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Research Note

An enhanced deconvolution procedure for retrieving the seismic moment tensor from a sparse network

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Summary. In theory, a single horizontally polarized seismometer can be used to find the six independent elements of the seismic moment tensor of a buried point source, provided that the instrument is neither longitudinally nor transversely polarized. Also, two vertically polarized seismometers can be used, provided that the epicentre does not lie on the great circle through the two instruments. These results form the theoretical basis for a procedure for retrieving the source mechanism from a sparse seismographic network.

Let the six independent elements of the seismic moment rate spectrum be $\mathbf{f}(\omega) = (f_1(\omega), \ldots, f_6(\omega))^T$ and suppose that P seismic spectra (records) $\mathbf{u}(\omega) = (u_1(\omega), \ldots, u_P(\omega))^T$ have been observed. The relationship between $\mathbf{u}(\omega)$ and $\mathbf{f}(\omega)$ is a linear one (Gilbert 1971, 1973; Dziewonski & Gilbert 1974; Gilbert & Dziewonski 1975, hereafter referenced as M)

$$\mathbf{u}(\omega) = \mathbf{H}(\omega) \cdot \mathbf{f}(\omega). \tag{1}$$

The $P \times 6$ matrix $\mathbf{H}(\omega)$ is a functional of the mechanical structure of the Earth and can be regarded as the spectral transfer matrix or system function that relates output $\mathbf{u}(\omega)$ to input $\mathbf{f}(\omega)$. Let the p-th row of $\mathbf{H}(\omega)$ be $\mathbf{h}_p^T(\omega)$.

The six-vector $\mathbf{h}_p(\omega)$ can be written as the sum of normal modes (M; 2.1.24, 2.1.28)

$$\mathbf{h}_p(\omega) = \sum_k \mathbf{A}_{kp} C_k(\omega) R_{kp}(\omega) \tag{2}$$

where $\mathbf{A}_{kp}$ specifies the excitation and amplitude of the k-th mode for the p-th record, C_k is the resonance function of the k-th mode, and R_{kp} represents the effect of truncation and the response of the p-th instrument. Each element of $\mathbf{h}_p(\omega)$ is the spectrum of a seismogram caused by a unit element of the moment rate tensor, a delta function in time. The observed seismogram at the p-th instrument is a linear combination of the six seismograms $\mathbf{h}_p(\omega)$ and the six coefficients in the linear combination are the six elements of $\mathbf{f}(\omega)$.

Suppose that our model of the Earth is good enough to permit us to ignore the difference between real and calculated $\mathbf{h}_p(\omega)$. Then we can seek to solve equation (1) for $\mathbf{f}(\omega)$. At low frequencies the spectral peaks in $\mathbf{h}_p(\omega)$ are sufficiently well separated to cause spectral gaps, frequencies where there is little or no information about $\mathbf{f}(\omega)$. However, it is generally believed that $\mathbf{f}(\omega)$ is a smooth function of ω at low frequencies, so smooth that it can be taken constant over a frequency band embracing many modes. Therefore, define set J of discrete frequencies ω_i

$$\omega_J - \delta\omega \leqslant \omega_i \leqslant \omega_J + \delta\omega; \quad i = i_J, i_J + 1, \ldots, i_J + I - 1 \tag{3}$$

and replace $\mathbf{f}(\omega)$ by $\mathbf{f}(\omega_J)$. There are now $I \cdot P$ equations for the six-vector $\mathbf{f}$

$$\mathbf{u}_J = \mathbf{H}_J \cdot \mathbf{f}(\omega_J) \tag{4}$$

and we solve (4) by applying the classical method of least squares

$$\mathbf{H}_J^{\mathrm{H}} \cdot \mathbf{u}_J = \mathscr{H}_J \cdot \mathbf{f}(\omega_J); \quad \mathscr{H}_J = \mathbf{H}_J^{\mathrm{H}} \cdot \mathbf{H}_J \tag{5}$$

where the superscript H denotes Hermitean transpose.

