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## AN ENUMERATION OF SURFACES IN FOUR-SPACE

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### 1. Introduction

In this paper, we consider the enumeration problem of knotted surfaces in Euclidean 4-space  $R^4$ . In classical case, there are many works on that of classical knots and links in Euclidean 3-space  $R^3$  since the 19th century (e.g. [12], [13], [14], [19]). Particularly, J.H. Conway gave a notation of classical knots and links in  $R^3$ , the so called tangle, which is suitable for machine computation, and he listed all classical knots of at most 11 crossings and all classical links of at most 10 crossings [4]. In 4-dimensional case, the author made a table of knotted surfaces in  $R^4$  with ch-index 10 or less [22], which will appear as an appendix of this paper. (The ch-index of a knotted surface will be defined in Section 2.) The purpose of this paper is to show a method of enumerating knotted surfaces in  $R^4$  which is used to make the table in [22].

We work in the piecewise-linear (or smooth) category. By a surface in  $R^4$  (or  $S^4$ ), we mean a closed and locally-flat (possibly disconnected or non-orientable) surface embedded in  $R^4$  (or  $S^4$ ) unless otherwise stated. In Section 2, we study a diagram of a surface  $F$  in  $R^4$  and define the ch-index of  $F$ , which is a numerical invariant of the knot type of surfaces in  $R^4$  and has a property that, for each  $n \geq 0$ , the number of the knot types of non-splittable surfaces in  $R^4$  with ch-index  $n$  is finite. Thus it is suitable for enumeration of surfaces in  $R^4$ . In Section 3, we introduce a graph of a surface in  $R^4$ . In Section 4, we explain how to list all surfaces with ch-index  $n$  in  $R^4$  for each  $n$  by using graphs. In Section 5, we give some remarks on surfaces of [22]. Appendix contains two tables. The first is the table of surfaces in  $R^4$  which was given in [22]. The other is that of their groups and first elementary ideals.

### 2. Diagrams of surfaces

In order to list surfaces in 4-space, we first need a convenient way of describing such surfaces. We use Fox's motion picture representation

of a surface in  $R^4$ , i.e., a representation in terms of a parametrized family of 3-dimensional cross-sections (cf. [5], [8], [9]). Moreover we know that essentially only one single 3-dimensional cross-section, the so called diagram, is needed (cf. [16]).

Two surfaces  $F$  and  $F'$  in  $R^4$  are said to be *of the same knot type* or *equivalent* if there exists an orientation preserving homeomorphism  $\Psi$  of  $R^4$  onto itself such that  $\Psi(F) = F'$ . We denote by  $R_t^3$  the hyperplane of  $R^4$  whose fourth coordinate  $x_4$ , is  $t$ , i.e.,  $R_t^3 = \{(x_1, x_2, x_3, x_4) \in R^4: x_4 = t\}$ .

**Proposition 2.1** ([5], [8], [9]). *For any surface  $F$  in  $R^4$ , there exists a surface  $\tilde{F}$  in  $R^4$  satisfying the following:*

(0)  *$\tilde{F}$  is equivalent to  $F$  and has only finitely many critical points, all of which are elementary.*

(1) *All maximal points of  $\tilde{F}$  are in the hyperplane  $R_1^3$ .*

(2) *All minimal points of  $\tilde{F}$  are in the hyperplane  $R_{-1}^3$ .*

(3) *All hyperbolic points of  $\tilde{F}$  are in the hyperplane  $R_0^3$ .*

We call such a representation  $\tilde{F}$  a *hyperbolic splitting* of  $F$ . A hyperbolic splitting of the spun 2-knot of the trefoil is shown in Fig. 2.1.

The entire surface can be completely reconstructed from the 0-level cross-section  $F_0 = \tilde{F} \cap R_0^3 \subset R_0^3$  and a set of labels (one for each hyperbolic point) indicating how the hyperbolic points open up above, i.e., for  $t > 0$  (cf. [9]). We thus obtain the following convenient representation of surfaces in  $R^4$ .

**DEFINITION 2.2.** Suppose that a surface  $F$  in  $R^4$  is described by a hyperbolic splitting. Then a *diagram* of  $F$  is the 0-level cross-section  $F_0 = \tilde{F} \cap R_0^3 \subset R_0^3$  with hyperbolic points labeled as shown in Fig. 2.2.

By Proposition 2.1, any surface in  $R^4$  can be represented by some diagram. Diagrams of the spun 2-knot of the trefoil, two projective planes  $P_+$  and  $P_-$  in  $R^4$ , called the *standard projective planes*, and the standard torus  $T_1$  of genus one are shown in Fig. 2.3. We usually describe a surface diagram by its regular projection on  $S^2$  with over and under crossings indicated in the standard way and with hyperbolic points labeled.

Let  $(S_i^4, F_i)$  be a pair of an oriented 4-sphere  $S_i^4$  and a surface  $F_i$  in  $S_i^4$  ( $i=1,2$ ). Consider the connected sum  $(S_1^4 \# S_2^4, F_1 \# F_2)$  of the pairs with respect to the orientations of  $S_1^4$  and  $S_2^4$ . The surface  $F_1 \# F_2$  in

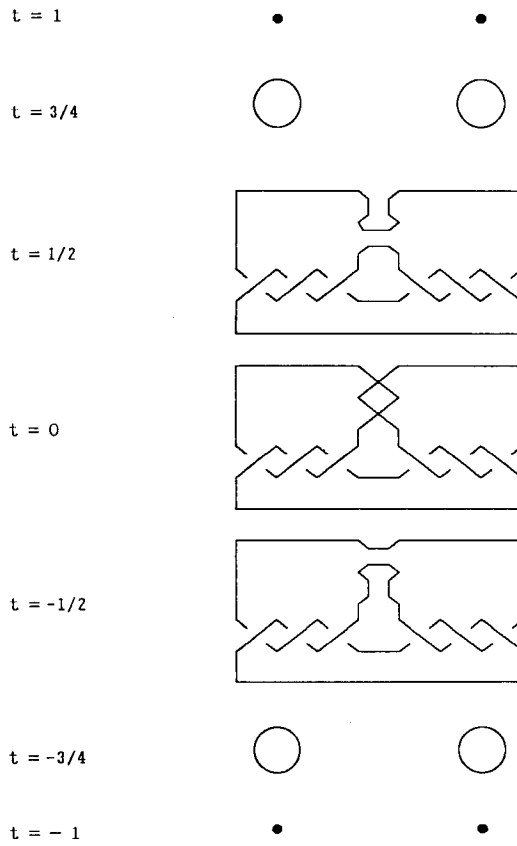


Fig. 2.1

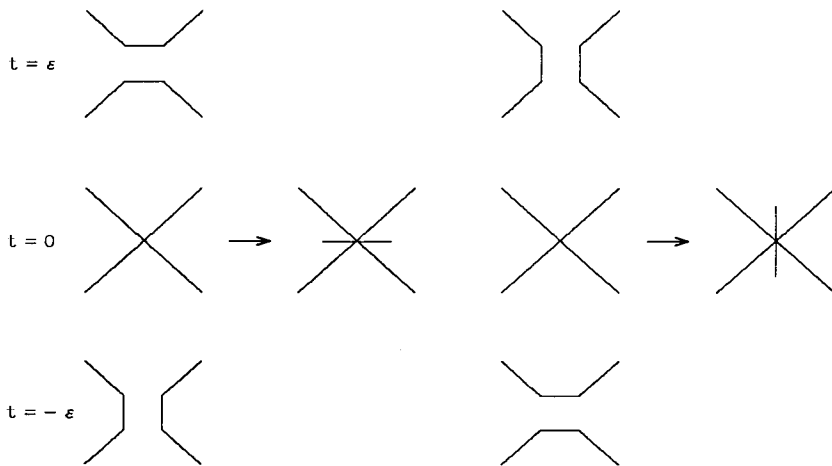
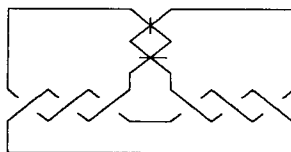


