

AN EPIDEMIOLOGICAL MODEL WITH A DELAY
AND A NONLINEAR INCIDENCE RATE

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Summary. An epidemiological model with both a time delay in the removed class and a nonlinear incidence rate is analysed to determine the equilibria and their stability. This model is for diseases where individuals are first susceptible, then infected, then removed with temporary immunity and then susceptible again when they lose their immunity. There are multiple equilibria for some parameter values, and, for certain of these, periodic solutions arise by Hopf bifurcation from the large nontrivial equilibrium state.

Key words: Epidemiological model – Hopf bifurcation – Nonlinear incidence – Time delay.

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1. Introduction

In the epidemiological model considered here a constant population is divided into susceptible, infectious and removed classes. Let $S(t)$ be the fraction of the population which is susceptible at time t , $I(t)$ be the infectious fraction and $R(t)$ be the fraction which is removed with immunity. The classes are disjoint so that these fractions add to 1. The model is called an SIRS model since susceptibles become infectious, then removed with temporary immunity and then susceptible again after the temporary immunity is lost.

Modellers need to know how various formulations of epidemiological models influence the global structure of the solutions. In particular, they need to know whether the endemic equilibrium state is stable or unstable and whether or not there are periodic solutions. Periodic solutions have been found in various epidemiological models with constant, nonperiodic coefficients. Hethcote *et al.* (1981a) have found that an SIRS model with bilinear mass action βIS , where β is the constant contact rate, and a time delay in the removed class can have periodic solutions for some parameter values. This time delay is reasonable when all individuals have approximately the same period of temporary immunity. Liu *et al.* (1986, 1987) have found that periodic solutions occur for some parameter values in an SIRS model with nonlinear incidence of the form $\beta I^p S^q$, $p > 1$. As described in Liu *et al.* (1987), deviations from the bilinear mass action incidence rate due to saturation or multiple exposures before infection could lead to a nonlinear incidence rate $\beta I^p S^q$ with p and q near 1.

It is natural to ask about the interaction of these two formulations, each of which leads to periodic solutions. Here an SIRS model with both a time delay in the removed class and a nonlinear incidence rate is analysed to determine the number of equilibria, their stability and the occurrence of periodic solutions by Hopf bifurcation. References to other epidemiological models with time delays are given in the papers cited above. A survey of mathematical epidemiology models has been given in Hethcote *et al.* (1981c). A more

recent survey of model formulations which can lead to periodic solutions is in Hethcote and Levin (1988).

2. Formulation of the Model

As noted in the introduction, the total population is assumed constant and the three classes are disjoint, thus

$$S(t) + I(t) + R(t) = 1. \quad (2.1)$$

The nonlinear incidence rate considered here is $\beta I^p S$ where p is a positive constant. The power of S has been chosen to be 1 since it has been shown in Liu *et al.* (1987) that the essential asymptotic behavior is independent of the power of S . Individuals recover and are removed from the infectious class at a rate proportional to the infective fraction with positive proportionality constant γ . Since this assumption corresponds to a negative exponential distribution for the infectious period, the average infectious period is $1/\gamma$. A constant period ω of temporary immunity is assumed so that the probability $P(t)$ of remaining immune t time units after becoming immune is 1 for $0 \leq t \leq \omega$, and 0 for $t > \omega$. The contact number $\sigma \equiv \beta/\gamma$ is defined as the product of the contact rate β and the average infectious period $1/\gamma$. The ratio of the period of temporary immunity ω to the average infectious period $1/\gamma$ is denoted by $r \equiv \omega\gamma$.

The initial susceptible and infective fractions are $S(0) = S_0 \geq 0$ and $I(0) = I_0 \geq 0$. Let $R_0(t)$ be the fraction of the initial population which is initially removed and is still removed at time t . The function $R_0(t)$ is assumed to be a continuous, nonincreasing function with $R_0(t) = 0$ for $t > \omega$. For all $t \geq 0$ the integral equation for $I(t)$ is

$$I(t) = I_0 e^{-\gamma t} + \int_0^t \beta I^p(y) S(y) e^{-\gamma(t-y)} dy, \quad (2.2)$$

where the first term on the right side corresponds to the removal of the initial infectives

and the second term is the sum up to time t of those who became infected at time y and are still infected at time t . The differential equation for $I(t)$, valid for all $t \geq 0$, is

$$I'(t) = -\gamma I(t) + \beta I^P(t)S(t). \quad (2.3)$$

For all $t \geq 0$ the integral equation for $R(t)$ is

$$R(t) = R_0(t) + \gamma \int_0^t I(x)P(t-x)dx, \quad (2.4)$$

so that

$$R(t) = \begin{cases} R_0(t) + \gamma \int_0^t I(x)dx & \text{for } t \leq \omega, \\ \gamma \int_{t-\omega}^t I(x)dx & \text{for } t > \omega. \end{cases} \quad (2.5)$$

The differential equation for $S(t)$ is

$$S'(t) = \begin{cases} -R_0'(t) - \beta I^P(t)S(t), & \text{for } t \leq \omega \\ \gamma I(t-\omega) - \beta I^P(t)S(t) & \text{for } t > \omega. \end{cases} \quad (2.6)$$

Our model is defined by equations (2.1), (2.3) and (2.5), but we can also write it as an integral equation or an integrodifferential equation in $I(t)$. Using the relationships above, the equations for $I(t)$ written as functions of I alone are

$$I(t) = I_0 e^{-\gamma t} + \int_0^t \beta I^P(y) \left[1 - I(y) - R_0(y) - \gamma \int_0^y I(x)P(y-x)dx \right] e^{-\gamma(t-y)} dy, \quad (2.7)$$

$$I'(t) = -\gamma I(t) + \beta I^p(t) \left[1 - I(t) - R_0(t) - \gamma \int_0^t I(x)P(t-x)dx \right]; \quad (2.8)$$

so that for $t \geq \omega$

$$\begin{aligned} I(t) = & I_0 e^{-\gamma t} + \int_0^\omega \beta I^p(y) \left[1 - I(y) - R_0(y) - \gamma \int_0^y I(x)dx \right] e^{-\gamma(t-y)} dy \\ & + \int_\omega^t \beta I^p(y) \left[1 - I(y) - \gamma \int_{y-\omega}^y I(x)dx \right] e^{-\gamma(t-y)} dy, \end{aligned} \quad (2.9)$$

and

$$I'(t) = -\gamma I(t) + \beta I^p(t) \left[1 - I(t) - \gamma \int_{t-\omega}^t I(x)dx \right] \equiv f(t, I). \quad (2.10)$$

The integrodifferential equation (2.10) with $p = 1$ has been analysed in Hethcote *et al.*(1981a). The model analysed in Liu *et al.* (1987) does not contain a delay, but it does contain features such as vital dynamics (births and deaths), a power q on the S factor in the nonlinear incidence term, and an exposed class. Since these three features were not essential for the existence of periodic solutions in the model of Liu *et al.* (1987), we have not included them.

