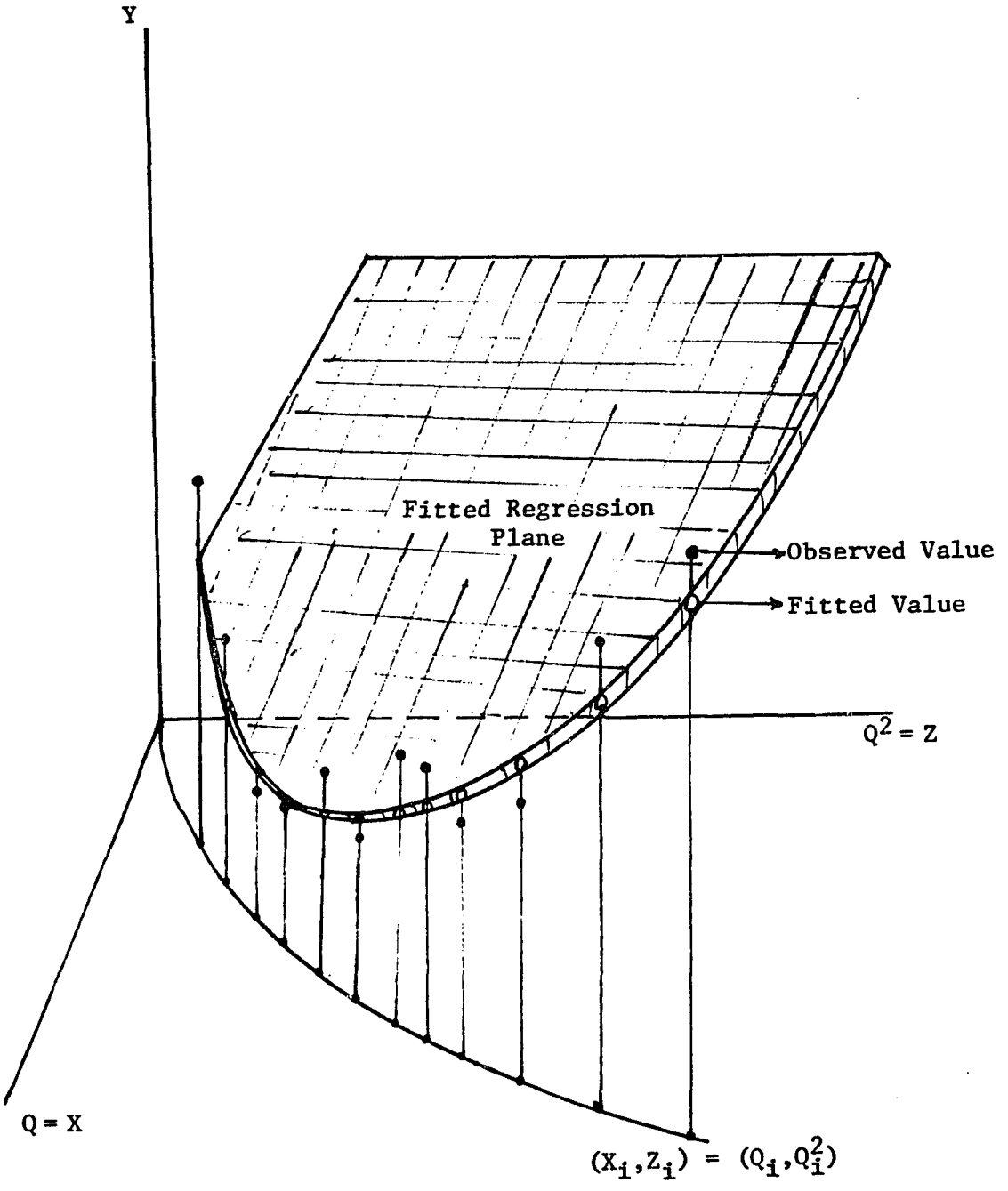

Figure 4-1

Polynomial Regression as a Special Case
of Multiple Regression [37].

determination, R^2. This coefficient is defined by:*

$$R^2 = \frac{\Sigma(\hat{Y}_i - \bar{Y})^2}{\Sigma(Y_i - \bar{Y})^2} = \frac{\text{Variation of Y explained by all regressors}}{\text{total variation of Y}} \tag{4-3}$$

Thus R^2 is seen to provide an overall index of how well Y can be explained by all the regressors. Consequently, since the matter of interest here is how many regressors (i.e. the order of the model) that will best "explain" the behavior of the plant, the problem is that of determining the order K for which R^2 is a maximum. Obviously there is a trivial solution which is, K = d for which $R^2 = 1$ (where d is the order of the plant). This situation is similar to the disapproved! method of Mahapatra [33] in which his criterion suggests that the optimal reduced order is the order of the original plant.

The selection of the optimal order can be done by the process of stepwise regression, given the argument of the last section. One of two criteria can be used to determine when the addition of any more regressors is no more necessary.

(i) Simultaneous hypothesis tests.

(ii) Minimizing the Mean Squared error (MSE).

Both of these methods shall be developed in this chapter using the R^2 criterion for goodness of fit for the later.

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* $0 \leq R^2 \leq 1$

! In a correspondence [32] Mahapatra derived a rule for selecting the order of the estimate. In a note [33] Rao et al have shown that Mahapatra's rule is not general and furthermore leads to the trivial solution of optimal order being the order of the plant.

4-4 Criterion For the Selection of Optimal Order by Minimizing the MSE

Consider a plant whose performance in the frequency domain is described by $H_p(s)$, a ratio of polynomials. Consider also an estimate of $H_p(s)$ which is another ratio of polynomials $H_R(s)$. The degree K of $H_R(s)$ is less than that of $H_p(s)$, d. There is an error, $A_p(s) - A_R(s)$, due to this estimation at every value of s. Where $A_p(s)$ and $A_R(s)$ are the magnitudes of $H_p(s)$ and $H_R(s)$, respectively. This error can be zero, negative, or positive. If several measurements of $(A_p(s) - A_R(s))$ are taken (at various values of s), one would expect that the sum of the errors

$$\Sigma\{A_p(s) - A_R(s)\} = e(s) \tag{4-4}$$

would be a good measure of how well $H_R(s)$ fits $H_p(s)$. But the positive errors might offset the negative errors thus leading to a wrong conclusion if Equation (4-4) is used as a criterion. One way of overcoming this sign problem is by comparing the sum of the squares of the errors

$$E(s) = e^2(s) = \Sigma\{A_p(s) - A_R(s)\}^2 \tag{4-5}$$

rather than the sum of errors. This is the bases of the so called least squared estimate LSE which asserts that to obtain the best estimate one should minimize Equation (4-5).

It is evident that the reliability of the above criterion increases as the number, n, of errors summed in Equation (4-5) increases. Also it is desirable to keep the bias and the variance both low. Thus a more appropriate criterion for selecting the best

estimate is to minimize the mean squared error MSE given by*

$$\text{MSE} = \frac{\{A_p(s) - A_R(s)\}^2}{n - k - 1} \tag{4-6}$$

Thus to determine how many regressors to retain (i.e. the order of the estimate) one might think of stopping the reduction process when Equation (4-6) is a minimum. This also results in the trivial solution which says that K = d is the optimal order.

It will be recalled that R^2, $(0 \leq R^2 \leq 1)$, is a measure of how well the k regressors (the estimate $H_R(s)$) explain (describes) the data (the plant $H_p(s)$). Consequently $1 - R^2$ is a measure of the unexplained variations. The degrees of freedom in the unexplained variation are n - k - 1. It should be expected that a good criterion for goodness of fit is minimizing $1 - R^2$. And for the same argument by which the minimization of the MSE was preferred to LSE, this criterion is replaced by a superior one namely:

$$\text{minimize} \quad \frac{1 - R^2}{(n - k - 1)^2} \tag{4-7}$$

It has been shown by Wonnacott and Wonnacott [37] that the criterion of expression (4-7) yields approximately the same result as minimizing the mean squared error MSE.

From Equation (4-3) the coefficient of indetermination $1 - R^2$ is given by:

$$1 - R^2 = 1 - \frac{\Sigma(\hat{Y}_i - \bar{Y})^2}{\Sigma(Y - \bar{Y})^2} = \frac{\Sigma(Y_i - \bar{Y})^2 - \Sigma(\hat{Y}_i - \bar{Y})^2}{\Sigma(Y - \bar{Y})^2}. \tag{4-8}$$

*The MSE is an unbiased estimate of the variance.

But

$$\Sigma(Y_i - \bar{Y})^2 = \Sigma(\hat{Y}_i - \bar{Y})^2 + \Sigma(Y_i - \hat{Y}_i)^2 . \qquad (4\text{-}9)$$

Therefore

$$1 - R^2 = \frac{\Sigma(Y_i - \hat{Y}_i)^2}{\Sigma(Y_i - \bar{Y})^2} . \qquad (4\text{-}10)$$

Equation (4-10) shows that $1 - R^2$ is proportional to the residual variation $\Sigma(Y_i - \hat{Y}_i)^2$. The criterion of expression (4-7) may therefore be rewritten as:

$$\text{minimize} \quad \frac{\Sigma(Y_i - \hat{Y}_i)^2}{(n - k - 1)^2} \qquad (4\text{-}11)$$

Thus to obtain the optimal order, letting $Y_i = A_p(s)_i$ and $\hat{Y}_i = A_R(s)_i$ the rule is: Stop reducing the order of model when

$$\frac{\Sigma\{A_p(s)_i - A_R(s)_i\}^2}{(n - k - 1)^2} \; ; \; k < d \qquad (4\text{-}12)$$

is a minimum, where k is the other of $H_R(s)$ and n is the number of frequencies for which the variance is obtained.

4.5 Criterion for the Selection of Model Order Using Simultaneous Hypothesis Tests

Suppose one assumes that the error due to estimating increases as the estimate order decreases. That is, suppose the reduction process is such that more information about the system is lost the more one tries to approximate it with a smaller system. Then the philosophy might be to stop reduction when the error due to the reduction process becomes statistically discernible (significant). The question that arises then is "discernible at what level?".

This is a question of engineering judgement. How much error that is tolerable depends on the application, and the purpose of the engineer; otherwise one might choose to fall back on the customary level of 5%.

Consider a transfer function $H_p(s)$, of order d, and two estimates of $H_p(s)$, $H_{Rq}(s)$ and $H_{Rr}(s)$. Let the orders of $H_{Rq}(s)$ and $H_{Rr}(s)$ be q and r, respectively, where $d > q > r$. Suppose the error due to the estimate $H_{Rq}(s)$ at a particular frequency is $e_q(s)$ given by

$$e_q(s)_i = |A_p(s)_i - A_{Rq}(s)_i| \tag{4-13}$$

and the error due to the estimate $H_{R2}(s)$ at that particular frequency is $e_r(s)_i$ given by

$$e_r(s)_i = |A_p(s)_i - A_{Rr}(s)_i| \tag{4-14}$$

The increase in error due to the reduction from model of order q to model of order r is

$$\Delta e_i = e_r(s)_i - e_q(s)_i \tag{4-15}$$

Thus a good test of the significance in reduction will be one that compares the sum of the error increases $\Sigma\Delta e_i$ (or a function of it) at all n frequencies where the errors were measured to some standard value. The chi-squared goodness of fit test has been shown to be appropriate for this purpose. Kendal and Stuart [47], 1985, have shown that the function

$$t = \sum_{i=1}^{n} \left\{ \frac{(e_r(s)_i - e_q(s)_i)^2}{e_q(s)_i} \right\} = \sum_{i=1}^{n} \left\{ \frac{(\Delta e_i)^2}{e_q(s)_i} \right\} \tag{4-16}$$

is a chi-squared distribution with n-1 degrees of freedom. It may be observed that the larger the increase in error gets the larger t is.