Fig. 2.2



the spun 2-knot of the trefoil

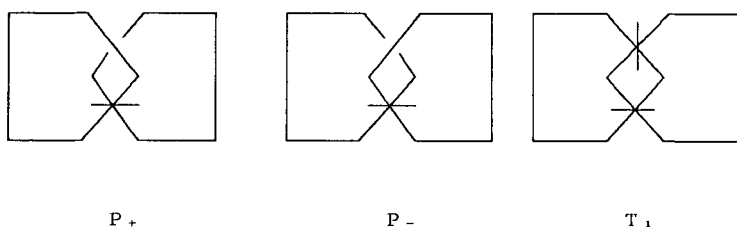


Fig. 2.3

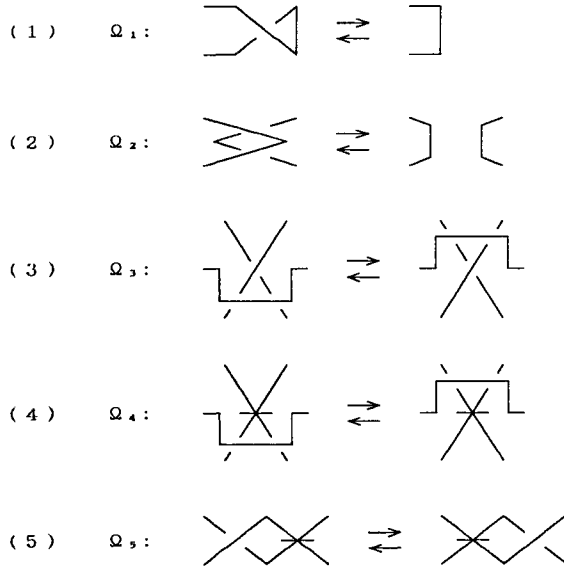
the oriented 4-sphere  $S^4 = S_1^4 \# S_2^4$  is called the *knot sum* of  $F_1$  and  $F_2$ . The knot sum of surfaces  $F_1$  and  $F_2$  in  $R^4$  is similarly defined. A connected orientable surface in  $R^4$  is said to be *unknotted* or *trivial* if it bounds a solid torus of the same genus in  $R^4$ . If a connected non-orientable surface  $F$  in  $R^4$  is equivalent to the knot sum of some copies of the standard projective planes  $P_+$  or  $P_-$ , we say that  $F$  is *unknotted* or *trivial*. The knot type of an unknotted, orientable surface in  $R^4$  is uniquely determined by only the genus, while that of an unknotted, non-orientable surface in  $R^4$  is uniquely determined by the (non-orientable) genus and the Euler number [7]. If a disconnected surface  $F$  in  $R^4$  is completely splittable and each (connected) component is unknotted, then  $F$  is called *unknotted* or *trivial*.

Equivalent surfaces in  $R^4$  may be described by many different diagrams, but some of them are connected by simple operations.

**DEFINITION 2.3.** Two surface diagrams are called *stably equivalent* if they are connected by a finite sequence of the operations  $\Omega_i, i=1, \dots, 8$ , described in Fig. 2.4 or their mirror image operations. (For the sake of simplicity, we omit the figures of the mirror image operations.)

The operations  $\Omega_i$  effect local changes in the diagram. It is known that all these operations can be realized by an ambient isotopy of  $R^4$

## TYPE I



## TYPE II

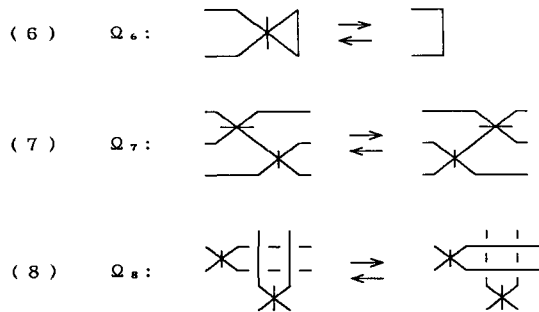
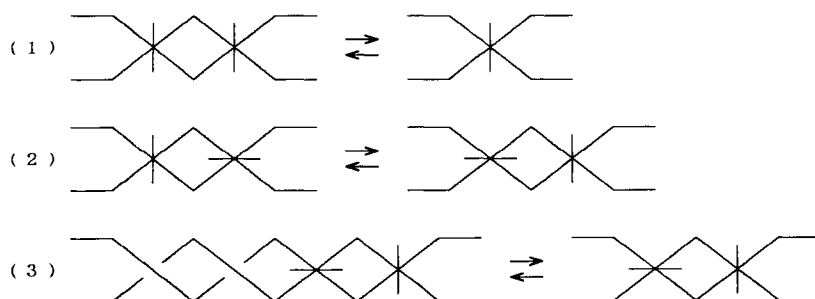


Fig. 2.4

(e.g. [9], [17]). Therefore, stably equivalent diagrams define equivalent surfaces in  $R^4$ . It remains open whether the converse is also true:

**QUESTION 2.4.** Are all diagrams of equivalent surfaces in  $R^4$  stably equivalent?

**Proposition 2.5.** *The following stable equivalences of surface diagrams hold:*



Proof. We first see that the operation  $\Omega'_7$  in Fig. 2.5 can be derived from the operations  $\Omega_i, i=1, \dots, 8$ , as shown in Fig. 2.6. Therefore, by (i), (ii) and (iii) of Fig. 2.7, we can verify that (1), (2) and (3) are stable equivalences of surface diagrams.