We now show that our model is mathematically and epidemiologically well-posed. We use the methods of Hale (1977) and give only a brief outline of proofs.

Theorem 2.1. *The integrodifferential equation (2.10) with initial condition $I(0) = I_0$ has a unique solution which exists on a maximal interval.*

Proof. For $p \geq 1$, the right side of (2.10) is Lipschitzian, since

$$\|f(t, I_1) - f(t, I_2)\| \leq (\gamma + \beta p(2 + \gamma\omega) + \beta(1 + \gamma\omega)) \|I_1 - I_2\|$$

where $\|I\| \equiv \sup_{-\omega \leq t \leq 0} |I(t)|$. For $p < 1$, define a new variable $U = I^{1-p}$, as in Liu *et al.*

(1986), then the equivalent integrodifferential equation for U is:

$$U'(t) = (1-p) \left\{ -\gamma U(t) + \beta [1 - U^s(t) - \gamma \int_{t-\omega}^t U^s(x) dx] \right\},$$

where $s \equiv (1-p)^{-1} > 1$. The right side of this is also Lipschitzian. Thus, by a standard theorem in Hale (1977, pg. 42) on retarded differential equations, there is a unique local solution of (2.10) (with initial condition $I(0) = I_0$) for all p , which exists on a maximal interval. ■

Theorem 2.2. *The region*

$$B \equiv \{(S, I, R): 0 \leq S \leq 1, 0 \leq I \leq 1, 0 \leq R \leq 1, S + I + R = 1\} \quad (2.11)$$

is positively invariant with respect to the system (2.1), (2.3), (2.5).

Proof. Considering separately the boundary points of B where $I_0 = 0$, where $S_0 = 0$ and where $R_0(0) = 0$, we see that no solution can exit the region from the boundary.

Also, if the initial point $(I_0, S_0, R_0(0))$ is an interior point of B , define

$T = \inf\{t > 0 \mid S(t)I(t)R(t) = 0\}$ and assume T is finite. Assuming $R(T) = 0$ (and so $I(t) > 0$ on $[0, T)$), contradicts (2.5). Similarly, assuming $I(T) = 0$ or assuming $S(T) = 0$ leads to a contradiction; thus T does not exist, and so the solution must stay in the interior of B for all finite time. By Hale (1977, pg. 42) the solution must be continuable for all t , thus the maximal interval in Th. 2.1 is $[0, \infty)$. ■

3. Equilibrium States

The system of equations derived in the previous section always has the trivial equilibrium which corresponds to the disease dying out. An endemic (nontrivial) equilibrium corresponds to persistence of the disease. Using the method as in Liu *et al.* (1986, 1987), the number of such equilibria can be determined. Recall $\sigma \equiv \beta/\gamma$, $r \equiv \omega\gamma$; the results are now summarized.

Theorem 3.1. *Let $\sigma^* \equiv p^p(1+r)^{p-1}/(p-1)^{p-1}$. The system (2.1), (2.3), (2.5) always has the disease-free equilibrium $(S,I,R) = (1,0,0)$, and if*

- (i) $p < 1$, then it has 1 nontrivial equilibrium,
- (ii) $p = 1$ and $\sigma \leq 1$, then it has no nontrivial equilibrium,
- (iii) $p = 1$ and $\sigma > 1$, then it has 1 nontrivial equilibrium,
- (iv) $p > 1$ and $\sigma < \sigma^*$, then it has no nontrivial equilibrium,
- (v) $p > 1$ and $\sigma = \sigma^*$, then it has 1 nontrivial equilibrium,
- (vi) $p > 1$ and $\sigma > \sigma^*$, then it has 2 nontrivial equilibria.

When a nontrivial equilibrium exists it is given by (S_e, I_e, R_e) , with $R_e = rI_e$, $S_e = 1 - (1+r)I_e$ and $\beta I_e^p S_e = \gamma I_e$. Thus $I_e \neq 0$ satisfies the equation

$$I_e^{p-1} - (1+r)I_e^p = 1/\sigma. \quad (3.1)$$

The surface $\sigma = \sigma^*$ in σ, p, r space is called the *saddle-node bifurcation surface*, by analogy with the model with no lag; it is shown in Figure 1. When $p = 1$, corresponding to the case of bilinear incidence as considered by Hethcote *et al.* (1981a), the unique nontrivial equilibrium for $\sigma > 1$ is given by $I_e = (1-1/\sigma)/(1+r)$. When $p > 1$ and $\sigma = \sigma^*$, then (3.1) has the unique solution $I_e = I_m = (1-1/p)/(1+r)$, giving $S_m = 1/p$. For $p > 1$ and $\sigma > \sigma^*$, (3.1) has two solutions, I_1, I_2 where $I_1 \in (0, I_m)$

and $I_2 \in (I_m, 1/(1+r))$; see Liu *et al.* (1987, Fig. 1).

4. Stability of the Disease-Free Equilibrium

To examine local stability of the disease-free equilibrium which is $(S,I,R) = (1,0,0)$, it is convenient to define $g(I) \equiv \beta I^p$ and to write (2.10) in the form (valid for all $t \geq \omega$)

$$I'(t) = -\gamma I(t) + g(I) \left[1 - I(t) - \gamma \int_{-\omega}^0 I(t+u) du \right]. \quad (4.1)$$

This is now linearized around the equilibrium value, $I = 0$, to give

$$I'(t) = (-\gamma + g'(0))I(t). \quad (4.2)$$

Examining this equation gives the following results.

Theorem 4.1. *Local asymptotic stability of the disease-free equilibrium of the model (2.1), (2.3), (2.5) is independent of the lag ω , but depends on p in the following way. If*

- (i) $p < 1$, then the disease-free equilibrium is unstable,
- (ii) $p = 1$, then the disease-free equilibrium is locally asymptotically stable if $\sigma \equiv \beta/\gamma < 1$,
- (iii) $p > 1$, then the disease-free equilibrium is locally asymptotically stable.

