# AN ENUMERATION OF THE PRIMITIVE RECURSIVE FUNCTIONS WITHOUT REPETITION

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In a theorem and its corollary [1] Friedberg gave an enumeration of all the recursively enumerable sets without repetition and an enumeration of all the partial recursive functions without repetition. This note is to prove a similar theorem for the primitive recursive functions. The proof is only a classical one. We shall show that the theorem is intuitionistically unprovable in the sense of Kleene [2]. For similar reason the theorem by Friedberg is also intuitionistically unprovable, which is not stated in his paper.

THEOREM. *There is a general recursive function $\psi(n, a)$ such that the sequence $\psi(0, a), \psi(1, a), \cdots$ is an enumeration of all the primitive recursive functions of one variable without repetition.*

PROOF. Let $\varphi(n, a)$ be an enumerating function of all the primitive recursive functions of one variable. (See [3].) We define a general recursive function $v(a)$ as follows.

$$v(0) = 0,$$

$$v(n+1) = \mu y, \text{ where } \mu y \text{ is the least } y \text{ such that for each } j < n+1,$$

$$\varphi(y, a) \neq \varphi(v(j), a) \text{ for some } a < n+1.$$

It is noted that the value $v(n+1)$ can be found by a constructive method, for obviously there exists some number $y$ such that the primitive recursive function $\varphi(y, a)$ takes a value greater than all the numbers $\varphi(v(0), 0), \varphi(v(1), 0), \ldots\ldots, \varphi(v(n), 0)$ for $a = 0$

Put $\psi(n, a) = \varphi(v(n), a)$. We first see that for any two numbers $j < i$, the two primitive recursive functions of variable $a$ $\psi(j, a)$ and $\psi(i, a)$ are not identically equal, for by definition, $\varphi(v(i), a) \neq \varphi(v(j), a)$ for some $a < i$. From this it also follows that $v(j) \neq v(i)$ for $j \neq i$. This is a fact which will be used later in the proof.

It remains to show that for any number $x$, there is a number $t$ such that $\varphi(x, a) = \psi(t, a)$. We distinguish two cases of $x$. Case 1. There is a number $p$ such that $v(p) = x$. In this case we have already a number $p$ such that $\varphi(x, a) = \varphi(v(p), a) = \psi(p, a)$. In the following we shall consider case 2, the opposite of case 1.

In case 2 $v(n) \neq x$ for all $n$. In this case we first see that for any number $n$, there is a number $r$ such that $\varphi(x, a) = \varphi(v(r), a)$ for $a < n$. Suppose this were false. Then there would be a number $n_0$ such that if $t$ is any number $> n_0$, then for each $j < t$, $\varphi(x, a) \neq \varphi(v(j), a)$ for some $a < n_0 < t$. Since $v(t) \neq x$, then according to the definition of $v(t)$, we would have $v(t) < x$. This implies that the infinitely many numbers $v(n_0 + 1)$, $v(n_0 + 2)$, ......, would all be less than $x$. This is impossible.

For each number $n$, let $r(n)$ be the least number $r$ such that $\varphi(x, a) = \varphi(v(r), a)$ for $a < n$. We can show that $v(r(n)) < x$ for all $n$. In case $r(n) > n$, we have that for each $j < r(n)$, $\varphi(x, a) \neq \varphi(v(j), a)$ for some $a < n < r(n)$, because $r(n)$ is the least number $r$ such that $\varphi(x, a) = \varphi(v(r), a)$ for $a < n$. Since in case 2 $v(r(n)) \neq x$, then according to the definition of $v(a)$, we have $v(r(n)) < x$. Now suppose $0 < r(n) \leqq n$. We have (1) $\varphi(x, a) = \varphi(v(r(n)), a)$ for $a < r(n) \leqq n$. According to the definition of $v(a)$, we have (2) for each $j < r(n)$, $\varphi(v(r(n)), a) \neq \varphi(v(j), a)$ for some $a < r(n)$. Again by the definition of $v(a)$, (1) and (2) implies that $v(r(n)) \leqq x$. In case $0 = r(n) \leqq n$, since $v(0) = 0$, we have also $v(r(n)) \leqq x$. Since $v(r(n)) \neq x$, we still have $v(r(n)) < x$.

Since $v(r(n)) < x$ for all $n$, and $v(j) \neq v(i)$ for $j \neq i$, then $r(n)$ takes only finitely many numbers as its values. Thus there must be a value, say, $q$ such that $q = r(n)$ for infinitely many values of $n$. According to the meaning of $r(n)$, this implies that $\varphi(x, a) = \varphi(v(q), a)$ for $a < n$, for infinitely many values of $n$. Thus in case 2 we also find a number $q$ such that $\varphi(x, a) = \varphi(v(q), a) = \psi(q, a)$ identically in $a$. This completes the proof.

That the theorem can not be proved intuitionistically in the sense of Kleene [2] can be seen from the following consideration. Suppose it could be so proved. Then we would have two general recursive functions $\psi(n, a)$ and $f(a)$ having the two properties: 1) $\psi(i, a) \neq \psi(j, a)$ for some $a$, if $i \neq j$; 2) for every number $x$, $\varphi(x, a) = \psi(f(x), a)$ identically in $a$. To show that this is impossible we let $p$ be such a number that $\varphi(p, a)$ is identically equal to zero. Then any primitive recursive function $\varphi(x, a)$ is identically equal to zero, if and only if $f(x) = f(p)$. This would imply that the predicate $(a)(\varphi(x, a) = 0)$ be effectively decidable. But it is well-known that this predicate is not effectively decidable. (This can also be seen from the fact that the predicate of Kleene $(a)\overline{T}_1(x, x, a)$ [4, p. 301] is not effectively decidable, while the decision problem for $(a)\overline{T}_1(x, x, a)$ can be reduced to that for $(a)(\varphi(x, a) = 0)$.) The same method can be adapted to show that Friedberg's Theorem 3 in [1] is also intuitionistically unprovable. To do this we only need to note that a primitive recursive function $\varphi(x, a)$ is identically equal to zero, if and only if the set $\hat{w}(Ey)$ $(w = \varphi(x, y))$

consists of the single element 0.

## REFERENCES

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