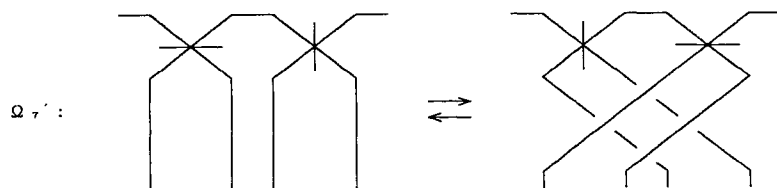


Fig. 2.5

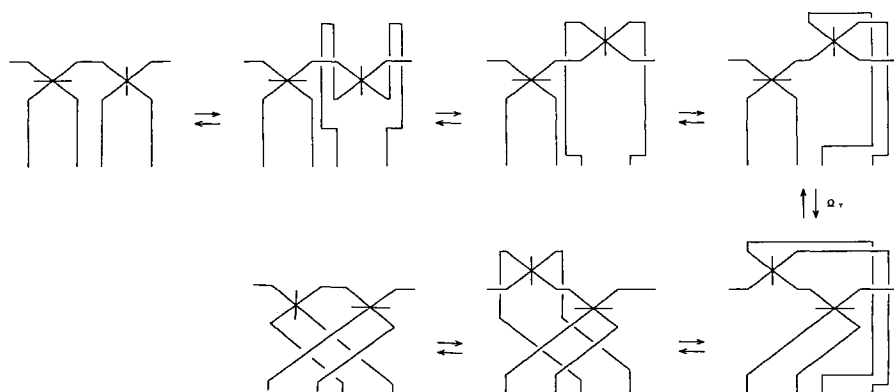


Fig. 2.6

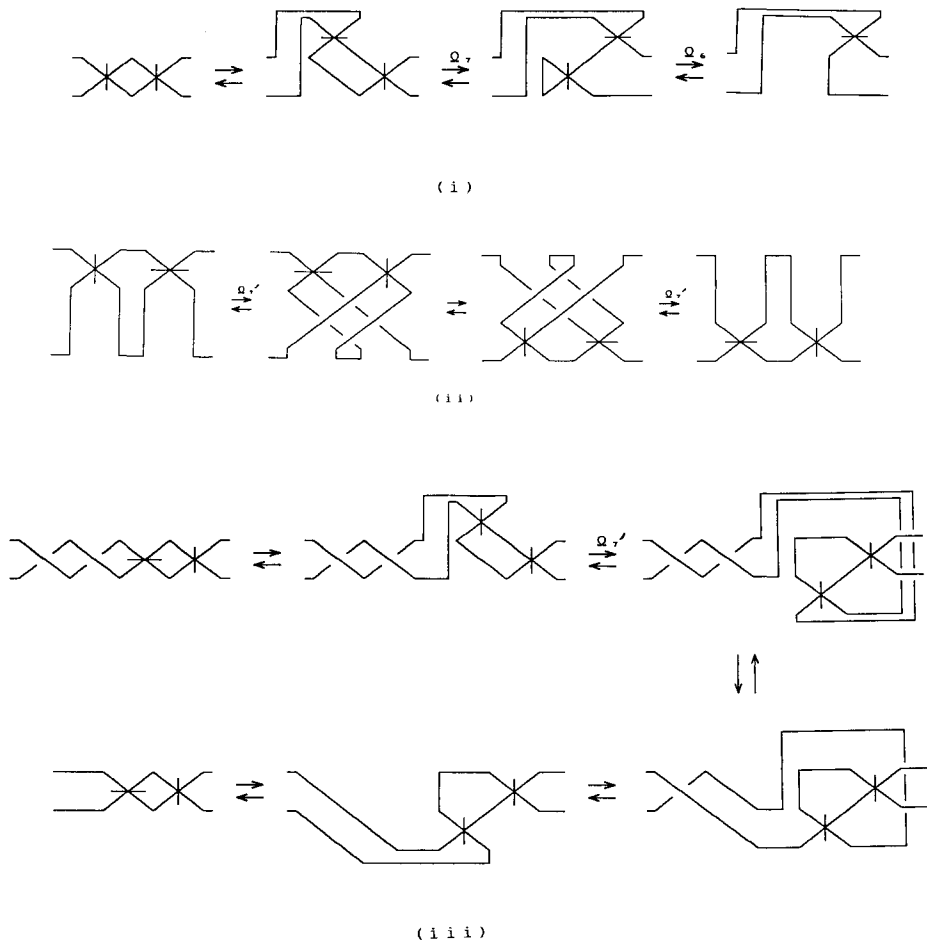


Fig. 2.7

**Proposition 2.6.** *Two surface diagrams in each of Fig. 2.8 (1) and (2) represent equivalent surfaces in  $R^4$ .*

**Proof.** To prove (1), it suffices to show the equivalence in Fig. 2.9. Let  $F$  be the surface in  $R^4$  represented by the diagram (i) of Fig. 2.9 and  $p$  the hyperbolic point of  $F$  shown in Fig. 2.9 (i). First pushing the hyperbolic point  $p$  into the level  $\epsilon_1$  (see Fig. 2.10 (i)), where  $0 < \epsilon_1 < \epsilon_2 < 1$ , we get a surface  $F'$  in  $R^4$ , which is equivalent to  $F$ . Let  $\alpha$  be the loop shown in Fig. 2.10 (i). Since  $F'$  is also locally flat, the  $\epsilon_2$ -level cross-section  $F' \cap R_{\epsilon_2}^3$  is a trivial 1-link. Therefore there exists a 2-disk  $D^2$  spanning  $\alpha$  in  $R_{\epsilon_1}^3$  such that  $D^2 \cap (F' \cap R_{\epsilon_1}^3) = \alpha$  (i.e.,  $\alpha$  bounds



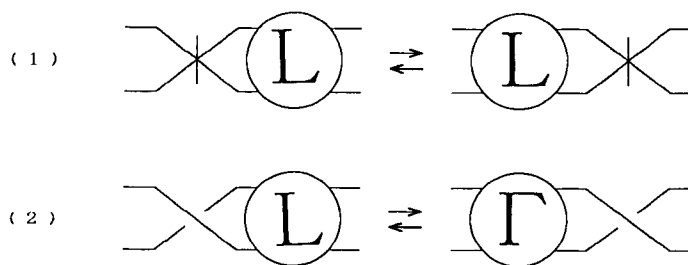


Fig. 2.8

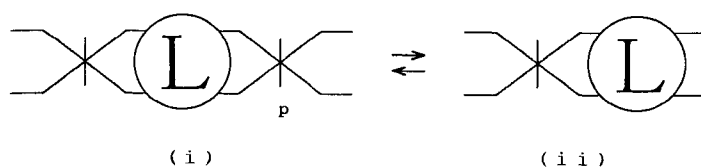


Fig. 2.9

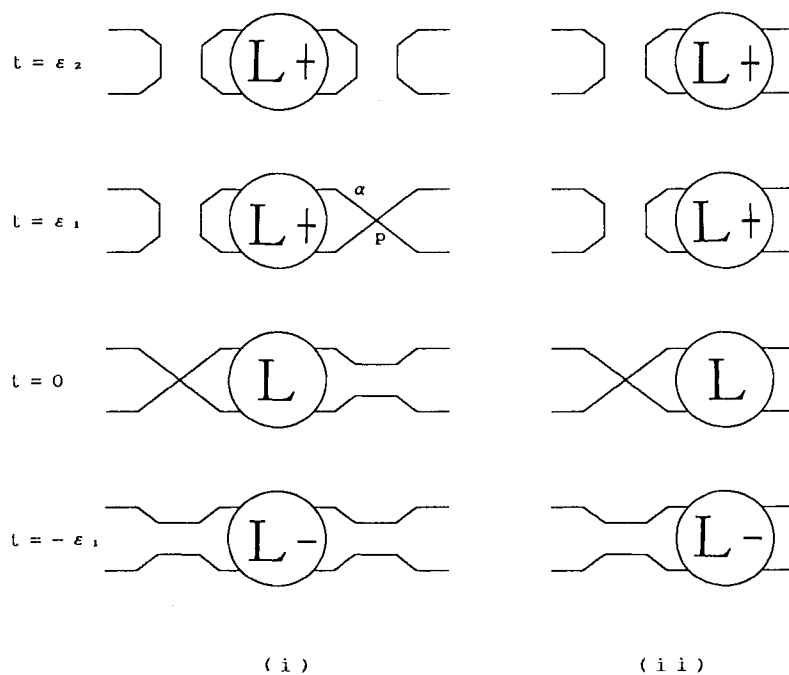


Fig. 2.10

a cusp of  $F$ ). Hence, by an ambient isotopy of  $R^4$  keeping  $R_t^3$ ,  $t \leq 0$ , fixed, we can slide  $p$  along  $D^2$  and remove it. In this deformation, we may assume that no new hyperbolic points appear (see Fig 2.10 (ii)). Thus we obtain the diagram (ii) in Fig. 2.9, which defines the surface equivalent to  $F$ . It is clear that (2) holds. This completes the proof.

**DEFINITION 2.7.** Denote by  $c(D)$  and  $h(D)$  the numbers of the crossings and the hyperbolic points of a diagram  $D$  of a surface in  $R^4$ , respectively. The sum of  $c(D)$  and  $h(D)$  is called the *ch-index* of  $D$ , denoted by  $ch(D)$ . On all diagrams representing a surface  $F$  in  $R^4$ , the minimal number of ch-indices is called the *ch-index* of  $F$ , denoted by  $ch(F)$ .

**REMARK 2.8.** (1) If a surface  $F$  in  $R^4$  is the split union of two surfaces  $F_1$  and  $F_2$ , then it holds that  $ch(F) = ch(F_1) + ch(F_2)$ .

(2) We consider the additivity of the ch-index with respect to the knot sum of surfaces in  $R^4$ . For the standard projective planes  $P_+$  and  $P_-$  in  $R^4$ , we have  $ch(P_+) = ch(P_-) = 2$  and  $ch(P_+ \# P_-) = 3$  (see Fig. 2.11). Hence we see that  $ch(P_+ \# P_-) < ch(P_+) + ch(P_-)$ . As another example, let  $K$  be a non-trivial 2-knot such that  $K \# P_+ = P_+$  (e.g., the 3-twist spun 2-knot of the trefoil). Then, since  $ch(K) > 0$ , it follows that

$$ch(K \# P_+) = 2 < ch(K) + ch(P_+).$$

Thus, in general, the additivity does not hold for non-orientable surfaces in  $R^4$ .