Note that in the case $p = 1$, σ is a threshold parameter which determines whether the disease dies out ($\sigma \leq 1$) or remains endemic ($\sigma > 1$). Such a threshold is common in epidemiological models (see, e.g. Hethcote (1976)); but this threshold does not occur in our model for $p \neq 1$.

For $p = 1$, a stronger result is true (Hethcote *et al.*, 1981a). The disease-free equilibrium is in fact globally asymptotically stable for $\sigma \leq 1$. A more restrictive result for $p > 1$ can be derived by similar use of a Lyapunov function, see also Liu *et al.* (1987).

Theorem 4.2. *If $p > 1$ and $\sigma < \sigma_1 \equiv p^p/(p-1)^{p-1}$, then all solutions of the model (2.1), (2.3), (2.5) which start in the feasible region B given by (2.11), approach the disease-free equilibrium, that is $(S,I,R) = (1,0,0)$, as $t \rightarrow \infty$.*

Proof. Consider the Lyapunov function I , then from (2.8)

$$\begin{aligned} I'(t) &= \gamma I(\sigma I^{p-1}[1 - I - R_0(t) - \gamma \int_0^t I(x)P(t-x)dx] - 1) \\ &\leq \gamma I(\sigma I^{p-1}[1 - I] - 1) \leq \gamma I(\sigma/\sigma_1 - 1), \end{aligned}$$

since $I^{p-1} - I^p$ has its maximum value $1/\sigma_1$ when $I = 1 - 1/p$. So for $p > 1$ and $\sigma < \sigma_1$, $I' \leq 0$. Since $\sigma < \sigma_1$, equality for $I' = 0$ holds only when $I = 0$, which implies that $R = 0$. Thus $(S,I,R) = (1,0,0)$ is the largest invariant subset in the set where $I' = 0$. By using a standard Lyapunov theorem (Hale, 1977, p. 119), the above result is proved. ■

We conjecture that the result is true for all $\sigma < \sigma^*$, but our proof technique limits the range to $\sigma < \sigma_1$.

5. Stability of Endemic Equilibria

The local stability of a nontrivial equilibrium I_e is examined by linearizing about the endemic equilibrium. This gives the characteristic equation

$$\lambda + \gamma + \beta I_e^p - \beta p S_e I_e^{p-1} + \gamma \beta I_e^p \frac{(1-e^{-\omega\lambda})}{\lambda} = 0.$$

Setting $\omega\lambda = z$, gives

$$\Delta(\sigma, z) \equiv z + a + c(1-e^{-z})/z = 0, \quad (5.1)$$

$$\text{where } a \equiv r + \omega\beta I_e^p - \omega\beta p S_e I_e^{p-1} = r(1-p + \sigma I_e^p), \quad (5.2)$$

$$\text{and } c \equiv r\omega\beta I_e^p = r^2 \sigma I_e^p > 0; \quad (5.3)$$

$$\text{thus } c = ar + r^2(p-1). \quad (5.4)$$

Equation (5.1) has a zero root iff $a + c = 0$, which (from (5.2), (5.3)) is equivalent to $1 - p + \sigma I_e^p(1+r) = 0$; which is true for $I_e = I_m$ on the saddle-node bifurcation surface where $\sigma = \sigma^*$. Complex (including nonzero purely imaginary) roots of (5.1) occur in conjugate pairs. To locate the purely imaginary roots, set $z = iy$ ($y > 0$) in (5.1) multiplied by z . Then equating real and imaginary parts gives

$$c = \frac{y^2}{1 - \cos y}, \quad a = -\frac{y \sin y}{1 - \cos y}, \quad y \in (0, 2\pi), \quad (5.5)$$

as parametric equations for the lowest neutral surface. The characteristic equation (5.1) is a special case of an equation considered by Hethcote *et al.* (1981b, Appendix) and by Brauer and Ma (1987). However, since it plays a vital role in subsequent analysis here, we redraw part of this imaginary root curve as our Figure 2, and restate the following results.

Theorem 5.1 (Hethcote *et al.*, 1981b).

- (a) $a + c = 0$ iff (5.1) has a zero root.
- (b) The roots of (5.1) with $\operatorname{Re} z \geq 0$ lie in a bounded domain, so no such root can come in from infinity.
- (c) If $a + c < 0$, then (5.1) has at least one positive real root.
- (d) If $a + c > 0$, $a > 0$ and $c < a^2/2$, then all roots of (5.1) have negative real parts.
- (e) If $a + c > 0$, $a < 0$ and $c < a^2/2$, then (5.1) has exactly two roots with nonnegative real parts.
- (f) If $a + c > 0$ and the point (a, c) lies below the imaginary root curve given parametrically by (5.5), then all roots of (5.1) have negative real parts.

We use Figure 2 and the above statements to prove the following local stability results.

Theorem 5.2. *If $p < 1$ and $r \leq 2$ or if $p < 1/2$, then the unique endemic equilibrium (S_e, I_e, R_e) is locally asymptotically stable.*

Proof. When $p < 1$, $a > 0$ and so $a + c > 0$. For a purely imaginary root the line given by (5.4) with slope r and negative intercept on the c axis must intercept the neutral curve. But the slope of the imaginary root curve is 2 at $a = 0$ and is greater than 2 for $a > 0$; see Figure 2. For $r \leq 2$ such an intersection is impossible. Thus by Theorem 5.1(f), all roots of (5.1) have negative real parts; thus the unique endemic equilibrium (as found in Section 3) is locally asymptotically stable. The condition $c < a^2/2$ in Theorem 5.1(d) in combination with the line (5.4) requires that $a^2 - 2ra + 2r^2(1-p) > 0$. This is certainly true for $p < 1/2$ (for any $r > 0$), which completes the proof. ■

As $p \rightarrow 1^-$ with $\sigma < 1$, $I_e \rightarrow 0$ so that the endemic equilibrium coalesces with the disease

free equilibrium. Note that in the model with no delay, Liu *et al.* (1987) find that the unique endemic state is locally asymptotically stable for all $p < 1$. The case $p = 1$ with delay is given in Hethcote *et al.* (1981a).