Consider then the null hypothesis H_o:

There is no significant increase in error due to the reduction in order from q to r.

H_o is tested against the alternative hypothesis H_A:

There is a significant increase in error due to the order reduction from q to r.

The null hypothesis may be rejected if $t \geq \chi^2_{\alpha,n-1}$ in which case the model $H_{Rr}(s)$ of order r is compared to the next lower order model (of order r-1).

If the null hypothesis is false then the model of order q should not be reduced any further. The decision rule may therefore be stated as: stop reduction when t of Equation (4-16) satisfies

$$t \geq \chi^2_{\alpha,n-1} \cdot \qquad (4\text{-}17)$$

Where α is the selected level of confidence to determine the discernibility of the increase in error, and n is the number of measurements.

However, it is not always true that lower-order estimates produce more mismatch errors than higher-order estimates. In fact some lower-order estimates may preserve more information about the plant than higher-order ones. This might be due to the fact that the lower-order model might consist of the 'right' set of regressors which models the plant best. It is therefore desirable to generalize the hypothesis test of the last paragraph so that it can be used to compare any two estimates as to determine the better one. This can be done by redefining the null hypothesis and rewriting the decision rule, bearing in mind that a lower-order estimate which has statistically equal error as a higher-order estimate is preferred to the higher-order one.

Consider, then, the null hypothesis H_o:

There is no significant difference between the error due to $H_{Rq}(s)$ and error due to $H_{Rr}(s)$.

If H_o is true (under the condition prescribed before) then the lower-order model $H_{Rr}(s)$ is considered better. If H_o is false, then the following rule is used to select the superior model:

If $\Sigma(e_r(s)_i - e_q(s)_i) < 0$; $H_{Rr}(s)$ is superior.

If $\Sigma(e_r(s)_i - e_q(s)_i) > 0$; $H_{Rq}(s)$ is superior.

The decision rule for selecting the best-order model using the hypothesis test can be summarized as follows:

For $t < \chi^2_{\alpha,n-1}$ select the lower-order model $H_{Rr}(s)$.

For $t \geq \chi^2_{\alpha,n-1}$ select the higher-order model $H_{Rq}(s)$ if $\Sigma(e_r(s)_i - e_q(s)_i) > 0$;

select the lower-order model $H_{Rr}(s)$ if $\Sigma(e_r(s)_i - e_q(s)_i) < 0$.

The selected model is then compared with another model until the 'best' model is obtained. This procedure offers an alternative to the one developed in Section 4-4.

CHAPTER V

ANALYSIS OF PROCEDURE

5.1 General

The algorithm developed in Chapter III is applicable to a large class of transfer functions. These transfer functions must have a certain structure to allow a complete implementation of the algorithm. However this structure is possessed by most practical systems thus it may be applied to a good number of practical situations.

This chapter analyzes the algorithm-pointing out its limitations, usefulness, and those properties that the transfer function must have to ensure complete implementation. Though the algorithm developed for the reduction of a transfer function (SISO system), in section 3.6 its limited application to MIMO systems was discussed. In this chapter the problems that might be encountered in its application to MIMO systems that are not diagonal dominant will be discussed.

Another important point is that nothing in the development of the algorithm guarantees that a stable estimate will be obtained if the plant exhibits that property. In the next section a way of obtaining a stable estimate is discussed, and this forms part of the whole reduction process. Lastly, the limit to the number of times Theorem VII (Chapter III) can be applied in the reduction of a given transfer function is discussed.

5.2 Obtaining a Stable Estimate

One of the requirements of Theorem I (Section 2.2) is that the plant and its approximant have the same number of poles in the right-half-plane. Generally, the algorithm of section 3.5 does not guarantee this condition. Nor is it certain that the reduced model will be stable if the plant is. However the algorithm allows one a good deal of freedom to make the necessary adjustments. The use of this freedom to obtain a stable estimate is the subject of this section.

Consider the transfer function of Equation (3-28) given by

$$H_p(s) = \cfrac{1}{P_2(s) + \cfrac{s}{P_4(s) + \cfrac{(s^2 + \omega_\pi^2)}{(P_6(s) + (s^2 + \omega_1^2)H_a(s)}}} \qquad (5\text{-}1)$$

where $P_6(s)$ is as shown in Step 12 of the algorithm.

$$P_6(s) = P_5(s) + \text{Im}\{H_6(j\omega_1)\}\, s/\omega_1 + \text{Re}\{H_6(j\omega_1)\} \qquad (5\text{-}2)$$

From Equation (5-1),

$$H_p(s) = \frac{P_4(s)\,P_6(s) + (s^2 + \omega_1^2)H_a(s) \quad + s^2 + \omega_\pi^2}{\{P_2(s)P_4(s)\,P_6(s) + (s^2 + \omega_1^2)H_a(s) + s\,P_6(s) + (s^2 + \omega_1^2)H_a(s) + P_2(s)(s^2 + \omega_\pi^2)\}.} \qquad (5\text{-}3)$$

The development of the algorithm ensures that for any choice of $H_a(s)$, an estimate $H_R(s)$ satisfies all the hard constraints.

At the end of the algorithm, i.e. after matching $H_p(s)$ and $H_R(s)$ at desired frequencies, a good initial choice for $H_a(s)$ is $H_a(s) = 0$. In this case Equation (5-3) will yield

$$H_R(s) = \frac{P_4(s)P_6(s) + s^2 + \omega_\pi^2}{P_2(s)P_4(s)P_6(s) + sP_6(s) + P_2(s)(s^2 + \omega_\pi^2)} \quad (5\text{-}4)$$

Suppose the estimate of Equation (5-4) is unstable. That is, suppose the equation

$$P_2(s)P_4(s)P_6(s) + sP_6(s) + P_2(s)(s^2 + \omega_\pi^2) = 0 \quad (5\text{-}5)$$

has positive roots. Then $H_a(s)$ in Equation (5-3) may be chosen to be

$$H_a(s) = K(s) \ . \quad (5\text{-}6)$$

Provided $H_a(s)$ has no poles at $\pm j\omega_1$. The resulting estimate from Equation (5-3) is

$$H_R(s) = \frac{P_4(s)P_6(s) + k(s)_4(s) + s^2 + \omega_\pi^2}{\{P_2(s)P_4(s)P_6(s) + k(s)_2(s)P_4(s) + P_2(s)(s^2 + \omega_\pi^2) + sP_6(s) + k(s)_2(s)\}} \quad (5\text{-}7)$$

To obtain a stable estimate, the Routh-Hurwitz criterion is used to choose what value of K stabilizes $H_R(s)$ of Equation (5-7). Similarly from the Routh array of the denominator polynomial

$$D_R(s) = P_2(s)P_4(s)P_6(s) + k(s)_2(s)P_4(s) + P_2(s)(s^2 + \omega_\pi^2) + SP_6(s) + k(s)_2(s) \quad (5\text{-}8)$$

the values of k(s) which makes any number of elements on the first column negative can be determined. Thus the estimate $H_R(s)$ of Equation (5-7) is in the general form. It will be noticed that since $H_a(s)$ is arbitrary then for any value of k(s) $H_R(s)$ will still retain the preserved characteristics of $H_p(s)$. For convenience k(s) may initially be chosen to be a constant k. If no constant k can stabilize the system then simple functions such as k(s+a) are tried until values of k, and a which stabilize the system are obtained.

In the development of Theorem I a stable system (Z = 0) was assumed. The reason is because most practical systems are designed to be stable. In fact [2] many practical systems are in the minimum phase category. However, for an unstable plant, $H_p(s)$, if it is required to obtain an estimate $H_R(s)$ with the same number of RHP zeros in its characteristic equation as $H_p(s)$ has, the outlined procedure is used to find what values of k(s) which will result in the required number of sign changes in the Routh array of the polynomial $N_R(s) + D_R(s)$. Where $N_R(s)$ is the numerator polynomial of $H_R(s)$ in Equation (5-7).

5.3 Minimum-Order Estimate

Some order reduction techniques [44] are capable of producing approximants of order two. The minimum order estimate obtainable from the algorithm developed in this thesis is a third-order estimate.

Consider the general form of the reduced-order model

$$H_R(s) = \frac{P_4(s)P_6(s) + KP_4(s) + s^2 + \omega_\pi^2}{\{P_2(s)P_4(s)P_6(s) + KP_2(s)P_4(s) + P_2(s)(s^2 + \omega_\pi^2) + sP_6(s) + KP_2(s)\}} \quad . \tag{5-7}$$

Since $H_p(s)$ is strictly proper, $P_2(s)$ must be a function of s, a polynomial with at least one term having a power of s greater than zero. Precisely the degree of $P_2(s)$ is equal to d - n; d > n. Therefore the term $P_2(s)(s^2 + \omega_\pi^2)$ in $H_R(s)$ is of the degree r, where r satisfies

$$r = 2 + d - n \tag{5-8}$$

and thus $r \geq 3$.

5.4 Minimum Required Order of the Plant's Numerator Polynomial

In order to implement all the essential steps of the algorithm it is necessary to have polynomials of sufficiently high order to allow for the calculations of the polynomials $P_2(s)$, $P_4(s)$ and $P_6(s)$ which ensure that the approximant satisfies the hard constraints. Each of these polynomials is obtained in part by a long division process which must result in a remainder polynomial which in turn becomes the divisor of the next stage of the continued fraction process and the former divisor then becomes the dividend. The present stage divisor must then be of sufficient degree so that a remainder polynomial of sufficient order will be produced to continue the process. The divisor for the first stage division is N(s) - the numerator polynomial of the plant. N(s) must then be of sufficiently high degree to meet this requirement.

Consider again the transfer function

$$H_p(s) = \frac{1}{D(s)/N(s)} \tag{3-1}$$

Following the process by which the algorithm was developed in Chapter III, Equation (3-9) can be written as

$$H_p(s) = \frac{1}{P_2(s) + \dfrac{s}{D_3(s)/N_3(s)}} \tag{5-10}$$

where the degree of $N_3(s)$ is 1 less than that of the polynomial N(s). The transfer function $1/\ D_3(s)/N_3(s)$ is given by

$$\frac{N_3(s)}{D_3(s)} = P_4(s) + (s^2 + \omega_\pi^2)\frac{N_5(s)}{D_5(s)} \tag{5-11}$$

where

$$\frac{N_5(s)}{D_5(s)} = \left\{ \frac{\dfrac{R_3(s)}{N_3(s)} - \mathrm{Im}\left\{\dfrac{R_3(j\omega_\pi)}{N_3(j\omega_\pi)}\right\}\dfrac{s}{\omega_\pi} - \mathrm{Re}\left\{\dfrac{R_3(j\omega_\pi)}{N_3(j\omega_\pi)}\right\}}{s^2 + \omega_\pi^2} \right\} \tag{5-12}$$

Equation (5-12) is of the form

$$\frac{(s^2 + \omega_\pi^2)N_5(s)}{D_5(s)} = \frac{R_3(s)}{N_3(s)} - (as + b) \tag{5-13}$$

Since $R_3(s)$ is the remainder polynomial of $D_3(s)/N_3(s)$, the degree of $R_3(s)$ must be less than the degree of $N_3(s)$.