(3) Let  $F$  be a connected trivial surface of genus  $g$  in  $R^4$ . If  $F$  is orientable, then we have  $ch(F) = 2g$ . If  $F$  is non-orientable, then we have  $ch(F) = g + 1$  or  $g + |e(F)/2|$  according as  $e(F) = 0$  or not, where  $e(F)$  is the Euler number of  $F$ .

We have the following question:

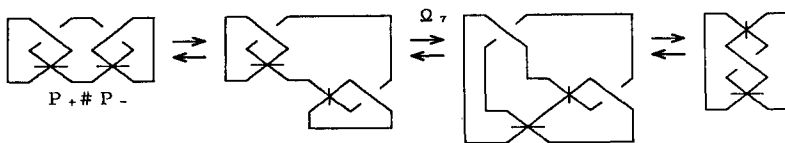


Fig. 2.11

QUESTION 2.9. For arbitrary orientable surfaces  $F_1$  and  $F_2$  in  $R^4$ , does it hold that

$$ch(F_1 \# F_2) = ch(F_1) + ch(F_2)?$$

**Theorem 2.10.** For any 2-knot  $K$  in  $R^4$ , it holds that

$$ch(K) \geq 2d + r + m - 3,$$

where  $d$  is the minimal degree of polynomials  $f(t)$  ( $\in Z[t^{\pm 1}]$ ) of the first elementary ideal of  $K$  such that  $f(1) = \pm 1$ ,  $r$  is the minimal number of generators of Wirtinger presentations of  $\pi_1(R^4 - K)$ , and  $m$  is the minimal number of elliptic points of 2-knots which are equivalent to  $K$ . In particular, if  $r = 2$ , it holds that

$$ch(K) \geq 2d + m.$$

Proof. Let  $D$  be a surface diagram of  $K$  such that  $ch(D) = ch(K)$ . By Euler's formula, we have

$$(1) \quad h(D) \geq m - 2.$$

The 2-knot  $K$  can be deformed into a 2-knot  $K'$  in the normal form (see[9]) such that

$$(2) \quad c(k) = c(D),$$

where  $c(k)$  is the number of crossings of the equatorial cross-sectional 1-knot  $k$ . Then, by Euler's formula, it holds that

$$(3) \quad c(k) \geq 2g(k) + b(k) - 1,$$

where  $g(k)$  and  $b(k)$  are the genus and the braid index of  $k$ . Moreover, for a 1-knot, it is well-known that

$$(4) \quad 2g(k) \geq \deg \Delta_k(t) \text{ and } b(k) \geq r_k,$$

where  $\Delta_k(t)$  is the Alexander polynomial of  $k$  and  $r_k$  is the minimal number of generators of Wirtinger presentations of the group of  $k$ . Since the the group of  $K$  is a quotient of the group of  $k$ , we see that

$$(5) \quad r_k \geq r.$$

Let  $K_+ = K' \cap R_+^4 \subset R_+^4$ , where  $R_+^4 = \{(x_1, x_2, x_3, x_4) \in R^4: x_4 \geq 0\}$ . Then  $K_+$  is a 2-disk properly embedded in  $R_+^4$  such that  $\partial K_+ = k$ . Hence  $\Delta_k(t)$  is equal to  $\Delta_{K_+}(t) \cdot \Delta_{K_+}(t^{-1})$  up to units of  $Z[t^{\pm 1}]$ , where  $\Delta_{K_+}(t)$  is

the Alexander polynomial of  $K_+ \in R_+^4$ . Therefore, since the group of  $K$  is a quotient of the group of  $K_+ \in R_+^4$ , we have

$$(6) \quad \deg \Delta_k(t) = 2 \deg \Delta_{K_+}(t) \geq 2d.$$

Thus, from (1) – (6), we see that

$$\begin{aligned} ch(K) &= ch(D) = c(D) + h(D) \\ &\geq (2g(k) + b(k) - 1) + (m - 2) \geq 2d + r + m - 3. \end{aligned}$$

Next assume that  $r = 2$ . Therefore, we have  $b(k) \geq 2$ . Since the 1-knot  $k$  is slice, it follows that  $b(k) \neq 2$ . Hence we get  $b(k) \geq 3$ . Therefore we see that

$$ch(K) \geq (2g(k) + b(k) - 1) + (m - 2) \geq 2d + m.$$

The proof is completed.

EXAMPLE 2.11. Let  $K$  be the spun 2-knot of the  $(2n+1, 2)$ -torus 1-knot ( $n > 0$ ). Then we have  $r = 2$ ,  $m = 4$  and  $d = 2n$ . By Theorem 2.10 and Fig. 2.12, it follows that

$$2(2n+1) + 2 \geq ch(K) \geq 2 \cdot 2n + 4.$$

Therefore we obtain  $ch(K) = 4(n+1)$ . Thus the second inequality of Theorem 2.10 is rough but best possible.

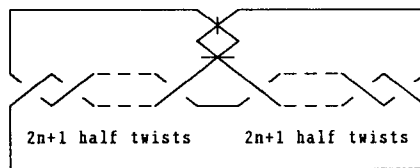


Fig. 2.12

REMARK 2.12. Let  $F$  be a connected surface in  $R^4$  and  $g(F)$  the genus of  $F$ . Then, in the same way as in the proof of Theorem 2.10, we can prove that

$$ch(F) \geq 2d + r + m + \gamma - 3,$$

and, if  $r = 2$ , then

$$ch(F) \geq 2d + m + \gamma,$$

where  $\gamma = 2g(F)$  or  $g(F)$  according as  $F$  is orientable or not.

### 3. Graphs of surfaces

In the way analogous to the definition of the graphs of 1-knots and 1-links([1], [21]), we will also introduce a graph of a surface in  $R^4$ .

Let  $D$  be a diagram of a surface  $F$  in  $R^4$ . Assume that  $D$  is described by its connected regular projection on a 2-sphere  $S^2$ . Then  $D$  divides  $S^2$  into several regions. Since the degree of each vertex of  $D$  is 4, the regions can be colored by two colors  $\alpha$  and  $\beta$  like a chess-board such that adjacent regions are never of the same color (see Fig. 3.1). Denote by  $\alpha_i, 1 \leq i \leq m$ , the  $\alpha$ -regions. Define a graph  $\Gamma$  whose vertices  $v_i$  correspond to the  $\alpha_i$ , and whose edges  $e_{ij}^k$  correspond to the double points and vertices  $A_k$  of  $D$ , where  $e_{ij}^k$  joins  $v_i$  and  $v_j$  (see Fig. 3.1). We label the edges as shown in Fig. 3.2. Then the graph  $\Gamma$  with labeled edges is called the *graph* of the surface  $F$  (with respect to the diagram  $D$ ). If a surface diagram  $D$  is described by a disconnected regular projection, we construct the graph  $\Gamma_i$  for each component  $D_i$  of  $D$  such that the  $\Gamma_i$  are pairwise disjoint. We then call  $\Gamma = \cup_i \Gamma_i$  the *graph* of  $F$  (with respect to  $D$ ). If the  $\beta$ -regions are used instead of the  $\alpha$ -regions, then another graph  $\Gamma^*$  is obtained from the diagram  $D$ . It is called the *dual graph* of  $\Gamma$ .

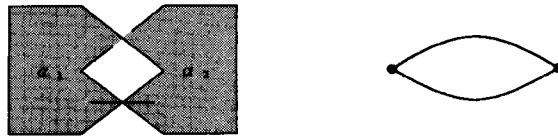


Fig. 3.1

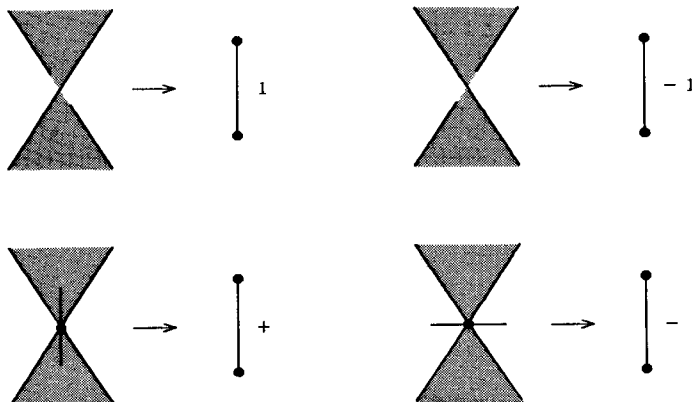


Fig. 3.2

Graphs of the spun 2-knot of the trefoil, the standard projective plane  $P_+$  and the standard torus  $T_1$  of genus one are illustrated in Fig. 3.3.