Theorem 5.3. *If $p > 1$ and $\sigma > \sigma^*$, then the smaller endemic equilibrium $I_1 \in (0, I_m)$ is unstable.*

Proof. When $p > 1$ and $\sigma > \sigma^*$ and $I_e = I_1 < I_m$, then

$$a + c < r(1-p) + r(r+1)I_m/S_m = r(1-p) + r(1-S_m)/S_m.$$

But $S_m = 1/p$, so $a + c < 0$. Thus by Theorem 5.1(c), equation (5.1) has at least one positive real root, and so the smaller equilibrium I_1 is always unstable. ■

As $p \rightarrow 1^+$, $I_1 \rightarrow 0$ so that the smaller endemic equilibrium coalesces with the disease free equilibrium. For $p > 1$ and $\sigma > \sigma^*$, the larger equilibrium value of I_e , namely $I_2 \in (I_m, 1/(1+r))$ makes $a + c > 0$ with $a > 0$ if $\sigma > \sigma^*(1+r)$. Thus the local stability of I_2 is more complicated. The following theorem gives a sufficient condition for stability which is proved by considering the slope of the neutral curve in Figure 2 and the line $c = ar + r^2(p-1)$ as in Theorem 5.2.

Theorem 5.4. *If $p > 1$, $\sigma > \sigma^*$ and $r < \min\{2, \pi/(2p-2)^{1/2}\}$, then the larger endemic equilibrium I_2 is locally asymptotically stable.*

If $r > \pi/(2p-2)^{1/2}$, the line $c = ar + r^2(p-1)$ always intersects the imaginary root curve, so the equilibrium I_2 is unstable for some parameter values and stable for others.

6. Hopf Bifurcation

For values of $p > 1$ when $\sigma > \sigma^*$, Hopf bifurcation may occur at the larger endemic equilibrium I_2 as a conjugate pair of roots of the characteristic equation cross the imaginary axis, and a similar possibility exists for the unique endemic equilibrium for $p \in (1/2, 1]$. The Hopf bifurcation surface in $p\sigma r$ space (*i.e.* where two conjugate roots of the characteristic equation (5.1) are purely imaginary) is denoted by $\sigma^{**}(p, r)$. Cross-sections of this Hopf bifurcation surface are shown in Figure 3. The results of numerically computing the Hopf bifurcation surface σ^{**} , the direction of bifurcation and the stability of the bifurcating periodic solutions are given in Section 7.

The following lemmas give results necessary to establish the existence of a Hopf bifurcation.

Lemma 6.1. *If iy with $y > 0$ is a root of (5.1), then it is a simple root, and no integral multiple of iy is a root.*

Proof. Assume that $z\Delta(\sigma, z) = 0$ has a double (multiple) root, then $2z + a + ce^{-z} = 0$. Together with (5.1) this gives

$$z^2 + (2+a)z + a + c = 0. \quad (6.1)$$

Setting $z = iy$, $y > 0$, equating real and imaginary parts gives $y^2 = a + c$ and $a = -2$. But $a = -2$ implies $c = 2$ (see Fig. 2), and so $y = 0$. This contradicts our assumption (5.5) on y , and proves that iy must be a simple root. From (5.5)

$$y^2 = 2c - a^2, \quad (6.2)$$

which is positive because the imaginary root curve in Fig. 2 is above the parabola

$c = a^2/2$. Given fixed parameters $p > 1$, $\sigma > \sigma^*$ and r , a value of I_2 is uniquely determined from (3.1), thus a unique value of $y > 0$ results from (5.2), (5.3) and (6.2). Similarly for $p \in (1/2, 1]$ because there is a unique nontrivial equilibrium. Thus no integral multiple of iy can be a root. ■

Lemma 6.2. (*Transversality Condition*) *The real part of a characteristic root of (5.1) changes sign as the imaginary root surface σ^{**} is crossed.*

Proof. Taking the partial derivative of $z\Delta(\sigma, z) = 0$ and using (5.1) to eliminate the exponential term, gives

$$\left[z^2 + (2+a)z + a + c \right] \frac{\partial z}{\partial \sigma} + \left[\frac{\partial a}{\partial \sigma} z - \frac{\partial c}{\partial \sigma} \left[\frac{z^2}{c} + \frac{az}{c} \right] \right] = 0.$$

From Lemma (6.1) the coefficient of the partial derivative of z is nonzero at an imaginary root. Dividing, taking conjugates, using (6.2) and (5.4) gives

$$\begin{aligned} \operatorname{sgn} x'(\sigma^{**}) &\equiv \operatorname{sgn} \left[\operatorname{Re} \frac{\partial z}{\partial \sigma} \right] \text{ at } z = iy, y > 0, \\ &= \operatorname{sgn} \left[\frac{\partial a}{\partial \sigma} (r(a+c) - c(2+a)) \right]. \end{aligned}$$

By differentiating (5.2) and (3.1), it can be shown that

$$\frac{\partial a}{\partial \sigma} = \frac{c}{r\sigma} + \frac{pc}{\sigma(a+c)} > 0$$

because $(a+c) > 0$ in the region of interest. Thus

$$\text{sgn } x'(\sigma^{**}) = \text{sgn}[(r(a+c)-c(2+a))]. \quad (6.3)$$

Now, considering p fixed, and differentiating (5.4) with respect to y , gives

$$\frac{\partial c}{\partial y} - r \frac{\partial a}{\partial y} = (a+2r(p-1)) \frac{\partial r}{\partial y}.$$

So $\frac{\partial r}{\partial \sigma} = 0$ iff $\frac{\partial r}{\partial y} = 0$ iff $\frac{\partial c}{\partial y} - r \frac{\partial a}{\partial y} = 0$. (Note that on the imaginary root curve the term $a + 2r(p-1)$ is positive, for all $p > 1/2$.) But, from the definitions of a and c , $\frac{\partial a}{\partial y} = \frac{a+c}{y}$ and $\frac{\partial c}{\partial y} = \frac{c(2+a)}{y}$. Hence $\frac{\partial r}{\partial \sigma} = 0$ iff $c(2+a) - r(a+c) = 0$. It follows that the minimum point of the imaginary root curve in the σ, r plane (for fixed p) occurs when $r = c(2+a)/(a+c)$. From (6.3), $x'(\sigma^{**})$ changes sign at this value of r , that is at the minimum point of the $r(\sigma)$ curve. The complex conjugate pair has negative real part below (outside) by continuity and Theorem 5.4, and so positive real part above (inside). Thus the transversality condition is verified. ■

The nontrivial equilibrium point (I_e for $\frac{1}{2} < p \leq 1$ and I_2 for $p > 1$) is locally asymptotically stable for parameter values outside the Hopf bifurcation surface σ^{**} and is unstable for parameter values inside the σ^{**} surface.