In forming (5) we cross-correlate u_p with each element of $\mathbf{h}_p$ – multiply by $\mathbf{h}_p^{\mathrm{H}}(\omega)$. This operation is conventionally termed matched filtering and is an operation to enhance the signal being sought. The result is summed over the frequencies in set J to give (5). Solving (5) is then the final step in the deconvolution procedure for retrieving $\mathbf{f}$. In order to solve (5) for $\mathbf{f}$ we require that $\mathscr{H}_J$ have rank-6. Thus $I \cdot P \geqslant 6$ is a necessary condition. For a dense network, $P \gg 1$, I, the number of discrete frequencies in set J, can be small. Alternatively if $I \geqslant 6$ it appears that we can have $P = 1$ and still maintain rank-6 for $\mathscr{H}_J$. To explore this possibility we examine the eigenvalues of $\mathscr{H}_J$. Without loss of generality we take $R_{kp}(\omega) = 1$.

Consider a single, vertically polarized accelerometer. In epicentral spherical coordinates the location of the receiver is (r, θ, ϕ). An inspection of (M; 2.1.30) shows that the six vector $\mathbf{A}_{kp}$ in (2) for vertical polarization (r-component) can be written

$$\mathbf{A} = (\mathbf{\Phi} \cdot \mathbf{S})\, U(r) \tag{6}$$

where $\mathbf{\Phi}$ is a 6×4 matrix whose non-zero elements are

$$\begin{aligned} &\Phi_{11} = \Phi_{22} = \Phi_{32} = 1, \quad \Phi_{23} = -\Phi_{33} = \cos 2\phi, \quad \Phi_{44} = \cos\phi, \quad \Phi_{54} = \sin\phi, \\ &\Phi_{63} = 2\sin 2\phi \end{aligned} \tag{7}$$

and $\mathbf{S}$ is a 4-vector with components (M; 2.1.30)

$$S_1 = \epsilon_1^0 X_l^0, \quad S_2 = \epsilon_2^0 X_l^0, \quad S_3 = 2\epsilon_2^2 X_l^2, \quad S_4 = 2\epsilon_4^1 X_l^1. \tag{8}$$

Substituting (6) into (2) gives,

$$\mathbf{h}(\mathbf{r}, \omega) = \mathbf{\Phi} \cdot \mathbf{P}(\mathbf{r}, \omega); \quad \mathbf{P}(\mathbf{r}, \omega) = \sum_k \mathbf{S}_k C_k(\omega)\, U_k(r) \tag{9}$$

where $\mathbf{P}(\mathbf{r}, \omega)$ is a 4-vector. For $\mathscr{H}_J$ we have

$$\mathscr{H}_J = \mathbf{\Phi} \cdot \mathscr{P} \cdot \mathbf{\Phi}^T \tag{10}$$

where

$$\mathscr{P} = \sum_i \mathbf{P}^*(\mathbf{r}, \omega_i)\, \mathbf{P}^T(\mathbf{r}, \omega_i) \tag{11}$$

and the asterisk denotes complex conjugate. In (10) the 4×4 Hermitean matrix $\mathscr{P}$ is limited to rank-4 and, therefore, so is $\mathscr{H}_J$. Consequently, the 6×6 Hermitean matrix $\mathscr{H}_J$ is singular, and, not surprisingly, the moment rate tensor cannot be retrieved from the spectrum of a single, vertically polarized accelerometer. However, if I, the number of discrete frequencies in set J, is large enough ($I \geqslant 4$ is necessary) then $\mathscr{P}$ can have rank-4. We shall assume that $\mathscr{P}$ is rank-4.