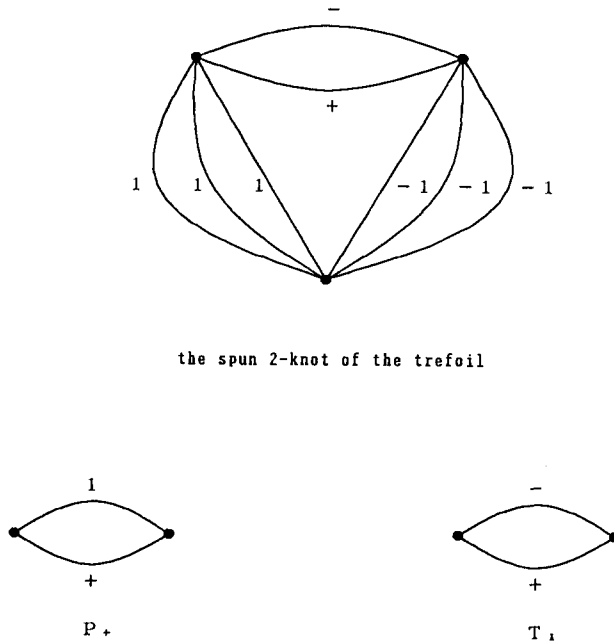


Fig. 3.3

The deformations  $\Omega_i$ ,  $i=1, \dots, 8$ , of surface diagrams can be translated into the operations on graphs as shown in Fig. 3.4. (For the sake of simplicity, we omit the figures of the mirror image operations.) Since there are two graphs  $\Gamma$  and  $\Gamma^*$  for a surface diagram  $D$ , we have two operations  $O_i$  and  $O_i^*$  on graphs for each deformation  $\Omega_i$ . Such operations will be called the *dual* operations of each other. In case that  $i=3, 4, 7$ , we note that  $O_i$  and  $O_i^*$  coincide.

TYPE I

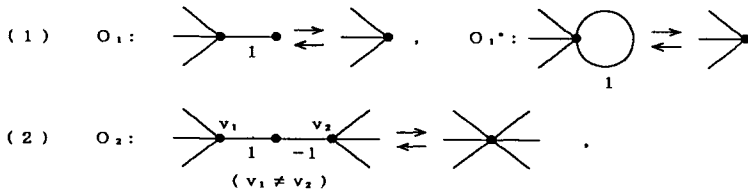


Fig. 3.4

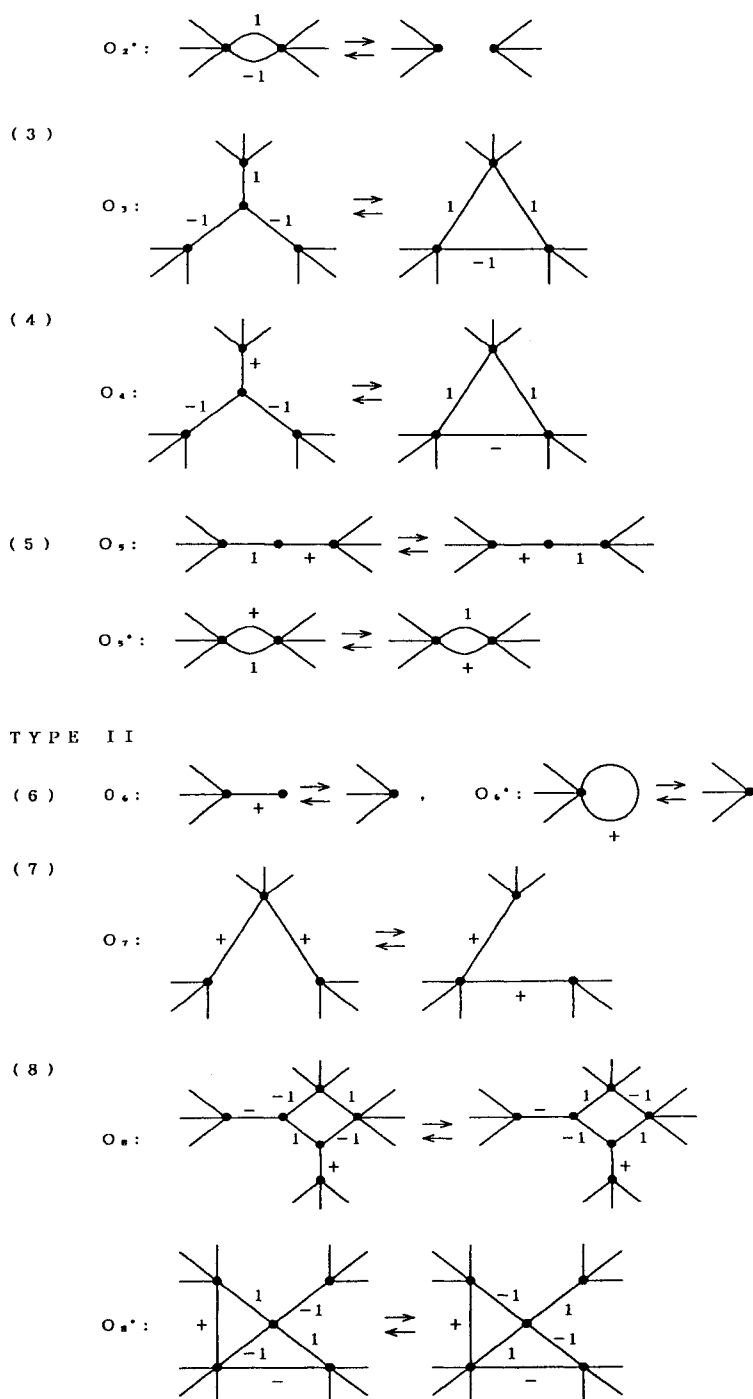


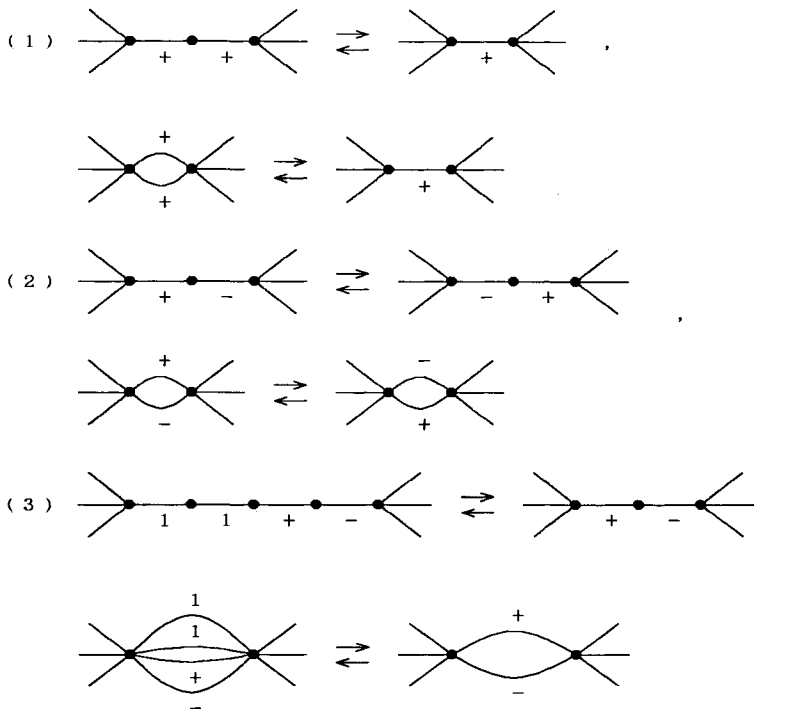
Fig. 3.4

By the similar argument as in (3.6) of [21, p. 159], we can prove the following:

**Proposition 3.1.** *Let  $\Gamma$  be the graph of a surface in  $R^4$ . Then the dual graph  $\Gamma^*$  can be obtained from  $\Gamma$  by a finite sequence of the operations  $O_i$  and  $O_i^*$ ,  $i=1, \dots, 8$ , in Fig. 3.4.*

From Propositions 2.5 and 2.6, we obtain the following two propositions, respectively:

**Proposition 3.2.** *The following operations on graphs of surfaces in  $R^4$  are derived from the operations  $O_i$  and  $O_i^*$ ,  $i=1, \dots, 8$ , in Fig. 3.4:*



**Proposition 3.3.** *Two graphs in each of (1) – (4) of Fig. 3.5 represent equivalent surfaces in  $R^4$ .*



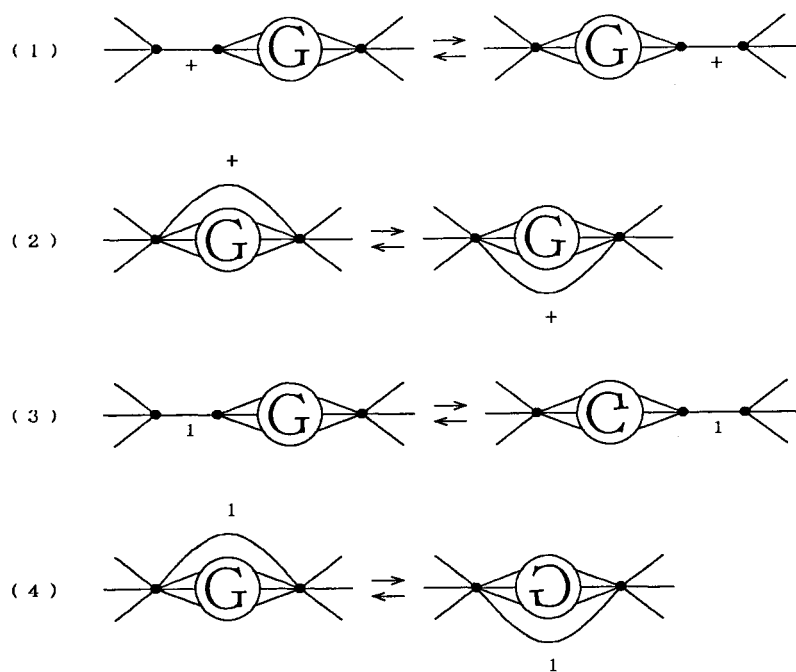


Fig. 3.5

Applying (2) and (4) of Proposition 3.3 to a loop edge, we have the following corollary:

**Corollary 3.4.** *Two graphs in each of (1) and (2) of Fig. 3.6 represent equivalent surfaces in  $R^4$ .*

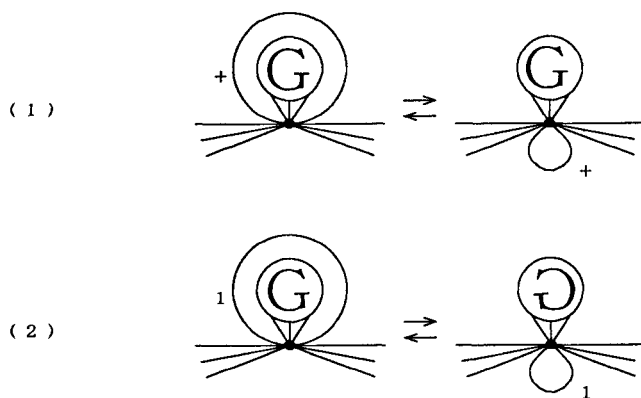


Fig. 3.6

From these results, we obtain a simple graph representation of a surface in  $R^4$  as follows (A graph is said to be *simple* if it has neither loops nor multiple edges.):

**Theorem 3.5.** *For any surface  $F$  in  $R^4$ , there exists a simple graph on  $S^2$  whose edges are labeled as shown in Fig. 3.7 and which represents  $F$ .*

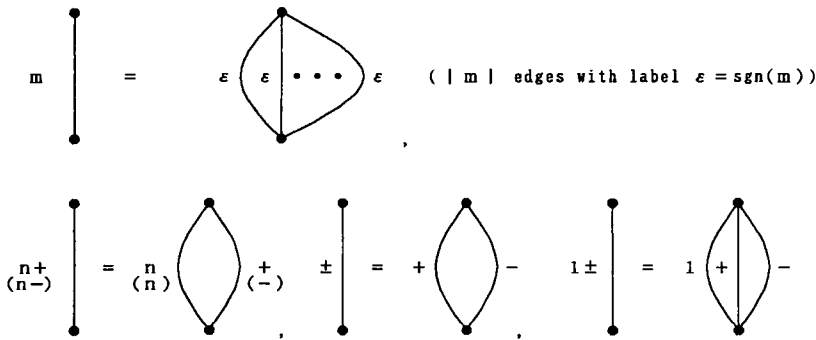


Fig. 3.7

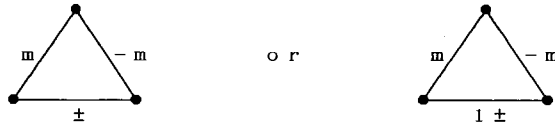
**Proof.** Let  $\Gamma$  be the graph of  $F$ . If  $\Gamma$  has a loop, then we remove it by using Corollary 3.4, the operations  $O_1^*$  or  $O_6^*$  in Fig. 3.4. Suppose that  $\Gamma$  has a multiple edge. Then, by the operations  $O_2^*$ ,  $O_5^*$  and Propositions 3.2, 3.3, we can replace it with a simple edge labeled as shown in Fig. 3.7. The proof is completed.

**REMARK 3.6.** Note that each deformation of graphs in the proof of Theorem 3.5 does not increase ch-index.

The following theorem holds [23]:

**Theorem 3.7.** *Let  $F$  be a surface in  $R^4$  which has a graph with at most three vertices. Then  $F$  is one of the following surfaces:*

- (1) *unknotted surfaces,*
- (2) *surfaces represented by graphs*



where  $m \geq 2$ .

**Corollary 3.8.** *Let  $F$  be a surface in  $R^4$ . If  $ch(F) \leq 5$ , then  $F$  is unknotted.*

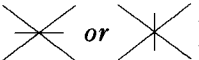
This corollary follows from Theorem 3.7 and the following lemma:

**Lemma 3.9.** *Let  $F$  be a non-splittable surface in  $R^4$ . Then  $F$  can be represented by a graph with at most  $[(ch(F) + 2)/2]$  vertices, where  $[x]$  denotes the greatest integer that does not exceed  $x$ .*

*Proof.* Let  $D$  be a diagram of  $F$  such that  $ch(D) = ch(F)$ , and let  $\Gamma$  be the graph of  $F$  with respect to  $D$ . Then, as a regular projection,  $D$  divides  $S^2$  into  $ch(F) + 2$  regions. Therefore one of the graph  $\Gamma$  and the dual graph  $\Gamma^*$  has at most  $[(ch(F) + 2)/2]$  vertices.

#### 4. Enumeration of surfaces

A diagram of a surface in  $R^4$  is considered as a 4-valent graph (possibly containing  $S^1$  as a component) with labeled vertices in  $R^3$ . Therefore, we have

**Proposition 4.1.** *A 4-valent graph  $D$  in  $R^3$  with labeled vertices (i.e., ) is a diagram of some surface in  $R^4$  if and only if  $L_+(D)$  and  $L_-(D)$  are trivial 1-links in  $R^3$ , where  $L_+(D)$  is the 1-link in  $R^3$  obtained from  $D$  by changing each vertex*



by



respectively, and similarly  $L_-(D)$  is the 1-link in  $R^3$  obtained from  $D$  by changing each vertex



by

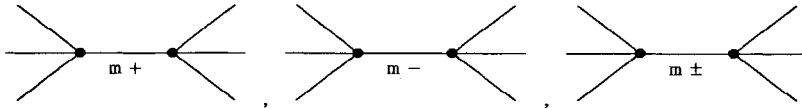


respectively.