We use the Hopf bifurcation theorem for functional differential equations as formulated by Stech (1985). To cast our model in the correct form, we take (2.10), and let $I = I_e(1+X)$, where I_e is the unique [resp. larger] nontrivial solution of (3.1) when $p \leq 1$ [resp. $p > 1$]. Rescaling time by setting $t = \omega\tau$ and defining $X_\tau(u) = X(\tau+u)$, gives

$$X'(\tau) = r(p-1)X_\tau(0) - \sigma r I_e^p (X_\tau(0) + r \int_{-1}^0 X_\tau(u) du) + H(\sigma, X_\tau). \quad (6.4)$$

The nonlinear part is

$$\begin{aligned}
H(\sigma, \psi) &= r[(1 + \psi(0))^P - 1 - p\psi(0)] \\
&\quad - r\sigma I_e^P [p\psi(0) + p(p-1)\psi^2(0)/2 + \dots][\psi(0) + r \int_{-1}^0 \psi(u) du] \\
&\equiv H_2(\psi, \psi) + H_3(\psi, \psi, \psi) + \dots,
\end{aligned} \tag{6.5}$$

where H_j is a σ -dependent, symmetric, bounded j -linear form (as in Stech 1985, (2.2)).

Here these forms are

$$\begin{aligned}
H_2(\psi_1, \psi_2) &= \frac{rp(p-1)}{2} \psi_1(0)\psi_2(0) - \frac{r\sigma I_e^P}{2} [p\psi_1(0)] \left[\psi_2(0) + r \int_{-1}^0 \psi_2(u) du \right] \\
&\quad - \frac{r\sigma I_e^P}{2} [p\psi_2(0)] \left[\psi_1(0) + r \int_{-1}^0 \psi_1(u) du \right],
\end{aligned}$$

$$\begin{aligned}
H_3(\psi_1, \psi_2, \psi_3) &= \frac{rp(p-1)(p-2)}{6} \psi_1(0)\psi_2(0)\psi_3(0) \\
&\quad - \frac{r\sigma I_e^P}{6} p(p-1) \left\{ \psi_1(0)\psi_2(0) \left[\psi_3(0) + r \int_{-1}^0 \psi_3(u) du \right] \right. \\
&\quad + \psi_1(0)\psi_3(0) \left[\psi_2(0) + r \int_{-1}^0 \psi_2(u) du \right] + \psi_2(0)\psi_3(0) \left[\psi_1(0) \right. \\
&\quad \left. \left. + r \int_{-1}^0 \psi_1(u) du \right] \right\}.
\end{aligned}$$

The linearized problem from (6.4) and the characteristic equation have been considered in Section 5. Since (6.4) is a scalar equation, ξ and ξ^* in Section 2 of Stech (1985) are chosen to be 1. We use Theorem 2.1 and Remarks 2.2 and 2.6 in Stech (1985) to obtain the theorem below. The Lemmas above show that the hypotheses are satisfied.

Theorem 6.3. *With the notation above, there is an $\epsilon > 0$ and a real solution $X(\tau, \sigma, C, \nu)$ defined for real C , $|C| < \epsilon$, $|\nu - y| < \epsilon$, $|\sigma - \sigma^{**}| < \epsilon$ and $\tau \in \mathbb{R}$, such that (6.4) has a $2\pi/\nu$ periodic solution, namely*

$$X(\tau, \sigma, C, \nu) = 2C \cos(\nu\tau) + \mathcal{O}(C^2)$$

where

$$\nu = y(\sigma) + \text{Im}\{M_3(\sigma, y(\sigma))\}C^2 + \mathcal{O}(C^4),$$

$M_3(\sigma, \nu) \equiv N_3(\sigma, \nu) / \frac{\partial \Delta}{\partial z}(\sigma, z)$, $N_3(\sigma, \nu) \equiv 3H_3(\varphi, \varphi, \bar{\varphi}) + 2H_2(\bar{\varphi}, A_{2,2})e^{2i\nu s} + 2H_2(\varphi, A_{2,0})$, with $\varphi(s) \equiv e^{i\nu s}$ for $s \leq 0$, $A_{2,2} \equiv H_2(\varphi, \varphi) / \Delta(\sigma, 2i\nu)$, and $A_{2,0} \equiv 2H_2(\varphi, \bar{\varphi}) / \Delta(\sigma, 0)$. Also $K_3(\sigma) \equiv \text{Re}\{M_3(\sigma, y(\sigma))\}$ and, in the generic case when $K_3(\sigma^{**}) \neq 0$,

$$C = \left| \frac{x'(\sigma^{**})(\sigma - \sigma^{**})}{K_3(\sigma^{**})} \right|^{\frac{1}{2}} + \mathcal{O}(|\sigma - \sigma^{**}|).$$

For $\text{sgn}\{x'(\sigma^{**})K_3(\sigma^{**})\} < 0$ [resp. > 0] the bifurcation exists for $\sigma > \sigma^{**}$ [resp. $\sigma < \sigma^{**}$]. When $K_3(\sigma^{**}) < 0$ [resp. > 0] the bifurcation is orbitally asymptotically stable [resp. unstable]. ■