Consider two, vertically polarized accelerometers with coordinates (θ_1, ϕ_1) and (θ_2, ϕ_2). In an obvious notation (10) becomes

$$\mathscr{H}_J = \mathbf{\Phi}_1 \cdot \mathscr{P}_1 \cdot \mathbf{\Phi}_1^T + \mathbf{\Phi}_2 \cdot \mathscr{P}_2 \cdot \mathbf{\Phi}_2^T. \tag{12}$$

If $\phi_1 = \phi_2$ then $\mathbf{\Phi}_1 = \mathbf{\Phi}_2$ and (11) becomes

$$\mathscr{H}_J = \mathbf{\Phi} \cdot (\mathscr{P}_1 + \mathscr{P}_2) \cdot \mathbf{\Phi}^T \tag{13}$$

making $\mathscr{H}_J$ singular. Also, if $|\phi_1 - \phi_2| = \pi$, $\mathbf{\Phi}_1$ and $\mathbf{\Phi}_2$ are the same except for the sign of column 4. If we change the sign of column 4 and row 4 of $\mathscr{P}_2$ we have (13) again. Therefore, if the epicentre lies on the great circle through the two vertical instruments, $\mathscr{H}_J$ is singular. Included here is the special case $\theta_1 = 0, \pi$ or $\theta_2 = 0, \pi$. By a proper choice of coordinates we can always have $-\phi_2 = \phi_1 = \phi$ in (12). Assuming that $\mathscr{P}_1$ and $\mathscr{P}_2$ both have rank-4 we deal with the matrix $\mathbf{\Phi}(\phi) \cdot \mathbf{\Phi}^T(\phi) + \mathbf{\Phi}(-\phi) \cdot \mathbf{\Phi}^T(-\phi)$ which has rank-6 unless $\phi = 0, \pi/2, \pi$. Therefore, if the epicentre does not lie on the great circle through the two instruments, $\mathscr{H}_J$ is non-singular and (5) can be solved for $\mathbf{f}(\omega_J)$.

This result is important because it shows that a sparse global network of vertically polarized instruments can be used to retrieve the seismic moment rate tensor. Buland & Gilbert (1976) have shown that using ten WWSSN stations leads to a satisfactory result for $m_b = 7$.

To consider horizontally polarized instruments we must take into account toroidal as well as spheroidal modes. We introduce the 6×2 matrix $\mathbf{\Psi}$ whose non-zero elements are

$$\Psi_{21} = -\Psi_{31} = \sin 2\phi, \quad \Psi_{61} = -2\cos 2\phi, \quad \Psi_{42} = \sin\phi, \quad \Psi_{52} = -\cos\phi \tag{14}$$

and the 2-vector $\mathbf{T}$ (M; 2.1.31)

$$T_1 = \epsilon_6^2 X_l^2 \qquad T_2 = 2\epsilon_5^1 X_l^1. \tag{15}$$

In terms of $\mathbf{\Psi}$ and $\mathbf{T}$, the dyadic, A, in (M; 2.1.28) for toroidal modes is

$$\mathbf{A} = -\hat{\boldsymbol{\theta}} \csc\theta\, \partial_\phi (\mathbf{\Psi} \cdot \mathbf{T})\, W(r) + \hat{\mathbf{\Phi}} \partial_\theta (\mathbf{\Psi} \cdot \mathbf{T})\, W(r) \tag{16}$$

and

$$\begin{aligned} \mathbf{h}_2(\mathbf{r}, \omega) &= -\partial_\phi \mathbf{\Psi} \cdot \sum_k \csc\theta\, \mathbf{T}_k C_k(\omega)\, W_k(r) = -\mathbf{\Psi}' \cdot \mathbf{Q} \csc\theta \\ \mathbf{h}_3(\mathbf{r}, \omega) &= \mathbf{\Psi} \cdot \sum_k \partial_\theta\, \mathbf{T}_k C_k(\omega)\, W_k(r) = \mathbf{\Psi} \cdot \mathbf{Q}'. \end{aligned} \tag{17}$$