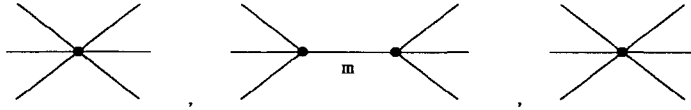
**Proof.** The definition of a diagram of a surface in  $R^4$  implies the proposition.

Similarly, for a graph of a surface in  $R^4$ , we have the following:

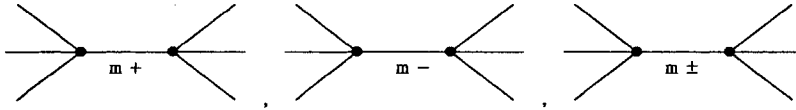
**Proposition 4.2.** A plane graph  $\Gamma$  in  $S^2$  with edges labeled as shown in Fig 3.6 is a graph of some surface in  $R^4$  if and only if  $G_+(\Gamma)$  and  $G_-(\Gamma)$  represent trivial 1-links in  $R^3$ , where  $G_+(\Gamma)$  is the graph in  $S^2$  obtained from  $\Gamma$  by changing each edge



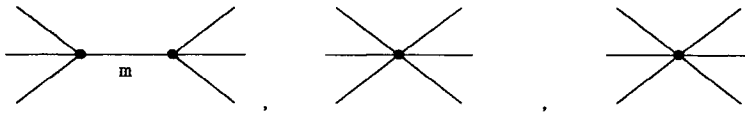
by



respectively, and similarly  $G_-(\Gamma)$  is the graph in  $S^2$  obtained from  $\Gamma$  by changing each edge



by



respectively.

For each  $n$ , since the number of connected planar graphs with  $n$  edges is finite, we can enumerate all non-splittable surfaces with ch-index  $n$  in  $R^4$  as follows:

*Step 1:* Enumerate all (abstract) graphs satisfying the following:

- (1) The numbers of the vertices and the edges are at most  $[(n+2)/2]$  and  $n$ , respectively.
- (2) They are planar, simple and connected.
- (3) They have no cut vertices.

Let  $\Gamma$  be such a graph.

*Step 2:* Label each edge of  $\Gamma$  by one of  $\alpha$ ,  $\beta+$ ,  $\beta-$ ,  $\pm$  and  $1\pm$  such that  $\text{ch-index} = n$ , where  $\alpha (\neq 0)$  and  $\beta$  are integers.

*Step 3:* Embed  $\Gamma$  in  $S^2$  and decide whether or not  $\Gamma$  represents a surface in  $R^4$ .

REMARK 4.3. (1) When the number  $v$  of the vertices is less than 4, we can apply Theorem 3.7.

(2) The condition (3) in Step 1 is needed for a surface to be prime.

(3) For a diagram  $D$  of a surface  $F$ , it holds that, if  $h(D) \leq 1$  or  $c(D) \leq 3$ , then  $F$  is unknotted. (For the first case, see [2], [10], [11], [18]. For the second, see [10].) Therefore, in Step 2, we may consider only the case  $4 \leq c(D) \leq n-2$ .

(4) In Step 3, to decide whether  $\Gamma$  represents a surface in  $R^4$ , we apply Proposition 4.2. There exists an algorithm for deciding whether a given projection of a 1-knot represents the trivial knot type [6], but its application is complicated. Thus it is more practical to use invariants of the graphs of 1-links (1-knots). The  $Q$  polynomial  $Q(l) \in Z[x^{\pm 1}]$  is an invariant of the knot type of an unoriented 1-link (1-knot)  $l$  in  $R^3$  which is calculated from the diagram [3]. It is also obtained from the graph  $G$  of  $l$  as follows:

(1)  $Q(G) = (2x^{-1} - 1)^{c-1}$  for the graph  $G$  of the trivial 1-link with  $c$  components.

(2)  $Q(G_+) + Q(G_-) = x(Q(G_0) + Q(G_\infty))$ ,

where  $G_+$ ,  $G_-$ ,  $G_0$  and  $G_\infty$  are graphs of 1-links as shown in Fig. 4.1, respectively.

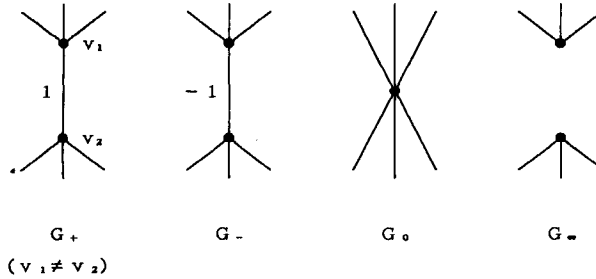


Fig. 4.1

Let  $G$  be the graph of a 1-link (1-knot) in  $R^3$  with the vertices  $v_1, \dots, v_v$ . Let  $E_{ij}$  ( $i, j = 1, \dots, v$ ) denote the set of all edges of  $G$  whose endpoints are the vertices  $v_i$  and  $v_j$ . The Goeritz matrix  $M$  of  $G$  is the  $v \times v$  matrix defined by the following:

$$m_{ij} = \sum_{e \in E_{ij}} \eta(e), \quad (i \neq j),$$

$$m_{jj} = - \sum_{i \neq j} m_{ij},$$

where  $\eta(e)$  denotes the label of the edge  $e$ . There is a unimodular matrix  $U$  such that  $UMU^T$  is diagonal. Let  $\tau(G) = |\prod_{a_{ii} \neq 0} a_{ii}|$ , where  $a_{ii} (i=1, \dots, v)$  is the  $ii$ -entry of the matrix  $UMU^T$ . It is known that  $\tau(G)$  and its nullity are invariants of the knot type of the 1-link  $L$ . Then, it holds that if a graph  $G$  represents the trivial 1-link with  $c$  components then  $\tau(G)=1$  and  $Q(G)=(2x^{-1}-1)^{c-1}$ . For 1-links of less than 9 crossings, the converse is also true. (It is unknown whether there exists a non-trivial 1-link whose  $Q$  polynomial is  $(2x^{-1}-1)^{c-1}$ .)

## 5. Some remarks on surfaces in Table I

A surface in  $R^4$  is called *irreducible* if it is not the knot sum  $F_1 \# F_2$ , where  $F_1$  is any surface and  $F_2$  is the standard projective planes  $P_+$ ,  $P_-$  or the standard torus of genus one. All surfaces in Table I except for  $2_1^1$  and  $2_1^{-1}$  are irreducible. A surface  $F$  in  $R^4$  is said to be *prime* if  $F$  is not the knot sum of any two surfaces in  $R^4$  that are not trivial 2-knots. The standard projective planes  $P_+$  and  $P_-$  are neither irreducible nor prime. In general, it is not easy to determine whether a given surface in  $R^4$  is prime. For example, it is unknown whether the trivial 2-knot is prime. We now introduce a weaker primeness for surfaces in  $R^4$ .

**DEFINITION 5.1.** A surface  $F$  in  $R^4$  is said to be *weakly prime* if  $F$  is not the knot sum of any two surfaces  $F_1$  and  $F_2$  in  $R^4$  such that  $ch(F_i) < ch(F)$ ,  $i=1, 2$ .

Any prime surface in  $R^4$  is weakly prime. We have the following:

**Proposition 5.2.** Any surface  $F$  in  $R^4$  is either a weakly prime one or the knot sum of finitely many weakly prime surfaces  $F_1, \dots, F_m$  in  $R^4$  such that  $ch(F_i) < ch(F)$ ,  $i=1, \dots, m$  ( $m \geq 2$ ).