Note that, to find the direction of the bifurcation, we require only the sign of $x'(\sigma^{**})$, which can be found from (6.3). Also $\Delta(\sigma, z)$ is given by (5.1) and

$$\partial \Delta / \partial z = 2 + a + z + (a+c)/z$$

(cf. Lemma 6.1) with $z = iy$ on the imaginary root curve. The H_j required to compute $K_3(\sigma^{**})$ are calculated as:

$$H_3(\varphi, \varphi, \bar{\varphi}) = \frac{rp(p-1)(p-2)}{6} - \frac{p(p-1)c}{6r} \left\{ 3 + r \left[\frac{1}{iy} + \frac{3 \sin y}{y} - \frac{\cos y}{iy} \right] \right\},$$

$$H_2(\varphi, \varphi) = iyp - rp(p-1)/2,$$

$$H_2(\bar{\varphi}, A_{2,2} e^{2iys}) = A_{2,2} p [-r(p-1) + iy - \Delta(\sigma^{**}, 2iy)]/2,$$

$$H_2(\varphi, \bar{\varphi}) = -rp(p-1)/2,$$

$$H_2(\varphi, A_{2,0}) = A_{2,0} p [-r(p-1) - a + iy]/2.$$

7. Numerical Calculations

The results in the previous sections allow us to calculate the Hopf bifurcation surface σ^{**} , the direction of bifurcation and the stability of the bifurcating periodic solutions. It is convenient to generate $\sigma^{**}(r,p)$ by calculating cross-sections for fixed p . For fixed $p > 1$, the quadratic equation (5.4) for r has one positive real root. Thus each $y \in (0, 2\pi)$ gives values of a and c in (5.5) which lead to a unique positive value of r . For $p = 1$ equation (5.4) has the unique root $r = c/a = -y/\sin y$ which is positive for $y \in (\pi, 2\pi)$. For $\frac{1}{2} < p < 1$, equation (5.4) has two real roots iff $a^2 - 4(1-p)c \geq 0$ which is equivalent to $\cos y \geq 3 - 4p$. In this case for $y \in (2\pi - \arccos(3-4p), 2\pi)$, the smaller r root of equation (5.4) generates the lower right part of the imaginary root curve and the larger r root generates the upper left part.

Thus for fixed p , each y value determines values of a , c and r . By combining (3.1) and (5.3) we find

$$\sigma^{**} = \left[1 + \frac{(1+r)c}{r^2} \right]^p \left[\frac{c}{r^2} \right]^{1-p}. \quad (7.1)$$

Hence the cross-sections in the σr plane of the Hopf bifurcation surface for fixed p are generated as y takes on different values. Cross-sections for fixed r are generated by varying $y \in (0, 2\pi)$, solving (5.4) for p and then using (7.1) to find σ^{**} . Cross-sections for fixed σ are generated by varying $y \in (0, 2\pi)$ and solving (7.1) iteratively for r while using (5.4) to find p in terms of r and y . These cross-sections are shown in Figure 3. The interior starting points of the Hopf bifurcation cross-section curves in Figure 3 correspond to the intersection of $\sigma = \sigma^{**}$ with the saddle-node bifurcation surface $\sigma = \sigma^*$. In Figure 3a the left part of the $p = 1$ curve approaches $\sigma = 1$ as r approaches infinity, but the left part of the $p = 0.75$ curve eventually curves back to the right as r increases (the asymptotic behavior is given in the Appendix).

The quantities $\text{sgn}\{x'(\sigma^{**})\}$ and $K_3(\sigma^{**})$ are evaluated numerically using (6.3) and the formulae in Section 6. The calculations verify the result proved in Lemma 6.2; namely, that $\text{sgn}\{x'(\sigma^{**})\}$ is negative for σ to the left of the minimum point of the imaginary root curve in the $\sigma\tau$ plane and is positive to the right. For $\frac{1}{2} < p \leq 1$, $K_3(\sigma^{**})$ is usually negative so that the bifurcating periodic solutions are orbitally asymptotically stable and occur for parameter values above (inside) the Hopf bifurcation surface, but K_3 can be positive. For example, if $p = 0.75$, then K_3 is negative along the cross-section curve in the $\sigma\tau$ plane except on the upper left portion between (1.35, 14.5) and (1.28, 53.5). For $p > 1$, $K_3(\sigma^{**})$ is negative except when y is near 2π . Thus for $p > 1$ the bifurcating periodic solutions are usually orbitally asymptotically stable and occur for parameter values above the Hopf bifurcation surface, but for large σ the periodic solutions are unstable and occur for parameter values below the Hopf bifurcation surface. Asymptotic values of K_3 for y near 2π are found in the Appendix. The stability constant K_3 is negative for all points shown in Figure 3.

To demonstrate limit cycle solutions, the system (2.1), (2.3), (2.6) with linear $R_0(t)$, $p = 1.5$, $r = 4.0$, $\sigma = 20.0$ was solved numerically for different initial data. Phase plane solutions are given in Figure 4.

8. Discussion

It is interesting to compare the results for this model which includes both a time delay in the removed class and nonlinear incidence with the results for the model which includes only a time delay in Hethcote *et al.* (1981a) and with the results for the model which includes only nonlinear incidence in Liu *et al.* (1987).

For $p < 1$ in our model there is an unstable disease-free equilibrium and a nontrivial equilibrium. The nontrivial equilibrium is stable for $0 < p \leq \frac{1}{2}$ and also for $\frac{1}{2} < p \leq 1$ with parameter values outside the Hopf bifurcation surface σ^{**} . For $\frac{1}{2} < p \leq 1$ with parameter values inside the surface σ^{**} , the nontrivial equilibrium is unstable and usually orbitally asymptotically stable periodic solutions appear for parameter values just inside the Hopf bifurcation surface σ^{**} . Note that bifurcated periodic solutions did not occur for the model of Liu *et al.* (1987) for $p \leq 1$ so that the periodic solutions here are due to the delay.

For $p > 1$ the disease-free equilibrium is always locally asymptotically stable and a saddle-node bifurcation occurs at $\sigma = \sigma^*$ so that there are 0, 1 or 2 nontrivial equilibria when $\sigma < \sigma^*$, $\sigma = \sigma^*$ and $\sigma > \sigma^*$, respectively. For $\sigma > \sigma^*$ the smaller equilibrium is always an unstable saddle and the larger equilibrium is locally asymptotically stable outside the σ^{**} surface and is unstable inside the surface. Although the periodic solutions arising near the Hopf bifurcation surface σ^{**} are orbitally unstable for large σ^{**} , they are usually orbitally asymptotically stable and occur around the unstable equilibrium for parameter values inside the σ^{**} surface. These results for $p > 1$ are similar to those for the model with nonlinear incidence of Liu *et al.* (1987) except that periodic solutions can occur here for all $p > 1$ instead of for $p > p_1 > 1$ and the stability of the periodic solutions is different here.