For the process to continue (to calculate $P_6(s)$) $N_5(s)/D_5(s)$ must be strictly proper. This condition is inherent as shown below. From Equation (5-13) one gets

$$(s^2 + \omega_\pi^2)N_5(s)N_3(s) = R_3(s)D_5(s) - (as + b)D_5(s)N_3(s) \tag{5-14}.$$

Since the degree of $R_3(s)$ is less than that of $N_3(s)$, Equation (5-14) can only be true if

$$\text{degree } \{D_5(s)\} - \text{degree } \{N_5(s)\} = 1 \tag{5-15}$$

Thus $N_5(s)/D_5(s)$ is strictly proper.

Now suppose $R_3(s)$ is zero, that is $D_3(s)/N_3(s)$ has no remainder then the process breaks down since $N_5(s)/D_5(s)$ will be zero, in Equation (5-13). Suppose again that $R_3(s)$ is a constant say c. Then $N_3(s)$ has the possibility of being in the form

$$N_3(s) = ds + e \tag{5-16}$$

and Equation (5-13) becomes

$$\frac{(s^2 + \omega_\pi^2)N_5(s)}{D_5(s)} = \frac{c - (as + b)(ds + e)}{ds + e} \tag{5-17}$$

or

$$(s^2 + \omega_\pi^2)(ds + e)N_5(s) = cD_5(s) - (as + b)(ds + e)D_5(s) \qquad (5\text{-}17)$$

where a, b, c, d and e are constants. Thus $N_3(s)$ must be at least a first degree polynomial. But the degree of N(s) the numerator polynomial of the plant satisfies

$$\text{degree } \{N(s)\} - \text{degree } \{N_3(s)\} = 2 \qquad (5\text{-}18)$$

Also once the polynomials $P_2(s)$, $P_4(s)$ and $P_6(s)$ are included in the construction of the estimate, the estimate will satisfy all the hard constraints.

It can therefore be concluded that in order to obtain an estimate which satisfies all the hard constraints the plants transfer function must be strictly proper, must be of order 3 or higher and the numerator polynomial must satisfy[!]

$$\text{degree } \{N(s)\} \geq 3 \qquad (5\text{-}19)$$

5.5 Maximum Number of Points That Can Be Matched

It was asserted that Theorem VII can be used to get exact matching of the response characteristics of $H_p(s)$ and $H_R(s)$ at any frequency $\bar{\omega}$. This theorem is necessarily used to preserve gain margin and phase margin. How many more times it can be used to achieve exact matching at other frequency points depends on the order of the plant because the order of the estimate thereby constructed increases as the number of points at which $H_p(s)$ and $H_R(s)$ are matched increases.

[!] degree {.} denotes the degree of the polynomial.

The reduction process yields an expression, for the transfer function, of the form

$$H_p(s) = \cfrac{1}{P_2(s) + \cfrac{s}{P_4(s) + \cfrac{s^2 + \omega_\pi^2}{P_6(s) + \cfrac{s^2 + \omega_1^2}{P_8(s) + \cfrac{s^2 + \omega_2^2}{P_{10}(s) + \cfrac{s^2 + \omega_3^2}{\ddots \; P_x(s) + (s^2 + \omega_y^2) H_a'(s)}}}}}} \qquad (5\text{-}20)$$

Where ω_π is the phase crossover frequency, ω_1 = the gain crossover frequency and ω_2, ω_3 ... ω_{y-1} are the particular frequencies at which it is chosen to match the response characteristics of $H_p(s)$ and $H_R(s)$ using Theorem VII. $H_a'(s)$ is arbitrary and as was pointed out in section 5.2 is always chosen to be zero initially. Setting $H_a'(s) = 0$, Equation (5-20) yields

$$H_p(s) = \cfrac{1}{P_2(s) + \cfrac{s}{P_4(s) + \cfrac{s^2 + \omega_\pi^2}{P_6(s) + \cfrac{s^2 + \omega_1^2}{P_8(s) + \cfrac{s^2 + \omega_2^2}{P_{10}(s) + \cfrac{s^2 + \omega_3^2}{\ddots \; P_x(s)}}}}}} \; . \qquad (5\text{-}21)$$

Consider an estimate construction process in which the responses of $H_p(s)$ and $H_R(s)$ is to be matched at only two frequency points, ω_π and ω_1. The estimate obtained from Equation (5-21) is of the form

$$H_R(s) = \frac{P_4(s)P_6(s) + s^2 + \omega_\pi^2}{P_2(s)P_4(s)P_6(s) + P_2(s)(s^2 + \omega_\pi^2) + sP_6(s)} \tag{5-22}$$

If three frequency points ω_π, ω_1 and ω_2 were used the estimate obtained will be of the form

$$H_R(s) = \frac{P_4(s)P_6(s)P_8(s) + P_4(s)(s^2 + \omega_1^2) + P_8(s)(s^2 + \omega_\pi^2)}{\{P_2(s)P_4(s)P_6(s)P_8(s) + P_2(s)P_4(s)(s^2 + \omega_1^2) + P_2(s)P_8(s)(s^2 + \omega_\pi^2) + sP_6(s)P_8(s) + s(s^2 + \omega_1^2)\}} \tag{5-23}$$

Matching responses at four frequency points ω_π, ω_1, ω_2 and ω_3 the estimate obtained, from Equation (5-21), is

$$H_R(s) = \frac{\begin{aligned}&\{P_4(s)P_6(s)P_8(s)P_{10}(s) + P_4(s)P_6(s)(s^2 + \omega_2^2) \\ &\quad + P_4(s)P_{10}(s)(s^2 + \omega_1^2) + P_4(s)P_6(s)(s^2 + \omega_\pi^2) \\ &\quad + (s^2 + \omega_\pi^2)(s^2 + \omega_2^2)\}\end{aligned}}{\begin{aligned}&\{P_2(s)P_4(s)P_6(s)P_8(s)P_{10}(s) + P_2(s)P_4(s)P_6(s)(s^2 + \omega_2^2) \\ &\quad + P_2(s)P_4(s)P_{10}(s^2 + \omega_1^2) + P_2(s)P_4(s)P_6(s)(s^2 + \omega_\pi^2) \\ &\quad + P_2(s)(s^2 + \omega_\pi^2)(s^2 + \omega_2^2) + sP_2(s)P_8(s)P_{10}(s) \\ &\quad + sP_6(s)(s^2 + \omega_2^2) + sP_{10}(s)(s^2 + \omega_1^2)\}\end{aligned}} \tag{5-24}$$

Similarly, by using five frequency points ω_π, ω_1, ω_2, ω_3 and ω_4 one obtains the estimate given by

$$H_R(s) = \frac{\begin{aligned}&\{P_4(s)P_6(s)P_8(s)P_{10}(s)P_{12}(s) + P_4(s)P_6(s)P_8(s)(s^2+\omega_3^2)\\&+ P_4(s)P_6(s)P_{12}(s)(s^2+\omega_2^2) + P_4(s)P_{10}(s)P_{12}(s)(s^2+\omega_1^2)\\&+ P_4(s)(s^2+\omega_3^2)(s^2+\omega_1^2) + P_8(s)P_{10}(s)P_{12}(s)(s^2+\omega_\pi^2)\\&+ P_8(s)(s^2+\omega_3^2)(s^2+\omega_\pi^2)\\&+ P_{12}(s)(s^2+\omega_2^2)(s^2+\omega_\pi^2)\}\end{aligned}}{\begin{aligned}&\{P_2(s)P_4(s)P_6(s)P_8(s)P_{10}(s)P_{12}(s)\\&+ P_2(s)P_4(s)P_6(s)P_{12}(s)(s^2+\omega_2^2) + P_2(s)P_4(s)P_6(s)P_8(s)(s^2+\omega_3^2)\\&+ P_2(s)P_4(s)P_{10}(s)P_{12}(s)(s^2+\omega_1^2) + P_2(s)P_4(s)(s^2+\omega_3^2)(s^2+\omega_1^2)\\&+ P_2(s)P_8(s)P_{10}(s)P_{12}(s)(s^2+\omega_\pi^2) + P_2(s)P_8(s)(s^2+\omega_3^2)(s^2+\omega_\pi^2)\\&+ P_2(s)P_{12}(s)(s^2+\omega_2^2)(s^2+\omega_\pi^2) + sP_6(s)P_8(s)P_{10}(s)P_{12}(s)\\&+ sP_6(s)P_8(s)(s^2+\omega_3^2) + sP_6(s)P_{12}(s)(s^2+\omega_2^2)\\&+ sP_{10}(s)P_{12}(s)(s^2+\omega_1^2) + s(s^2+\omega_3^2)(s^2+\omega_1^2)\}\end{aligned}} \quad (5\text{-}25)$$

The polynomial $P_1(s)$ is the quotient of $D(s)/N(s)$ and thus is of degree $d - n$. The polynomial $P_2(s)$ is obtained by adding a constant to $P_1(s)$ thus

$$\text{degree } \{P_2(s)\} = \text{degree } \{P_1(s)\} = d - n \quad (5\text{-}26)$$

The polynomial $P_4(s)$ is obtained by adding a first degree polynomial to $P_3(s)$, the quotient of $D_3(s)/N_3(s)$. Since degree $\{D_3(s)\}$ - degree $\{N_3(s)\} = 1$, the degree $\{P_3(s)\} = 1$ and hence degree $\{P_4(s)\} = 1$. It was shown in section 5.4 (see Equation (5-15)) that the degree of $P_5(s)$, the quotient of $D_5(s)/N_5(s) = 1$, thus the degree of $P_6(s)$ is also unity. In the same manner it can be shown that all the polynomials $P_x(s)$, $x > 2$, appearing in the expressions for the reduced-order transfer function are first degree polynomials.

Examining the transfer functions of Equation (5-22) through (5-25) it will be observed that the order of any of the transfer functions is given by the degree of the term in the denominator which is a product of all the P's. Consequently, the order of the T.F. of Equation (5-22) is 2 + d - n; the order of the T.F. of Equation (5-23) is 3 + d - n; the order of the T.F. of Equation (5-24) is 4 + d - n; and the order of the T.F. of Equation (5-25) is 5 + d - n. Thus, in general, the lowest order model that can be constructed when the response characteristics of plant and estimate are matched at y frequency points has an order r given by

$$r = y + d - n \ . \tag{5-27}$$

Since the purpose of the whole exercise is to obtain a lower-order model the number of frequency points y to be used must satisfy the relation

$$y = r + n - d; \ y \leq d \tag{5-28}$$

where,

$$d - r > 0, \tag{5-29}$$

r is the desired order of estimate.

5.6 Application to MIMO Systems

In section 3.6 it was poined out that the application of the algorithm to the individual transfer function of the transfer function matrix $M_p(s)$, may result in a transfer function Matrix $M_R(s)$ whose elements have lower order than the corresponding elements of $M_p(s)$ but whose state-space realization matrix A_R is of higher order than the state-space realization Matrix A_p of $H_p(s)$. Since this method of order-reduction does not require the knowledge of the poles of the

transfer functions, it is not possible to find the monic lowest common multiple of the denominator of the constituent transfer functions of $M_p(s)$, so that this could be used as the common denominator of the elements of $M_p(s)$ to circumvent the problem.