**Proof.** We use induction on the ch-index of  $F$ . If  $ch(F)=0$ , then  $F$  is weakly prime. Assume that  $ch(F)>0$ . If  $F$  is not weakly prime, then there exist surfaces  $W_1$  and  $W_2$  in  $R^4$  with  $ch(W_i) < ch(F)$ ,  $i=1, 2$ , such that  $F=W_1 \# W_2$ . If  $W_i$ ,  $i=1, 2$ , are weakly prime, then the proposition holds. Therefore, suppose that  $W_i$  is not weakly prime. Since  $ch(W_i) < ch(F)$ , it follows from the inductive hypotheses that  $W_i$  is the knot sum of finitely many weakly prime surfaces  $W_{i1}, \dots, W_{in_i}$  in  $R^4$  such that  $ch(W_{ij}) < ch(W_i)$ ,  $j=1, \dots, n_i$ . This completes the proof.

Thus it is reasonable to list all weakly prime surfaces in  $R^4$ . All

surfaces in Table I are weakly prime.

All surfaces in Table I are distinct esch other. (The surfaces  $8_1^{1,1}$  and  $10_1^{1,1}$  are the spun surface and the 1-twist spun surface of Hopf 1-link, respectively. Thus they have the same group but are not equivalent (cf. [15], [20]).) Table I contains six 2-knots and two tori; the trivial 2-knot  $0_1$ , the spun 2-knot  $8_1$  of the trefoil, the ribbon 2-knot  $9_1$  associated with  $6_1$  1-knot, the spun 2-knot  $10_1$  of the figure eight, the 2-twist spun 2-knot  $10_2$  of the trefoil and the 3-twist spun 2-knot  $10_3$  of the trefoil; the standard torus  $2_1^1$  of genus one and the spun torus  $10_1^1$  of the trefoil.

It remains open whether or not there exists an irreducible projective plane in  $R^4$ . On the other hand, we have

**Proposition 5.3.** *There exists an irreducible surface in  $R^4$  with two components each of which is homeomorphic to a projective plane.*

Proof. The surfaces  $8_1^{-1,-1}$  and  $10_1^{-1,-1}$  are such examples because the order of the meridian of each component of them is 4.

## Appendix

Table I is the table of all weakly prime surfaces with up to 10 ch-index in  $R^4$ , and Table II is that of their groups and first elementary ideals. In the tables, by  $I_k^{g_1, \dots, g_c}$ , we mean the  $k$ th surface with ch-index  $I$  and  $c$  components whose genera are  $g_1, \dots, g_c$ . (For a 2-knot,  $I_k^0$  is written  $I_k$ .) Here, if  $g_i < 0$ , then it means non-orientable genus. Table I was given in [22], but we omit  $10_3^{0,1}$  because of a duplication  $6_1^{0,1} = 10_3^{0,1}$ . In Table II, the columns headed  $\pi_1(R^4 - F)$  and  $E_1$  give the fundamental group of the complement  $R^4 - F$  and the first elementary ideal of a surface  $F$ . We denote the infinite cyclic group, the quaternion group and the dicyclic group of order 16 by  $Z$ ,  $Q$  and  $\langle 2, 2, 4 \rangle$ , respectively.

*Note:* Recently, T. Yasuda made a table of ribbon  $n$ -knots ( $n \geq 2$ ).




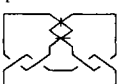
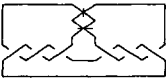
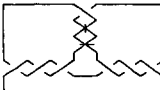
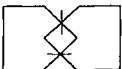
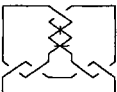
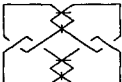
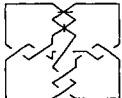

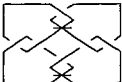
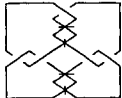
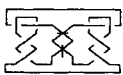


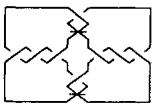

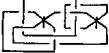
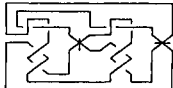
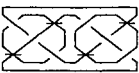
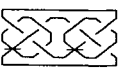
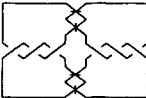
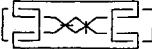
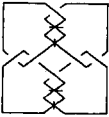
$0_1$ 	$6^{0,1}_1$ 	$8_1$ 	$9_1$ 
$2_1$ 	$7^{q,-2}_1$ 	$8_{1,1}$ 	$9^{q,1}_1$ 
$2_{1^{-1}}$ 		$8_{1^{-1},-1}$ 	$9_{1,-2}$ 
$10_1$ 	$10^{q,1}_1$ 	$10^{q,-2}_1$ 	
$10_2$ 	$10^{q,1}_2$ 	$10^{q,-2}_2$ 	
$10_3$ 	$10_{1,1}$ 	$10_{1^{-1},-1}$ 	
$10_{\frac{1}{2}}$ 	$10^{q,0,1}_1$ 	$10_{1^{-2},-2}$ 	

Table I

Table II

$F$	$\pi_1(R^4 - F)$	$E_1$
$0_1$	$Z$	(1)
$2_1^1$	$Z$	(1)
$2_1^{-1}$	$Z_2$	(1)
$6_1^{0,1}$	$Z \oplus Z$	$(x-1, y-1)$
$7_1^{0,-2}$	$\langle x, y: yxyx^{-1} \rangle$	$(x+1, y-1)$
$8_1$	$\langle x_1, x_2: x_1x_2x_1 = x_2x_1x_2 \rangle$	$(x^2 - x + 1)$
$8_1^{1,1}$	$Z \oplus Z$	$(x-1, y-1)$
$8_1^{-1,-1}$	$Q$	$(x+1, y+1, 2)$
$9_1$	$\langle x_1, x_2: x_1x_2^{-1}x_1x_2x_1^{-1}x_2^{-1} \rangle$	$(x-2)$
$9_1^{0,1}$	$\langle x, y: x^{-1}y^{-1}xyx^{-1}yxy^{-1} \rangle$	$(x-1, y-1)(y-1)$
$9_1^{1,-2}$	$\langle x, y: yxyx^{-1} \rangle$	$(x+1, y-1)$
$10_1$	$\langle x_1, x_2: x_1^{-1}x_2x_1x_2^{-1}x_1x_2x_1^{-1}x_2^{-1}x_1x_2^{-1} \rangle$	$(x^2 - 3x + 1)$
$10_2$	$\langle x_1, x_2: x_1x_2x_1x_2^{-1}x_1^{-1}x_2^{-1}, x_1^2x_2x_1^{-2}x_2^{-1} \rangle$	$(x+1, 3)$
$10_3$	$\langle x_1, x_2: x_1x_2x_1x_2^{-1}x_1^{-1}x_2^{-1}, x_1^3x_2x_1^{-3}x_2^{-1} \rangle$	$(x^2 + x + 1, 2)$
$10_1^1$	$\langle x_1, x_2: x_1x_2x_1 = x_2x_1x_2 \rangle$	$(x^2 - x + 1)$
$10_1^{0,1}$	$\langle x, y: x^{-1}y^{-1}x^{-1}yxyxy^{-1} \rangle$	$(x-1, y-1)(xy+1)$
$10_2^{0,1}$	$\langle x, y: x^2yx^{-2}y^{-1} \rangle$	$(x-1, y-1)(x+1)$
$10_1^{1,1}$	$Z \oplus Z$	$(x-1, y-1)$
$10_1^{0,0,1}$	$\langle x, y, z: y^{-1}x^{-1}zxyz^{-1} \rangle$	(0)
$10_1^{0,-2}$	$\langle x, y: x^{-1}y^{-1}xyx^{-1}yxy \rangle$	$(2x+y-1, 4)$
$10_2^{0,-2}$	$\langle x, y_1, y_2: xy_1x^{-1}y_2^{-1}, y_1^2=y_2^2=(y_1y_2)^2 \rangle$	$(2x+y-1, 4)$
$10_1^{-1,-1}$	$\langle 2, 2, 4 \rangle$	$(x+1, y+1, 4)$
$10_1^{-2,-2}$	$Q$	$(x+1, y+1, 2)$

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