For $p = 1$ the model here is the same as in Hethcote *et al.* (1981a) and the results here agree with the results there. For $p = 1$, the contact number σ is the average number of adequate contacts per infective during the infectious period. If $\sigma \leq 1$, then

each infective spreads the infection to at most one new infective, and the disease dies out. If $\sigma > 1$, then each infective has more than one adequate contact so that the disease remains endemic, and the infective replacement number σS_e is 1 at the endemic equilibrium. For the model here and the model of Liu *et al.* (1987), this intuitive threshold interpretation fails for $p \neq 1$. For $0 < p < 1$ there is no threshold since the disease always remains endemic and approaches a unique equilibrium. For $p > 1$ there is no clear threshold interpretation since the disease dies out for $\sigma < \sigma^*$ and also for some initial conditions when $\sigma > \sigma^*$.

APPENDIX

Here we calculate the asymptotic values of the stability constant K_3 in Theorem 6.3 as $y \rightarrow 2\pi^-$. Let $y = 2\pi - \epsilon$ where $\epsilon \rightarrow 0^+$. From equation (5.5) we obtain

$$a \sim \frac{4\pi}{\epsilon} - 2 - \frac{\pi\epsilon}{3}, \quad c \sim \frac{8\pi^2}{\epsilon^2} - \frac{8\pi}{\epsilon} + 2 + \frac{2\pi^2}{3}.$$

From (5.4) we find $r \sim K/\epsilon$ where $K = 2\pi(1 - \sqrt{2p - 1})/(1-p)$ for $p > 1$ and for the lower branch of the Hopf bifurcation curve when $\frac{1}{2} < p < 1$, $K = 2\pi$ for $p = 1$ and $K = 2\pi(1 + \sqrt{2p - 1})/(1-p)$ for the upper branch of the Hopf bifurcation curve when $\frac{1}{2} < p < 1$. Using (7.1) and $r \sim K/\epsilon$ yields

$$r \sim [K^2 \sigma^{**}/8\pi^2]^{1/p}.$$

Asymptotic expressions for quantities in Theorem 6.3 are

$$\Delta(\sigma^{**}, 2iy) \sim 2\pi\epsilon,$$

$$H_3 \sim \frac{p(p-1)}{6\epsilon} \left[K(p-2) - \frac{24\pi^2}{K} + 12\pi \right] + \frac{i p(p-1)\pi}{3},$$

$$A_{2,2} \sim -Kp(p-1)/(4\pi\epsilon^2),$$

$$H_2(\bar{\varphi}, A_{2,2} e^{2i\nu s}) = pA_{2,2}[-K(p-1)/\epsilon + 2\pi i]/2,$$

$$A_{2,0} \sim -Kp(p-1)\epsilon/(8\pi^2),$$

$$H_2(\varphi, A_{2,0}) = pA_{2,0}[-K(p-1) - 4\pi]/(2\epsilon).$$

The expression for N_3 is dominated by its second term so that

$$N_3 \sim K^2 p^2 (p-1)^2 / (4\pi\epsilon^3) - K p^2 (p-1) i / (2\epsilon^2).$$

Then

$$\frac{\partial \Delta}{\partial z} (\sigma^{**}, iy) \sim 4\pi/\epsilon - i 4\pi/\epsilon^2,$$

$$K_3 = \operatorname{Re} \left[N_3 / \frac{\partial \Delta}{\partial z} \right] \sim \frac{K p^2 (p-1)}{16\pi\epsilon^2} \left[\frac{K}{\pi} (p-1) + 2 \right].$$

Thus as $y \rightarrow 2\pi^-$, K_3 is positive for $p > 1$. For $p = 1$ we calculated in Hethcote *et al.* (1981a) that the stability constant is negative. For $\frac{1}{2} < p < 1$ the bracketed expression above is positive so that K_3 is asymptotically negative.

Figure Captions

- Figure 1. The saddle–node bifurcation surface $\sigma = \sigma^*$ (chopped at $r = 0$ and 15).
- Figure 2. The imaginary root curve (5.5) for (5.1) with $y \in (0, 2\pi)$. As $y \rightarrow 2\pi^-$, the neutral curve approaches $c = a^2/2$, shown here as a broken curve.
- Figure 3. Cross–sections of the Hopf bifurcation surface $\sigma = \sigma^{**}$ truncated at $r = 10$ and $\sigma = 40$. The interior starting points correspond to the intersection with the saddle–node bifurcation surface $\sigma = \sigma^*$.
- (a) p fixed at 0.75, 1.0, 1.25, 1.5, 2.0, 2.5.
 - (b) r fixed at 2.0, 4.0, 10.0.
 - (c) σ fixed at 5.0, 10.0, 20.0.
- Figure 4. Phase plane solutions to system (2.1), (2.3), (2.6) with linear $R_0(t)$ and with $p = 1.5$, $r = 4.0$, $\sigma = 20.0$, giving $S_e \simeq 0.12$, $I_e \simeq 0.18$.
- (a) $S(0) = 0.2$, $I(0) = 0.8$; limit cycle approached from outside.
 - (b) $S(0) = 0.1$, $I(0) = 0.2$; limit cycle approached from inside.