The application of this method to MIMO system is therefore restricted to diagonal dominant systems, as was explained in Chapter III, for practical purposes. The method will preserve the diagonal dominance if all the elements of $H_p(s)$ are reduced by the same degree.

5.7 Other Considerations

In section 5.3 it is shown that the minimum-order estimate obtainable using this technique is an estimate of the third-order. It is also discovered in section 5.4 that the technique is applicable only to transfer function whose numerator is a third degree polynomial or higher. Given that the transfer function must be strictly proper, it was then concluded that the technique is applicable to fourth-order transfer functions. Also, considering the fact that a third-order system cannot be reduced to any lower-order system using this technique one concludes that this technique is useful for reducing transfer functions of the form

$$H_p(s) = \frac{N(s)}{D(s)}$$

where $D \in R^d$, $N \in R^n$;

$$n \geq 3,$$
$$d \geq 4.$$

This limitation cannot in any way be considered serious because it is almost trivial to reduce a third-order system as many examples in the

literature did. The purpose of order reduction is to reduce the order of a highly complex system. A third order system can hardly be considered highly complex. Thus this technique is applicable to a large class of systems that require approximation for easy studies.

Also, the majority of practical systems are of the minimum phase category. Thus the requirement of Theorem I that the system and estimate have the same number of poles with positive real part will often moderate to "the estimate has no poles on the right-half-plane". This makes the solution for K in the Routh array as discussed in section 5.2 even easier. Hence this technique may find a wide application in practical situations.

The criterion of Equation (5-27) will yield a large y, the number of frequency points that can be matched, if d the order of the system is high. In this case one has a good deal of freedom to construct an estimate which will preserve the systems characteristic response in a desired frequency range, according as the intended application.

The claim that the algorithm preserves the order difference d - n of the plant may be justified by examining the reduced-order transfer functions of Equations (5-22) through (5-25). It will be observed that the degree of the numerator of each of those transfer functions is equal to the degree of the first terms. That is, the degree of the numerator of the T.F. of Equation (5-22) is

$$\text{degree } \{H_R(s)\} = \text{degree } \{P_4(s)P_6(s)\} = 2 \qquad (5\text{-}30)$$

For the T.F. of Equation (5-23);

$$\text{degree } \{H_R(s)\} = \text{degree } \{P_4(s)P_6(s)P_8(s)\} = 3 \qquad (5\text{-}31)$$

For the T.F. of Equation (5-24);

$$\text{degree } \{H_R(s)\} = \text{degree } \{P_4(s)P_6(s)P_8(s)P_{10}(s)\} = 4 \qquad (5\text{-}32)$$

and the numerator of the T.F. of Equation (5-25) is

$$\text{degree } H_R(s) = \text{degree } P_4(s)P_6(s)P_8(s)P_{10}(s)P_{12}(s) = 5. \qquad (5\text{-}33)$$

In each case the difference $(d_R - n_R)$ between the denominator degree and numerator degree is the degree of $P_2(s)$ which is equal to $d - n$.

Finally, it should be noted that if an unstable estimate results from the algorithm, the application of the method of Section 5-2 to obtain a stable one produces an estimate of order higher than the unstable one. This new estimate, if desired, may then be reduced further.

CHAPTER VI

ILLUSTRATIONS AND EVALUATION OF PROCEDURE

6.1 General

The work presented in this dissertation may result in two main contributions, namely: (1) an alternative way of obtaining an estimate of a high-order transfer function which is much simpler than many of the existing methods; (2) mathematical criteria for selecting the optimal order of the estimate. The purpose of this chapter is to evaluate the usefulness and adequateness of these contributions. This is done both by comparing results from the present work to those obtained using other well-known methods, and by using examples to illustrate the usefulness of the new procedure.

To evaluate the usefulness of the new model reduction technique, reduced-order models obtained from this new technique are compared with those obtained using some of the numerous existing techniques, with the original, high-order, model as a reference. Further validation is done by the use of the estimate obtained via this new algorithm to design a controller for the high-order plant. It is hoped that these illustrations shall show some evidence of the advantages of this method over many others in the existing literature.

Not much interest has been shown in the problem of optimal order selection. An extensive survey of the existing literature reveals that the number of suggestions in this matter [32], [33], [34], [35], [41], [42] is negligible compared to the contributions in model order

reduction. In the literature criteria for selecting the low-order model are given and examples are used to show the computational processes and their results. But the resultant low-order models are not shown as to compare their responses to those of the original plants as to justify the claims. In this chapter the optimal-order model response is compared to that of the plant along side with those of some of the 'non-optimal-order' reduced models as to justify its claims to optimality.

6.2 Order Reduction Examples.

An example which shows the calculations involved in the construction of the reduced-order model is given in this section. This example will follow the steps of the algorithm and further is extended to include the case of 'fine-tuning'. The example selected has been used by other authors [17, 31] as to compare the results.

Consider a system represented by the transfer function

$$H_p(s) = N(s)/D(s)$$

$$= \frac{\{s^5 + 6.93195s^4 - 4.8805413s^3 - 0.9768258s^2 + 0.18728s + 0.0014062\}}{\{s^6 + 3.61115s^5 + 2.1117625s^4 + 0.4161319s^3 + 0.0256456s^2 + 0.0001788s\}} \quad (6\text{-}1)$$

which is strictly proper.

$$H_p(s) = \left\{\frac{D(s)}{N(s)}\right\}^{-1} = \frac{1}{P_1(s) + H_1(s)} \quad (6\text{-}2)$$

where

$$P_1(s) = s - 3.3208, \text{ and } H_1(s) = R_1(s)/N(s) \quad (6\text{-}3)$$

$$R_1 = 30.011924s^4 - 14.814344s^3 - 3.1632087s^2 + 0.6231468s + 0.0046697 \quad (6\text{-}4)$$

$$N(s) = s^5 + 6.93195s^4 - 4.8805413s^3 - 0.9768258s^2 + 0.18728s + 0.0014062. \quad (6\text{-}5)$$

$$H_1(0) = \frac{0.0046697}{0.0014062} = 3.3208 \quad (6\text{-}6)$$

Consequently,

$$H_2(s) = H_1(s) - H_1(0) = \frac{\{-3.3208s^5 + 6.9923038s^4 + 1.3929577s^3 - 0.1616344s^2 - 0.0012274s\}}{\{s^5 + 6.93195s^4 - 4.8805413s^3 - 0.9768258s^2 + 0.18728s + 0.0014062\}} \quad (6\text{-}7)$$

and

$$H_3(s) = H_2(s)/s = \frac{\{-3.3208s^4 + 6.9923038s^3 + 1.3929577s^2 - 0.1616344s - 0.0012274\}}{\{s^5 + 6.93195s^4 - 4.8805413s^3 - 0.9768258s^2 + 0.18728s + 0.0014062\}} \quad (6\text{-}8)$$

$$P_2(s) = P_1(s) + H_1(0) = s \quad (6\text{-}9)$$

$$1/H_3(s) = D_3(s)/N_3(s) = \frac{\{s^5 + 6.93195s^4 - 4.8805413s^3 - 0.9768258s^2 + 0.18728s + 0.0014062\}}{\{-3.3208s^4 + 6.9923038s^3 + 1.3929577s^2 - 0.1616344s - 0.0012274\}} \quad (6\text{-}10)$$

$P_3(s)$ the quotient of $1/H_3(s)$ is

$$P_3(s) = -0.3011323s - 2.7215003 \quad (6\text{-}11)$$

$$H_4(s) = R_3(s)/N_3(s) = \frac{14.56848s^3 + 2.7654357s^2 - 0.2529777s - 0.0019342}{\{-3.3208s^4 + 6.9923038s^3 + 1.3929577s^2 - 0.1616344s - 0.0012274\}} \quad (6\text{-}12)$$

Where,

$$R_3(s) = 14.56848s^3 + 2.7654357s^2 - 0.252977s - 0.0019342 \quad (6\text{-}13)$$

is the remainder polynomial of $1/H_3(s)$, and

$$N_3(s) = -3.3208s^4 + 6.9923038s^3 + 1.3929577s^2 - 0.1616344s - 0.0012274. \quad (6\text{-}14)$$

The plant's ($H_p(s)$) phase crossover frequency ω_π is

$$\omega_\pi = 0.08 \text{ rad/sec} \quad (6\text{-}15)$$

Thus, for $s = j\omega_\pi$,

$$H_4(j\omega_\pi) = \frac{0.019633 + j0.0276973}{0.0102783 + j0.0165109} = 1.742268 - j0.104415 \quad (6\text{-}16)$$

Therefore,

$$\text{Im}\{H_4(j\omega_\pi)\}s = -1.3051875s \quad (6\text{-}17)$$

ω_π

and the real part of $H_4(j\omega_\pi)$ is $\text{Re}\{H_4(j\omega_\pi)\} = 1.742268.$ (6-18)

Consequently, putting the value of $P_3(s)$ from Equation (6-11) in

$$P_4(s) = P_3(s) + \frac{\text{Im}\{H_4(j\omega_\pi)\}s}{\omega_\pi} + \text{Re}\{H_4(j\omega_\pi)\} \quad (3\text{-}26)$$

one gets

$$P_4(s) = -1.6063136s - 0.9792323 \quad (6\text{-}19).$$

The next step (Step 9 of the algorithm) is to calculate $H_5(s)$ using Equation (3-24). From Equation (6-12)

$$\omega_\pi H_4(s) = \frac{\{1.1654784s^3 + 0.2212349s^2 - 0.0202382s - 0.0001547\}}{\{-3.3208s^4 + 6.9923038s^3 + 1.3929577s^2 - 0.1616344s - 0.0012274\}} \quad (6\text{-}20)$$

$$\text{Im}\{(H_4(j\omega_\pi)\}s = -0.104415s. \quad (6\text{-}21)$$

$$\omega_\pi \text{Re}\{H_4(j\omega_\pi)\} = 0.1393814. \quad (6\text{-}22)$$

Thus from Equation (3-24),

$$\omega_\pi(s^2+\omega_\pi^2)H_5(s) = \frac{\{-4.3342667s^5 + 14.911991s^4 + 3.5678201s^3 + 0.01275669s^2 + 0.0270314s + 0.0002043\}}{\{-3.3208s^4 + 6.9923038s^3 + 1.3929577s^2 - 0.1616344s - 0.0012274\}} \quad (6\text{-}23)$$

and

$$H_5(s) = \frac{\{-4.3342667s^5 + 14.911991s^4 + 4.204083s^3 + 0.1275669s^2 + 0.0270314s + 0.0002043\}}{\{-3.3208s^6 + 6.9923038s^5 + 1.3717046s^4 - 0.1168837s^3 + 0.0076875s^2 - 0.0010345s - 0.0000079\}} \quad (6\text{-}24)$$

$P_5(s)$, the quotient polynomial of $1/H_5(s)$ is

$$P_5(s) = 0.7661734s + 1.0227491 \quad (6\text{-}25)$$

and the remainder polynomial $R_5(s)$ is

$$R_5(s) = -17.10059s^4 + 4.5143523s^3 + 0.1434918s^2 + 0.0288373s + 0.0002168 \quad (6\text{-}26)$$