For spheroidal modes $\mathbf{h}_2$ and $\mathbf{h}_3$ are

$$\begin{aligned} \mathbf{h}_2(\mathbf{r}, \omega) &= \mathbf{\Phi} \cdot \partial_\theta \mathbf{D} = \mathbf{\Phi} \cdot \mathbf{D}'; \quad \mathbf{D}(\mathbf{r}, \omega) = \sum_k \mathbf{S}_k C_k(\omega)\, V_k(r) \\ \mathbf{h}_3(\mathbf{r}, \omega) &= \partial_\phi \mathbf{\Phi} \cdot \mathbf{D} \csc\theta = \mathbf{\Phi}' \cdot \mathbf{D} \csc\theta. \end{aligned} \tag{18}$$

Thus the complete 'synthetic seismograms' are

$$\begin{aligned} \mathbf{h}_2(\mathbf{r}, \omega) &= \mathbf{\Omega}_2 \cdot \mathbf{R}_2(\omega), \quad \mathbf{\Omega}_2 = \mathbf{\Phi} \oplus -\mathbf{\Psi}', \quad \mathbf{R}_2 = \mathbf{D}' \oplus \mathbf{Q} \csc\theta \\ \mathbf{h}_3(\mathbf{r}, \omega) &= \mathbf{\Omega}_3 \cdot \mathbf{R}_3(\omega), \quad \mathbf{\Omega}_3 = \mathbf{\Phi}' \oplus \mathbf{\Psi}, \quad \mathbf{R}_3 = \csc\theta\, \mathbf{D} \oplus \mathbf{Q}' \end{aligned} \tag{19}$$

where the 6×6 matrices $\mathbf{\Omega}_2$ and $\mathbf{\Omega}_3$ are functions only of ϕ and the 6-vectors $\mathbf{R}_2$ and $\mathbf{R}_3$ are functions of r, θ and ω. For both $\mathbf{\Omega}_2$ and $\mathbf{\Omega}_3$ the fifth column is proportional to the third and the sixth to the fourth. Also, columns 1 and 2 of $\mathbf{\Omega}_3$ are zero. Thus $\mathbf{\Omega}_2$ has rank-4 and $\mathbf{\Omega}_3$ has rank-2. When we sum over set J to obtain $\mathscr{H}_J$ we use the 6×6 matrices

$$\mathscr{R}_{\beta\gamma} = \sum_i \mathbf{R}_\beta^*(\omega_i)\, \mathbf{R}_\gamma^T(\omega_i); \quad \beta, \gamma = 2, 3 \tag{20}$$

and we have $\mathscr{R}_{\beta\gamma}^{\mathrm{H}} = \mathscr{R}_{\gamma\beta}$. We assume $\mathscr{R}_{\beta\gamma}$ to have rank-6. Although this assumption will be supported for ω-bands that include multiplets for several values of l, $\mathscr{R}_{\beta\gamma}$ approaches singularity for large l. This is a result of $\| \mathbf{D}' \| / \| \mathbf{D} \| = 0(l)$ for large l. The same is true for $\mathbf{Q}$. This means that $\mathscr{R}_{22}$ approaches rank-4 as an upper left 4×4 block, $\mathscr{R}_{33}$ approaches rank-2 as a lower right 2×2 block, and $\mathscr{R}_{23}$ approaches rank-2 as an upper right 4×2 block. Physically, this decomposition is a result of spheroidal modes dominating the θ-component and of toroidal modes dominating the ϕ-component for large l.

We now consider a single, horizontally polarized instrument oriented at an angle α with respect to the $\hat{\boldsymbol{\theta}}$-vector. In terms of (19) and (20) $\mathscr{H}_J$ is

$$\mathscr{H}_J = \cos^2\alpha\, \mathbf{\Omega}_2 \cdot \mathscr{R}_{22} \cdot \mathbf{\Omega}_2^T + \cos\alpha \sin\alpha\, (\mathbf{\Omega}_2 \cdot \mathscr{R}_{23} \cdot \mathbf{\Omega}_3^T + \mathbf{\Omega}_3 \cdot \mathscr{R}_{32} \cdot \mathbf{\Omega}_2^T) + \sin^2\alpha\, \mathbf{\Omega}_3 \cdot \mathscr{R}_{33} \cdot \mathbf{\Omega}_3^T. \tag{21}$$