Step 11 (of the algorithm) is the calculation of $H_6(j\omega_1)$. Where,

$$H_6(s) = R_5(s)/N_5(s) = \frac{\{-17.10059s^4 + 4.5143523s^3 + 0.1434918s^2 + 0.0288373s + 0.0002168\}}{\{-4.3343667s^5 + 14.911991s^4 + 4.204083s^3 + 0.1275669s^2 + 0.0270314s + 0.0002043\}} \quad (6\text{-}27)$$

and ω_1, the gain crossover frequency of the plant is

$$\omega_1 = 2.27 \text{ rad/sec.} \quad (6\text{-}28)$$

Therefore

$$H_6(j\omega_1) = \frac{-453.32218 + j52.739275403}{395.29202 - j310.35738}$$

$$= -0.774268 - j0.474486 \quad (6\text{-}29).$$

The next step (Step 12 of the algorithm) is to construct the polynomial $P_6(s)$. To do this one requires:

$$\operatorname{Im}\{H_6(j\omega_1)\}_{\omega_1} s = -0.2090247s \tag{6-30}$$

and

$$\operatorname{Re}\{H_6(j\omega_1)\} = -0.774268 \tag{6-31}$$

Putting Equations (6-25), (6-30), and (6-31) in the Equation of Step 12 of the algorithm yields

$$P_6(s) = 0.5571487s + 0.2484811 \tag{6-32}$$

At this stage enough data for constructing a good estimate, $H_R(s)$, of $H_p(s)$ is available. According to Step 13 of the algorithm

$$H_R(s) = \cfrac{1}{P_2(s) + \cfrac{s}{P_4(s) + \cfrac{s^2 + \omega_\pi^2}{P_6(s) + (s^2 + \omega_1^2)H_a(s)}}} \tag{6-33}$$

substituting values from Equations (6-9), (6-15), (6-19), (6-28) and (6-32) in Equation (6-33) gives

$$H_R(s) = \cfrac{1}{s + \cfrac{s}{-1.6063136s - 0.9792323 + B}} \tag{6-34}$$

Where,

$$B = \frac{s^2 + 0.0064}{0.5571487s + 0.2484811 + (s^2 + 5.1529)H_a(s)} \tag{6-35}$$

and $H_a(s)$ is arbitrary if one does not wish to match $H_p(s)$ and $H_R(s)$ at any specific frequencies other than at $s = 0$, $s = j\omega_\pi$, $s = j\omega_1$, $s = j\infty$. Let $H_a(s) = 0$, then Equation (6-35) reduces to

$$B = \frac{s^2 + 0.0064}{0.5571487s + 0.2484811} \tag{6-36}$$

and $H_R(s)$ becomes a third-order estimate of the system $H_p(s)$ given by

$$H_{R3}(s) = \frac{0.1050445s^2 - 0.944604s - 0.2368621}{0.1050445s^3 - 0.3874553s^2 + 0.011619s} \quad (6\text{-}37)$$

This estimate satisfies the following essential conditions: $H_{R3}(0) = H_p(0)$, $H_{R3}(j\omega_\pi) = H_p(j\omega_\pi)$, $H_{R3}(j\omega_1) = H_p(j\omega_1)$ and $H_{R3}(j\infty) = H_p(j\infty)$.

But this estimate is unstable, having two poles on the right-half-plane, whereas the plant $H_p(s)$ is stable, with no poles on the right-half-plane. To stabilize the estimate $H_{R3}(s)$ let Equation (6-35) be of the form

$$B = \frac{s^2 + 0.0064}{0.55714878 + 0.02484811 + K(s^2+5.1529)} \quad (6\text{-}38)$$

Where K is a constant. This implies choosing the arbitrary transfer function to be

$$H_a(s) = K \quad (6\text{-}39)$$

In this case a fourth-order estimate is obtained and is given by.

$$H_{R4}(s) = \frac{\{-(0.8949555+1.6063136k)s^3 + (0.055284-0.979232k)s^2 + (0.024321+8.2771733k)s + (0.0064-5.0458861k)\}}{\{-(0.894955+1.6063136k)s^4 + (0.6124327+0.0207677k)s^3 + (0.0491802+8.2771733k)s^2 + (0.0064+0.1070139k)s\}}. \quad (6\text{-}40)$$

For $H_{R4}(s)$ to be stable there must be no poles of $H_{R4}(s)$ on the RHP. This is true if there is no sign change on the Routh array of the denominator of $H_{R4}(s)$. The values of k which will satisfy this condition are

$$k < -29.5 \quad (6\text{-}41)$$

Let

$$k = -30 \quad (6\text{-}42)$$

Then Equation (6-40) becomes

$$H_{R4}(s) = \frac{\{47.29445s^3 + 29.43225s^2 - 248.291s + 151.38298\}}{\{47.29445s^4 - 0.0106s^3 - 248.266s^2 - 3.20402s\}} \quad (6\text{-}43)$$

which is stable and satisfies all the hard constraints that $H_{R3}(s)$ satisfies. But what is required is a third-order estimate. Thus the estimate $H_{R4}(s)$ is reduced to its third-order approximant using the reduction procedure. A third-order estimate of this transfer function is

$$H_{R3}(s) = \frac{0.97268s^2 + 0.01225s + 0.01022}{0.97268s^3 + 0.01792s^2 + 0.00648s} \quad (6\text{-}44)$$

or

$$H_{R3}(s) = \frac{s^2 + 0.0125941s + 0.0105071}{s^3 + 0.0184233s^2 + 0.00666^2s} \quad (6\text{-}45)$$

This estimate satisfies the conditions $H_{R3}(0) = H_p(0)$, $H_{R3}(j\omega_\pi) = H_p(j\omega_\pi)$, $H_{R3}(j\omega_1) = H_p(j\omega_1)$, $H_{R3}(j\infty) = H_p(j\infty)$ and is also stable with poles at $-0.00872 + j0.0811$. Consequently, $H_{R3}(s) \simeq H_p(s)$. The response of $H_{R3}(s)$ of Equation (6-45) is compared to those of a third-order estimate of $H_p(s)$ obtained by another well-known method [31] and the plant $H_p(s)$ to validate the new reduction algorithm. These responses are shown in Figure 6-1.

However, if it is demanded to exactly match $H_R(s)$ and $H_p(s)$ at any particular frequency, $H_a(s)$ must be calculated using a relation similar to that of Equation (3-24) and the reduction process continues. The following example is used to illustrate the fine-tuning process. Suppose the purpose (the application for which the estimate is intended) requires that the estimate and plant be exactly matched at $\omega = 10$ rad. per sec. Then one would proceed as follows in constructing the estimate:

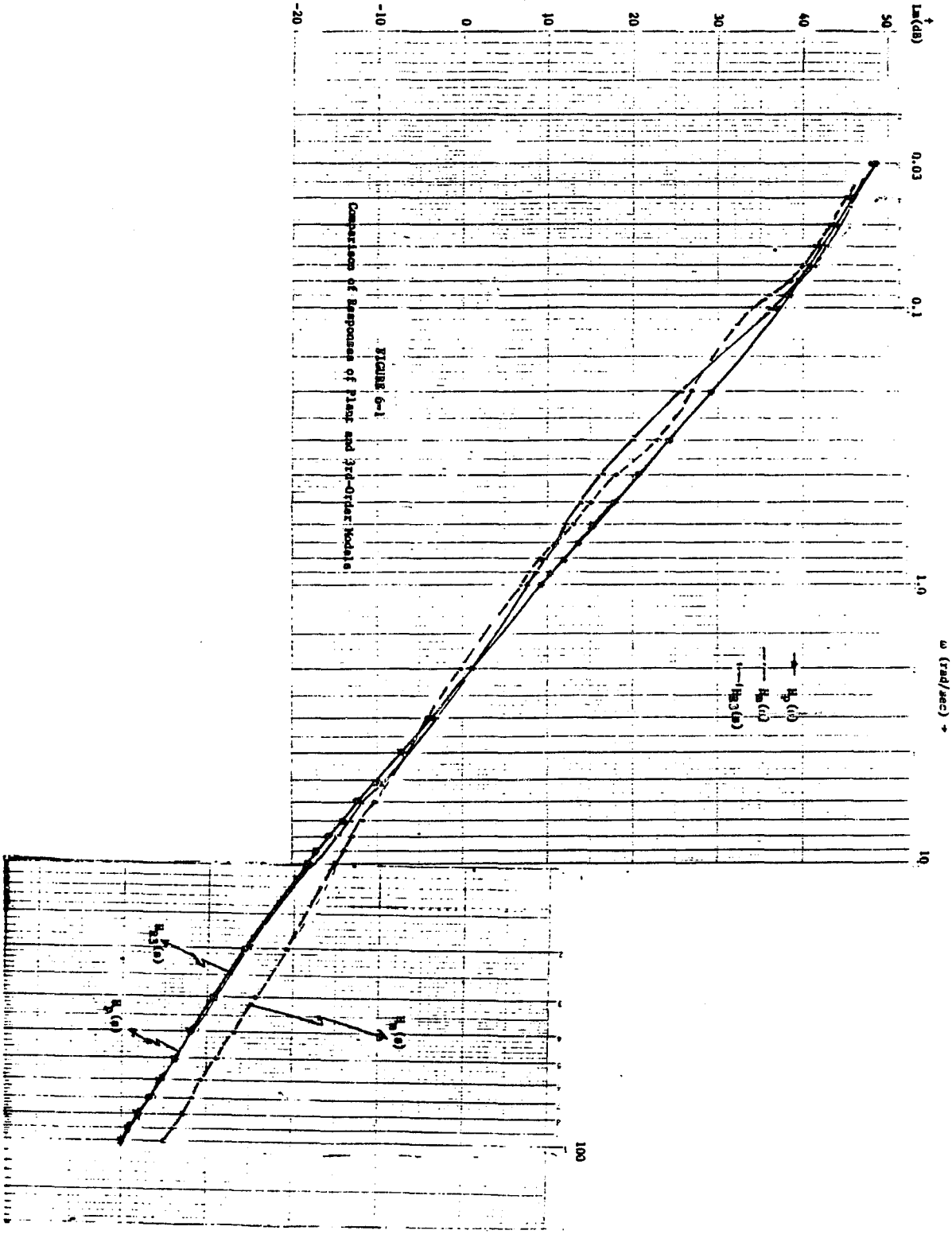

FIGURE 6-1 Comparison of Responses of Plant and 3rd-Order Models

$$H_a(s) = \frac{\omega_1 H_6(s) - \mathrm{Im}\{H_6(j\omega_1)\} - \omega_1 \mathrm{Re}\{H_6(j\omega_1)\}}{s^2 + \omega_1^2} \qquad (6\text{-}46)$$

substituting values from equations (6-28), (6-30) and (6-31) gives

$$H_a(s)(s^2 + 5.1529) = H_6(s) + 0.2090247s + 0.774268 \qquad (6\text{-}47)$$

and from Equation (6-27)

$$H_a(s) = \frac{\{-2.0565326s^6 + 0.5422996s^5 + 10.614686s^4 + 2.7985441s^3 + 0.1773084s^2 - 0.0154913s + 0.001154\}}{\{-0.3467413s^7 + 1.1929593s^6 - 1.4503962s^5 + 6.1574\text{-}54s^4 + 1.7352218s^3 + 0.0527514s^2 + 0.0111426s + 0.0008451\}} \qquad (6\text{-}48)$$

Next the quotient polynomial, $P_7(s)$, of $1/H_a(s)$ is calculated.

$$P_7(s) = 0.1686048s - 1.024687 \qquad (6\text{-}49)$$

and

$$1/H_a(s) = P_7(s) + H_7(s). \qquad (6\text{-}50)$$

where

$$H_7(s) = R_7(s)/N_7(s),$$

$R_7(s)$ is the remainder polynomial of $1/H_a(s)$ and $N_7(s)$ is the numerator polynomial of $H_a(s)$.

$$H_7(s) = \frac{\{0.2163964s^5 + 4.2474776s^4 + 1.102515s^3 + 0.1263223s^2 + 0.0045365s + 0.0003374\}}{\{2.0565326s^6 + 0.5422996s^5 + 10.614686s^4 + 2.7985441s^3 + 0.1773084s^2 + 0.0154913s + 0.001154\}} \qquad (6\text{-}51)$$

The next step is the calculation of the Imaginery Part and the real part of $H_7(j10)$ used for calculating $P_8(s)$ - the component that preserves plant behavior at $\omega = 10$ (or $s = j10$).

$$H_7(j10) = \frac{42462.144 + j20537.17}{-1950403.501154 + j51431.571} = -0.021478 - j.011096 \qquad (6\text{-}52)$$

Therefore

$$\mathrm{Im}\{H_7(j10)\}s = -0.001096s \tag{6-53}$$

and

$$\mathrm{Re}\{H_7(j10)\} = -0.021478 \tag{6-54}$$

But

$$P_8(s) = P_7(s) + \frac{\mathrm{Im}\{H_7(j10)\}s}{10} + \mathrm{Re}\{H_7(j10)\} \tag{6-55}$$

Thus combining Equations (6-49), (6-53) and (6-54) gives

$$P_8(s) = 0.167.5088s - 1.046165. \tag{6-56}$$

properly augumenting the previously obtained transfer function by $P_8(s)$ guarantees that the approximant will exactly match the plant's behavior at $s = j10$.

Emulating Step 13 of the algorithm the approximant so constructed is given by -

$$H_R(s) = \cfrac{1}{P_2(s) + \left\{\cfrac{s}{P_4(s) + \left\{\cfrac{s^2 + \omega_\pi^2}{P_6(s) + \left\{\cfrac{s^2 + \omega_1^2}{P_8(s) + (s^2 + 100)H_a'(s)}\right\}}\right\}}\right\}} \tag{6-57}$$

Where $H_a'(s)$ is arbitrary and can be used to match $H_p(s)$ and $H_R(s)$ at any other desired frequency.

Let $H_a'(s) = 0$. Equation (6-57) then reduces to

$$H_R(s) = \cfrac{1}{P_2(s) + \left\{\cfrac{s}{P_4(s) + \left\{\cfrac{s^2 + \omega_\pi^2}{P_6(s) + \left\{\cfrac{s^2 + \omega_1^2}{P_8(s)}\right\}}\right\}}\right\}} \tag{6-58}$$

Substituting values from Equations (6-9), (6-15), (6-19), (6-28), (6-32) and (6-48) a fourth-order estimate is obtained and is given by

$$H_{R4}(s) = \frac{3.3127217s^3 - 7.653835s^2 - 3.5684935s + 1.8707044}{3.3127217s^4 - 9.804297s^3 + 3.1971181s^2 - 0.0328367s} \quad (6\text{-}59)$$

$H_{R4}(s)$ satisfies the following conditions:

$H_{R4}(0) = H_p(0)$, $H_{R4}(j\omega_\pi) = H_p(j\omega_\pi)$, $H_{R4}(j\omega_1) = H_p(j\omega_1)$, $H_{R4}(j10) = H_p(j10)$ and $H_{R4}(j\infty) = H_p(j\infty)$.

A third-order model of the system $H_p(s)$ of Equation (6-1) obtained using Marshall's reduction technique [31] is given by

$$H_{M3}(s) = \frac{2.5232982s^2 - 2.2482588s + 0.2637553}{s^3 + 0.4252s^2 + 0.0321114s} \quad (6\text{-}60)$$

The response of this approximant is compared to those of $H_{R3}(s)$ and $H_p(s)$ in figure (6-1) to evaluate the adequateness of $H_{R3}(s)$.

6.3 Validating the Mathematical Criteria for Selecting the Optimal-Order Estimates

Many model-order reduction techniques are used to obtain an approximant of any desired order. Examples of this group include the eigenvalue retention methods [15], and the dominant poles methods [30]. In some other methods such as the method using the Routh Stability criterion [44] an approximant is further reduced to another approximant of lower-order until the desired order is obtained. In both cases it is important to know the lowest-order model estimate which preserves all the essential plant characteristics. The order of such estimate may be considered the optimal-order.

To validate the order selection criterion of section 4.4 several estimates of a high-order transfer function are obtained using the same technique. Consider the transfer function [44] $H_p(s)$ given by

$$H_p(s) = \frac{\{35s^7 + 1086s^6 + 13285s^5 + 82402s^4 + 27876s^3 + 511812s^2 + 482964s + 194480\}}{\{s^8 + 33s^7 + 437s^6 + 3017s^5 + 11870s^4 + 27470s^3 + 37492s^2 + 28880s + 9600\}} \quad (6\text{-}61)$$

The reduced-order models of $H_p(s)$ obtained via the method of Krishnamurthy and Seshadri [44] are: -

$$H_{R7}(s) = \frac{\{1086.0s^6 + 10629.3s^5 + 82402s^4 + 261881.1s^3 + 511812.s^2 + 476696.1s + 194480\}}{\{330^7 + 345.6s^6 + 3017s^5 + 11037.6s^4 + 27470s^3 + 36616.8s^2 + 28880s + 9600\}} \quad (6\text{-}62)$$

$$H_{R6}(s) = \frac{\{10629.3s^5 + 55645.5s^4 + 261881.1s^3 + 463107.8s^2 + 476696.1s + 194480\}}{\{345.6s^6 + 1963s^5 + 11037.6s^4 + 23973.4s^3 + 3661.8s^2 + 27963.3s + 9600\}} \quad (6\text{-}63)$$

$$H_{R5}(s) = \frac{\{55645.5s^4 + 173419.1s^3 + 463107.8s^2 + 439546.9s + 194480\}}{\{1963s^5 + 6817.2s^4 + 23973.4s^3 + 31694s^2 + 27963.3s + 9600\}} \quad (6\text{-}64)$$

$$H_{R4}(s) = \frac{173419.1s^3 + 322069s^2 + 439546.9s + 194480}{6817.2s^4 + 14847.1s^3 + 31694s^2 + 25199s + 9600} \quad (6\text{-}65)$$

TABLE 6-1

Responses (Log Magnitude) of $H_p(s)$ as a Function of Frequency

n	Frequency (rad/sec)	$A_p(s)$(dB) k = 8	$A_{R7}(s)$(dB) k = 7	$A_{R6}(s)$(dB) k = 6	$A_{R5}(s)$(dB) k = 5
1	.1	-17.14156	-11.33553	-25.6660	-75.37344
2	.2	-17.18194	-11.37406	-25.15655	-76.3015
3	.3	-17.24071	-11.40478	-25.67408	-77.28947
4	.4	-17.30774	-11.39231	-26.26669	-78.23486
5	.5	-17.37338	-11.31001	-26.95666	-79.04518
6	.6	-17.43135	-11.15076	-27.73315	-79.6623
7	.7	-17.48079	-10.93	-28.554	-80.07258
8	.8	-17.52658	-10.67984	-29.35952	-80.29979
9	.9	-17.57802	-10.43657	-30.08939	-80.38711
10	1	-17.64629	-10.22741	-30.69654	-80.37992
11	2	-19.8906	-9.355202	-31.49749	-79.16673
12	3	-22.67867	-10.81076	-34.01045	-75.98841
13	4	-24.85003	-13.73939	-36.49061	-73.69975
14	5	-26.57377	-16.02418	-36.01691	-72.31403
15	6	-28.0029	-17.46648	-35.11432	-71.6452
16	7	-29.22605	-18.40391	-35.12202	-71.95508
17	8	-30.29672	-19.29264	-36.14085	-73.1908
18	9	-31.24957	-20.2579	-37.72321	-74.90951
19	10	-32.10842	-21.22816	-39.46338	-76.72625
20	20	-37.89046	-27.82412	-52.08131	-88.8715
21	30	-41.35057	-31.48135	-59.23391	-94.87849
22	40	-43.82546	-34.02771	-64.24167	-98.42245

TABLE 6-1 (cont'd)

n	Frequency (rad/sec)	$A_p(s)$ (dB) k = 8	$A_{R7}(s)$ (dB) k = 7	$A_{R6}(s)$ (dB) k = 6	$A_{R5}(s)$ (dB) k = 5
23	50	-45.75213	-35.98793	-68.10923	-100.4693
24	60	-47.32936	-37.5835	-71.26424	-101.284
25	70	-48.66439	-38.92964	-73.93024	-100.711
26	80	-49.82168	-40.09415	-76.23929	-97.77137
27	90	-50.84297	-41.1204	-78.27613	-84.53912
28	100	-51.75685	-42.03785	-80.09835	-98.13281

n	Frequency (rad/sec)	$A_{R4}(s)$ (dB) k = 4	$A_{R3}(s)$ (dB) k = 3	$A_{R2}(s)$ (dB) k = 2
1	.1	-6.158022	3.467913E-02	1.733785
2	.2	-5.747127	-.2930943	1.725441
3	.3	-5.367345	-.6993831	1.65413
4	.4	-5.126004	-1.027133	1.475221
5	.5	-5.099247	-1.12844	1.172375
6	.6	-5.296665	-.9083664	.7581916
7	.7	-5.663417	-.3705608	.2612111
8	.8	-6.110021	.3586462	-.2874831
9	.9	-6.540886	1.051923	-.8617848
10	1	-6.869236	1.461815	-1.442699
11	2	-5.491518	-4.594183	-6.364435
12	3	-9.24528	-8.91146	-9.691539
13	4	-12.02183	-11.68982	-12.12427
14	5	-14.04564	-13.75583	-14.03235
15	6	-15.65181	-15.40831	-15.59975

TABLE 6-1 (cont'd)

n	Frequency (rad/sec)	$A_{R4}(s)$ (dB) k = 4	$A_{R3}(s)$ (dB) k = 3	$A_{R2}(s)$ (dB) k = 2
16	7	-16.99212	-16.78854	-16.92895
17	8	-18.14625	-17.97509	-18.08249
18	9	-6.540886	-19.01641	-19.10123
19	10	-20.06897	-19.94461	-20.01329
20	20	-26.04635	-26.0068	-26.02408
21	30	-29.55529	-29.53631	-29.54409
22	40	-32.04885	-32.03777	-32.04223
23	50	-33.98447	-33.97722	-33.98014
24	60	-35.56663	-35.56152	-35.5636
25	70	-36.90466	-36.90086	-36.90245
26	80	-38.0639	-38.06097	-38.06222
27	90	-39.08653	-39.0842	-39.08523
28	100	-40.00137	-39.99947	-40.00035

TABLE 6-2

Deviations of the Responses of $H_{Rx}(s)$ From the Responses of $H_p(s)$ at Various Frequencies (n = 10)