In general, $\mathscr{H}_J$ will be non-singular. However, if $\alpha = 0$, longitudinal polarization, or $\pi/2$, transverse polarization, $\mathscr{H}_J$ will be singular because $\mathbf{\Omega}_2$ and $\mathbf{\Omega}_3$ are singular. Also, the matrices $\mathscr{R}_{\beta\gamma}$ have rank-2 for $\theta = 0$, π. This means that a source directly beneath the receiver or its antipode cannot be retrieved. Otherwise, the moment rate tensor can be retrieved from a *single*, horizontally polarized instrument. As in the previous example, for two vertical instruments, it is necessary to sum over an ω-band containing multiplets for several values of l, in order that $\mathscr{R}_{\beta\gamma}$ have full rank, and it is assumed that $\mathrm{f}(\omega)$ is nearly constant in each ω-band. This result remains true for large l even though $\mathscr{R}_{\beta\gamma}$ becomes singular. Since $\mathscr{R}_{22}$ becomes an upper left 4×4 block we can replace $\mathbf{\Omega}_2$ by $\mathbf{\Phi}$ in (21). Similarly, we can replace $\mathbf{\Omega}_3$ by $\mathbf{\Psi}$. Let

$$\begin{aligned} \mathscr{R} &= \cos^2\alpha\, \mathscr{R}_{22} + \sin^2\alpha\, \mathscr{R}_{33} + \cos\alpha \sin\alpha\, (\mathscr{R}_{23} + \mathscr{R}_{32}) \\ \mathbf{\Omega} &= \mathbf{\Phi} \oplus \mathbf{\Psi}. \end{aligned} \tag{22}$$

For large l (21) becomes

$$\mathscr{H}_J = \mathbf{\Omega} \cdot \mathscr{R} \cdot \mathbf{\Omega}^T. \tag{23}$$

In (23) we assume that I, the number of discrete frequencies in set J, is large enough to make $\mathscr{R}$ have rank-6. Since $\det \mathbf{\Omega} \neq 0$ (actually $\det \mathbf{\Omega} = 8$) we see that $\mathscr{H}_J$ has rank-6. Thus, even at short periods, the moment rate tensor can be retrieved from a single horizontal instrument unless $\alpha = 0$, $\pi/2$ or $\theta = 0$, π.

In practice, a seismographic station has two horizontal instruments, in which case it is clear that the moment rate tensor can be retrieved. Moreover, a standard installation, consisting of one vertical and two horizontal instruments certainly enables the retrieval of the moment rate tensor. Here, the only exclusion is $\theta = 0$, π.

The foregoing examples demonstrate theoretically that, except in special circumstances, the moment rate tensor of a buried point source can be retrieved from the spectra of *two* vertical accelerometers or from the spectrum of *one* horizontal accelerometer. From these theoretical results we can easily infer that a *network* of a small number of instruments can be used to retrieve source mechanisms on a routine basis. The ability to achieve such retrievals makes possible some interesting research projects.

The method presented here is an extension of the concept of matched filtering (see, e.g. Robinson 1967, pp. 259–264). The matched filters, $\mathbf{h}(\mathbf{r}, \omega)$ in (2), are the best linear filters in that they maximize the signal-to-noise ratio. For Gaussian noise they are optimum.

Although we have obtained the transfer functions $\mathbf{h}(\mathbf{r}, \omega)$ in (2) by summing normal mode multiplets, it should be emphasized that the method of retrieval is independent of the procedure used to obtain them. Any procedure for generating synthetic seismograms can be used to obtain $\mathbf{h}(\mathbf{r}, \omega)$. Therefore, matched filtering for the seismic moment tensor can be done globally, regionally or locally, depending on the magnitude of the seismic source and the configuration of the network.

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