Frequency (rad/sec)	$\lvert A_p(s)-A_{R7}(s)\rvert$	$\lvert A_p(s)-A_{R6}(s)\rvert$	$\lvert A_p(s)-A_{R5}(s)\rvert$	$\lvert A_p(s)-A_{R4}(s)\rvert$
0.1	5.80626	8.52444	58.23188	10.983538
0.5	6.06337	9.58322	61.6718	12.274133
.7	6.55079	10.4431	62.59258	11.8173
1	7.41888	13.05025	62.73363	10.777054

TABLE 6-2 (cont'd)

Frequency (rad/sec)	$\lvert A_p(s)-A_{R7}(s)\rvert$	$\lvert A_p(s)-A_{R6}(s)\rvert$	$\lvert A_p(s)-A_{R5}(s)\rvert$	$\lvert A_p(s)-A_{R4}(s)\rvert$
5	10.54959	9.44314	45.74026	12.52813
7	10.82214	5.89597	42.72903	12.23393
10	10.88026	7.35496	44.6	12.03945
50	9.77213	22.3571	54.71717	11.76766
70	9.73475	25.26585	52.04661	11.75973
100	9.719	28.3415	46.37596	11.75548

Frequency (rad/sec)	$\lvert A_p(s)-A_{R3}(s)\rvert$	$\lvert A_p(s)-A_{R2}(s)\rvert$
0.1	17.176239	18.895345
0.5	16.24494	18.545755
.7	17.11023	17.219579
1	19.108105	16.203591
5	12.81794	12.54142
7	12.43751	12.2971
10	12.16381	12.09513
50	11.77491	11.77199
70	11.76353	11.76194
100	11.75738	11.7565

TABLE 6-3

Values of the Criterion of Equation (4-12)
(n = 10)

k	Value of Criterion
7	199.928
6	284.81459
5	1794.8463
4	55.727
3	58.30
2	43.464

TABLE 6-4

Values for the Hypothesis Test[!]

$\alpha = 0.01$, $n = 10$, $\chi^2_{\alpha,n-1} = 21.666$

q	r	t	$\Sigma(e_r(s)_i - e_q(s)_i)$	Choice
7	6	90.05	52.9197	$H_{R7}(s)$
7	5	t > 473	438	$H_{R7}(s)$
7	4	18.524	*	$H_{R4}(s)$
4	3	13.628	*	$H_{R3}(s)$
3	2	1.001	*	$H_{R2}(s)$

The 'best' estimate is $H_{R2}(s)$.

! $\Sigma(e_r(s)_i - e_q(s)_i)$ is not computed if H_o is not rejected i.e. if $t < 21.666$.

$$H_{R3}(s) = \frac{322069s^2 + 334828.5s + 194480}{14847.1s^3 + 20123.7s^2 + 25199s + 9600} \qquad (6\text{-}66)$$

$$H_{R2}(s) = \frac{334828.5s + 194480}{20123.7s^2 + 18116.2s + 9600} \qquad (6\text{-}67)$$

The magnitudes of $H_p(s)$ and those of its approximants at various frequency levels are shown in Table 6-1. The deviations of the approximants' magnitudes from those of $H_p(s)$ at some frequencies are shown in Table 6-2. Using the criterion of expression (4-12) it is seen that the approximant of Equations (6-65) and (6-67) are good choices, and the optimal-order model is $H_{R2}(s)$.

Alternatively, using the hypothesis test of Section 4.5, the results are summarized in Table 6-4. It will again be seen that $H_{R2}(s)$ is the best choice at 1% confidence level. It should be observed from Tables 6-3 and 6-4 that the criteria of Sections 4.4 and 4.5 yield identical results. Both indicate that $H_{R2}(s)$ is the best choice and that the most inferior estimate is $H_{R5}(s)$.

6.4 Application of the Estimate in the Design of Controllers

The following example shall illustrate the usefulness of the reduced-order model, obtained by the new method, in control system design.

A state-space realization of Equation (6-45) is of the form

$$A_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -0.006662 & -0.0184233 \end{bmatrix} \qquad (6\text{-}68)$$

$$B_R = [0 \quad 0 \quad 1]^T \tag{6-69}$$

$$C_R = [0.0105071 \quad 0.0125941 \quad 1] \tag{6-70}$$

$$D_R = [0]. \tag{6-71}$$

Suppose it is required to design a stabilizing gain compensator K. Let the feedback gain K be such that the poles -3, -5, -10 are assigned to the system. Then the matrix K is given by

$$K^T = \begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix}^T = \begin{bmatrix} 18 - 0.0184233 \\ 95 - 0.006662 \\ 150 - 0 \end{bmatrix}^T = \begin{bmatrix} 17.98 \\ 94.99 \\ 150 \end{bmatrix}^T \tag{6-72}$$

The control input U is given by

$$U = - KX_R. \tag{6-73}$$

Where X_R is given by

$$\begin{aligned} Y_R &= C_R X_R \\ &= [0.0105071 \quad 0.0125941 \quad 1] \begin{bmatrix} x_{R1} \\ x_{R2} \\ x_{R3} \end{bmatrix} \end{aligned} \tag{6-74}$$

Solving for X_R;

$$\begin{aligned} X_R &\cong C_R^T \left[C_R C_R^T\right]^{-1} Y_R \\ &= [0.010507 \quad 0.012594 \quad 1]^T Y_R \end{aligned} \tag{6-75}$$

Thus, from Equations (6-72) and (6-73)

$$U = - [17.98 \quad 94.99 \quad 150] \begin{matrix} 0.010507 \\ 0.012594 \\ 1 \end{matrix} Y_R \tag{6-76}$$

After taking the Laplace transform this yields

$$U(s) = -151.385 Y_R(s). \tag{6-77}$$

Applying this control law to the plant $H_p(s)$ of Equation (6-1) gives a forward path transfer function given by

$$151.385H_R(s) \tag{6-78}$$

The closed-loop system

$$G(s) = \frac{151.385H_p(s)}{1 + 151.385H_p(s)} \tag{6-79}$$

is stable with a gain margin of 1.322 and a phase margin of 58°. This shows that the original system can be controlled by the control system designed using the estimate.

CHAPTER VII

CONCLUSIONS

The last decade has seen a lot of suggestions on how to approximate a high-order system by a lower-order model. Many of the suggested technique are of mathematical nature - not specifying any engineering implications. They cannot therefore be applied to engineering problems with certainty. The development of some of the methods such as the singular perturbational method has been mostly on an ad-hoc basis. All the theoretical implications of the method have not been fully explained or understood.

Also, no methods exist yet which can be used for all applications. Some of the existing approximation techniques yield approximants with overall frequency response similar to that of the plant but many of them lack simplicity. Finally, errors commonly committed by the techniques known today include mismatch of steady-state responses, producing an unstable estimate even though the plant is stable and lack of generality.

In this thesis an alternative method of model-order reduction in frequency domain is introduced which overcomes much of the deficiencies of some of the other existing methods. The importance of preserving performance characteristics at zero frequency and high frequencies is pointed out. The engineering implications of constructing an estimate with the same degree of relative stability with the plant

is discussed. The algorithm developed thus ensures that the method produces estimates that may be useful for engineering applications.

The method introduced here though efficient is both computationally and conceptually simpler than most others in the existing literature. It does not require the knowledge of the poles a priori as most fairly efficient methods do. Furthermore the level of mathematics employed is very ordinary.

Particularly attractive is the versatility of the approach. The relative importance of system characteristics depend on the particular application, it is important therefore that a reduction technique allow the engineer to preserve any desired plant characteristic in the estimate according to the application that the estimate is intended for. The technique introduced here offers such opportunity. Finally, the limitations of the method cannot be considered a serious handicap since it is applicable to a large class of practical systems.

Another area of model reduction which has received far less attention is the determination of how small the order can be. A statistical approach to this order determination is suggested. The thesis gives two criteria for selecting the order, one of which depends on the engineering judgment of the user.

Finally, an example has been used to show the strength of this method. It is expected that this alternative and simple method of model-order reduction shall be useful to the engineering world.

CITED REFERENCES

1. Genesio, R. and M. Milanese; "A Note on the Derivation and Use of Reduced-Order Model," IEEE Trans. Automatic. Contr., Vol. AC-21, pp. 118-122, 1976.

2. Melsa, J. L. and D. J. Schultz. Linear Control Systems, McGraw-Hill, New York, 1969.

3. Moore, Bruce C.; "Principle Component Analysis in Linear Systems: Controllability, Observability, and Model Reduction." IEEE Trans. on Automatic Control, Vol. AC-26, No. 1, February 1981, pp. 17-32.

4. Decoster, M. et al.; "A Comparative Study of Different Reduction Methods (Part 1 & 2)," Journal A, Vol. 17, pp. 68-74; 125-134, 1976.

5. Chen, C.F. and L. S. Shieh; "An Algebraic Method for Control System Design," International Journal of Control, Vol. 11, pp. 717-739, 1970.

6. Lal, M. and R. Mitra; "Simplification of Large System Dynamics Using a Moment Evaluation Algorithm," IEEE Transactions on Automatic Control, Vol. AC-19, pp. 602-603, 1974.

7. Shamash; "Stable Reduced Order Models Using Pade' Type Approximations," IEEE Transactions on Automatic Controls, Vol. AC-19, pp. 615-616, 1974.

8. Zakian, V.; "Simplication of Linear Time-Invariant Systems by Moment Approximants," International Journal on Control, Vol. 18, pp. 455-460, 1973.

9. Hutton, M. F. and B. Friedland; "Routh Approximations for Reducing Order of Linear Time-Invariant Systems," IEEE Transaction on Automatic Control, Vol. AC-20, pp. 329-337, 1975.

10. Langholz, G. and D. Finmester; "Model Reduction to Routh Approximations," International Journal on System Science, Vol. 9, pp. 493-496, 1978.

11. Eitelberg, Edward; "Iterative Model Reduction by Minimizing the Weighted Equation Error, Regelungstechnik, Vol. 26, pp. 320-322, 1978.

12. Obinata, G. and H. Inoka; "A Method for Modeling Linear Time - Invariant Systems by Linear Systems of Low Order," IEEE Transactions on Automatic Control, Vol. AC-21, pp. 602-603, 1976.

13. Obinata, G. and H. Inoka; "A Method for Reducing the Order of Multivariable Stochastic Systems," IEEE Transactions on Automatic Control, Vol. AC-22, pp. 676-677, 1977.

14. Eitelberg, E.; "Comments on Model Reduction by Minimizing the Equation Error," IEEE Transactions on Automatic Control, Vol. AC-27, No. 4, pp. 1000-1002, 1982.

15. Davidson, E. J.; "A Method for Simplifying Linear Dynamic Systems," IEEE Transactions on Automatic Control, Vol. AC-11, pp. 93-101, 1966.

16. Wilson, D. A.; "Optimum Solution of Model Reduction Problem," Proceedings of IEEE, Vol. 117, pp. 1161-1165, 1970.

17. Marshall, S. A; "An Approximate Method for Reducing the Order of a Large System," Control Engineering, Vol. 10, pp. 642-648, 1966.

18. Hickin, J. and N. K. Sinha; "Model Reduction for Linear Multivariable Systems," IEEE Transactions on Automatic Control, Vol. AC-25, No. 6, pp. 1121-1126, 1980.

19. Chen, C. F.; "Model Reduction of Multivariable Control System by Means of Matrix Continued Fraction," International Journal of Control, Vol. 20, pp. 225-238, 1974.

20. Shieh, L. S. and Y. G. Wei; "A Mixed Method for Multivariate System Reduction," IEEE Transaction on Automatic Control, Vol. AC-20, pp. 429-432, 1975.

21. Chen, C. F. and L. S. Shieh; "A Novel Approach to Linear Model Simplification," International Journal of Control, Vol. 8, pp. 561-570, 1968.

22. Anderson, J. H.; "Geometrical Approach to Reduction of Dynamical Systems," IEEE Proceedings, Vol. 114, pp. 1014-1018, 1967.

23. Sanjuti, P. and P. V. Kototovic; "Near Optimum Design of Linear Systems Using Singular Perturbation Method," IEEE Transaction on Automatic Control, Vol. AC-14, pp. 15-21, 1969.

24. Rozsa, P. and N. K. Sinha; "Minimal Realization of a Transfer Function Matrix in Canonical Forms," International Journal of Control, Vol. 21, pp. 273-284, 1975.

25. Akoi, M. and J. R. Auddle; "Estimation of the State Vector of a Linear Stochastic System with a Constrained Estimator," IEEE Transaction on Automatic Control, Vol. AC-12, pp. 432-433, Aug. 1967.

26. Elrazaz, F. and N. K. Sinha; "Modeling and Simulation of Power Systems for Dynamic Analysis," In Proc. IFAC Symposiums on Computer Application to Large Scale Power System, New Delhi, India, 1979, pp. 204-211.

27. Milanese, M. and A. Negro; "Min-Max Feed-back Control of Systems Described by Approximate Models," Presented at the IEEE Conference on Decision Control, Phoenix, Arizona, 1974.

28. Lamba, S. S. and S. Vittal Rao; "Derivation of Aggregation Matrices for Simplified Models of Linear Dynamic Systems and Their Application for Optimal Control," In Proc. Joint Automatic Control Conference, 1972, pp. 498-503.

29. Towill, D. R.; "Low Order Modeling Techniques: Tools and Toys?" IEEE Conference Publication 96, pp. 206-212, 1973.

30. Chidambara, M. R.; "On a Method for Simplifying Linear Dynamic Systems," E. G. Davidson, Author's Reply, IEEE Transaction Automatic Control, Vol. AC-12, pp. 119-121, 1967.

31. Marshall, S. A.; "A Design of Reduced-Order Systems," International Journal on Control, Vol. 31, No. 4, pp. 677-690, 1980.

32. Mahapatra, G. B.; "A Note on Selecting Low-Order Systems by Davidson Simplification Technique, IEEE Transactions on Automatic Control, Vol. AC-22, pp. 677-674, Aug. 1977.

33. Rao, A. D. S. S. Lamba, and S. V. Rao; "Comments on 'A Note on Selecting Low-Order System by Davidson's Model Simplification Technique'", IEEE Transactions on Automatic Control, Vol. AC-24, No. 4, Feb. 1979, pp. 141-142.

34. Malhapatra, G. B.; "A Further Note on Selecting a Low-Order System Using the Dominant Eigenvalue Concept", IEEE Transactions on Automatic Control, Vol. AC-24, No. 1, Feb. 1979, pp. 135-136.

35. Rao, A. S., S. S. Lambo, and S. V. Rao; "Comments on 'A Further Note on Selecting a Low-Order System Using the Dominant Eigenvalue Concept'", IEEE Transactions on Automatic Control, Vol. AC-26, No. 2, pp. 604-605.

36. Nejals, Ashoor and Vimal Singh; "A Note on Low-Order Modeling", IEEE Transactions on Automatic Control, Vol. AC-27, No. 5, October 1982.

37. Wonnacott, R. J. and T. H. Wonnacott. Regression: A Second Course in Statistics, John Wiley and Sons, Inc., 1981.

38. Singh, V.; "Nonuniqueness of Model Reduction Using Routh Approach", IEEE Transactions on Automatic Control, Vol. AC-24, No. 4, August 1979, pp. 650-651.

39. Pal, J.; "Systems Reduction by a Mixed Method", IEEE Transactions on Automatic Control, Vol. AC-25, No. 5, October 1980, pp. 973-976.

40. Ideka, M., D. D. Siljah, and D. E. White; "Decentralized Control with Overlapping Information Sets", Journal of Optimization Theory and Applications, Vol. 34, June 1981, pp. 279-310.

41. Enright, W. H. and M. S. Kamel; "On Selecting a Low-Order Model Using the Dominant Mode Concept", IEEE Transactions on Automatic Control, Vol. AC-25, No. 5, October 1980, pp. 977-978.

42. Elraza, Z. and K. Sinha; "On the Selection of the Dominant Poles of a System to be Retained in a Low-Order Model", IEEE Transactions on Automatic Control, Vol. AC-24, No. 5, October 1979, pp. 792-793.

43. Gopal, M. and S. I. Mehta; "On the Selection of the Eigenvalues to be Retained in the Reduced-Order Model", IEEE Transactions on Automatic Control, Vol. AC-27, No. 3, June 1982.

44. Alexandro, F. J. Jr.; "Stable Partial Pade' Approximations for Reduced-Order Transfer Functions", IEEE Transactions on Automatic Control, Vol. AC-29, No. 2, February 1984, pp. 159-160.

45. Bistritz, Y. and G. Langholz; "Model Reduction by Chebyshev Polynomial Techniques", IEEE Transactions on Automatic Control, Vol. AC-24, No. 5, October 1979, pp. 741-746.

46. Sinha, P. K.; Multivariable Control, John Wiley and Sons, Inc., 1984.

47. Kendal, M. G. and A. Stuart; The Advanced Theory of Statistics, Vol. 1, Griffin, London, 1962.

BIBLIOGRAPHY

Anderson, B.D.O. "External and Internal Stability of a Linear System - A New Connection," IEEE Trans. on Automatic Control, Vol. AC-17, No. 1. Feb. 1972, pp. 107-111.

Antsaklis, P. J. "Some New Matrix Methods Applied to Multivariable System Analysis and Design," Ph.D. Dissertation. Brown University, 1977.

Benson; Levine; Goldstein. Intermediate Statistical Methods and Applications. Prentice-Hall, Inc., Englewood Cliff, New Jersey, 1983.

Bierman, G. J. "Weighted Least-Squares Stationary Approximation to Linear Systems," IEEE Trans. on Automatic Control, Vol. AC-11, No. 3. July 1966, pp. 596-606.

Brockett, R. W. "The Status of Stability Theory for Deterministic Systems," IEEE Trans. on Automatic Control, Vol. AC-17, No. 2. April 1972, pp. 232-234.

Brogan, W. L. Modern Control Theory. Quantum Publishers, Inc., New York, 1974.

Bryson, A. E. and Y. C. Ho. Applied Optimal Control. Blaisdell, Waltham, Massachusetts, 1969.

Cochran, W. G. and G. W. Snedecor. Statistical Methods, 7th Edition. The Iowa University Press, Ames, Iowa, 1980.

Daniel, C. Application of Statistics to Industrial Experimentation. Wiley, New York, 1976.

Dazzo, J. and C. H. Houpis. Linear Control System Analysis and Design: Conventional and Modern. McGraw-Hill, Inc., New York, 1975.

DeJong, L. S. "Numerical Aspects of Realization Algorithms in Linear Systems Theorem," Ph.D. Dissertation, Eindhoven University of Tech. Eindhoven, Netherlands, 1975.

Hickin, J. "Model-Order Reduction Using Singular-Perturbations," Ph.D. Dissertation, McMaster University, Hamilton, Ontario, Canada, 1978.

Luenberger, D. G. "Determining the State of a Linear System with Observer of Low Order," Ph.D. Dissertation, Stanford, California, 1963.

Mallows, C. L. "Choosing a Subset Regression." Presented at the annual meeting of the American Statistical Association, Los Angeles, August 1966.

Neter, J. and W. Wasserman. Applied Linear Statistical Models. Irwin, Homewood, Illinois, 1974.

Nims, P. T. "Some Design Criteria for Automatic Controls," Trans. AIEE, Part II, Vol. 70, pp. 606-611, 1951.

Ogata, K. Modern Control Engineering. Prentice-Hall, Inc., Englewood Cliff, New Jersey, 1970.

Pinquet, P. J. M. "State Space Formulation of a Class of Model Reduction Methods," Ph.D. Dissertation, Cambridge University, Cambridge, England, 1978.

Pipes, L. A. Applied Mathematics for Engineers and Physists, 3rd Edition. McGraw-Hill, New York, 1971.

Rao, C. R. Linear Statistical Inference and Its Application, 2nd Edition. Wiley, New York, 1973.

Shamash, Y. "Linear System Reduction Using Pade' Approximation," Ph.D. Dissertation, Imperial College of Science and Technology, The University of London, London, England, 1973.

Shinners, S. M. Modern Control System Theory and Application. Addison-Wesley Publishing Company, Inc., Philippines, 1978.

Walls, J. E., Jr. "Estimation and Control for Large Scale Systems Having Spatial Symetry," Ph.D. Dissertation, Massachusetts Institute of Tech., Cambridge, Massachusetts, 1978.

Wylie, C. R. Advanced Engineering Mathematics. McGraw-Hill, Inc., New York, 1975.

APPENDIX A

DEFINING THE REGRESSORS OF A TRANSFER FUNCTION

The illustration of Section 4-2 shows that an appropriate definition of the regressors of a polynomial results in a model with k regressors, where k is the degree of the polynomial, which is equal to the number of the roots of the polynomial. Similarly the transfer function which is a ratio of two polynomials can be treated as a multiple regression model having k regressors, where k is the order of the transfer function.

It has been established that from a mathematical point of view the physical or economic significance of the regressors are irrelevant. Thus for the purpose of comparing models using step wise regression approach, a uniform way of defining the regressors of the models (transfer functions) may be in such a way that the number of regressors equals the order of the model. The appropriateness of this can be seen if one examines the partial fraction expansion of the transfer function. It will be observed that there are exactly k terms in the partial fraction expansion of the transfer function, where k is the order of the transfer function and each term can be defined as a regressor given a total of k regressors as asserted.

VITA

Mr. Alexius O. Kalu is the son of Rev. James O. Kalu and Mrs. Peace Udo Kalu of Ihechiowa in Nigeria. Mr. Kalu attended Agrey Memorial Secondary School, Arochukwu in Imo State, Nigeria. Upon graduation from secondary school he attended Alvan Ikoku College of Education, majoring in Mathematics/Physics from where he continued to the University of Texas at Arlington. He received the B.S.E.E. degree from UTA and subsequently a M.S. degree in Electrical Engineering from Louisiana Tech University, Ruston, Louisiana. Mr. Kalu came to L.S.U., Baton Rouge, in the fall of 1982. Here in L.S.U. he was enrolled in the interdepartmental program in Engineering - Engineering Science, majoring in Electrical and Industrial Engineering with Nuclear Engineering and Quantitative Business Analysis being his minor areas.

DOCTORAL EXAMINATION AND DISSERTATION REPORT

Candidate: Alexius Ogbusua Kalu

Major Field: Engineering Science
(Electrical/Industrial Engineering)

Title of Dissertation: An Engineering Approach to Model Order Reduction and Its Application to Controller Design

Approved:

Major Professor and Chairman

Dean of the Graduate School

EXAMINING COMMITTEE:

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