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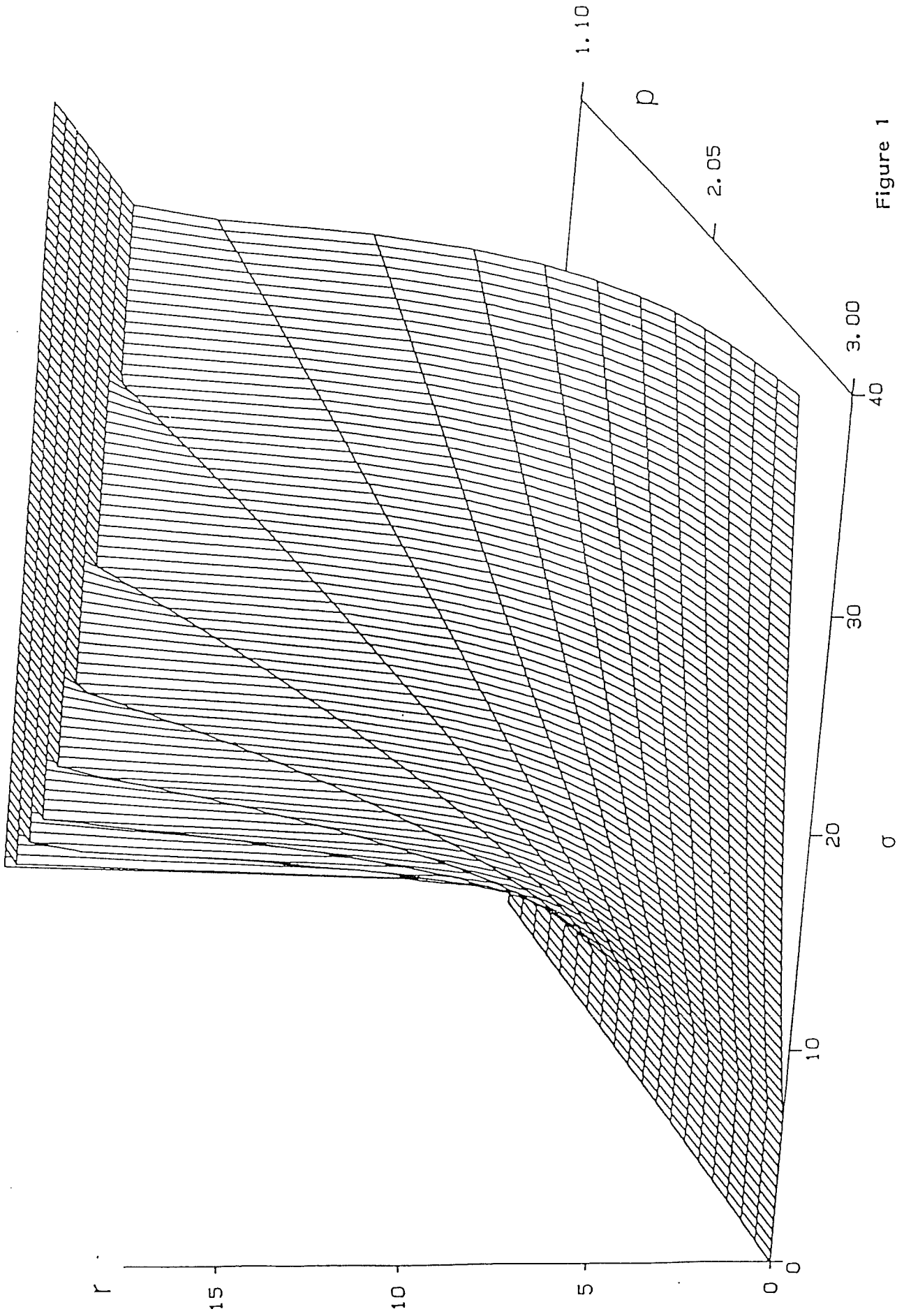


Figure 1

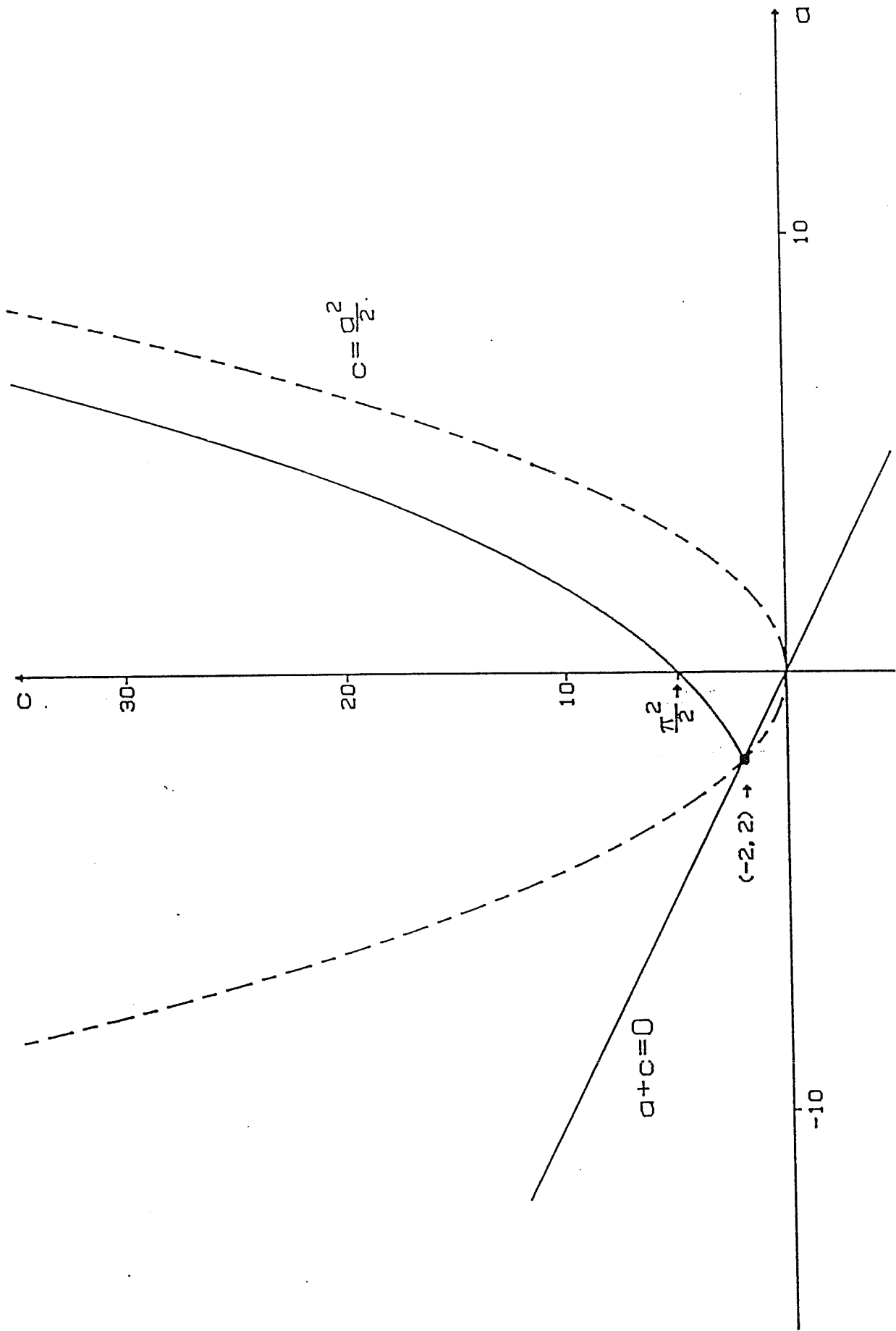


Figure 2

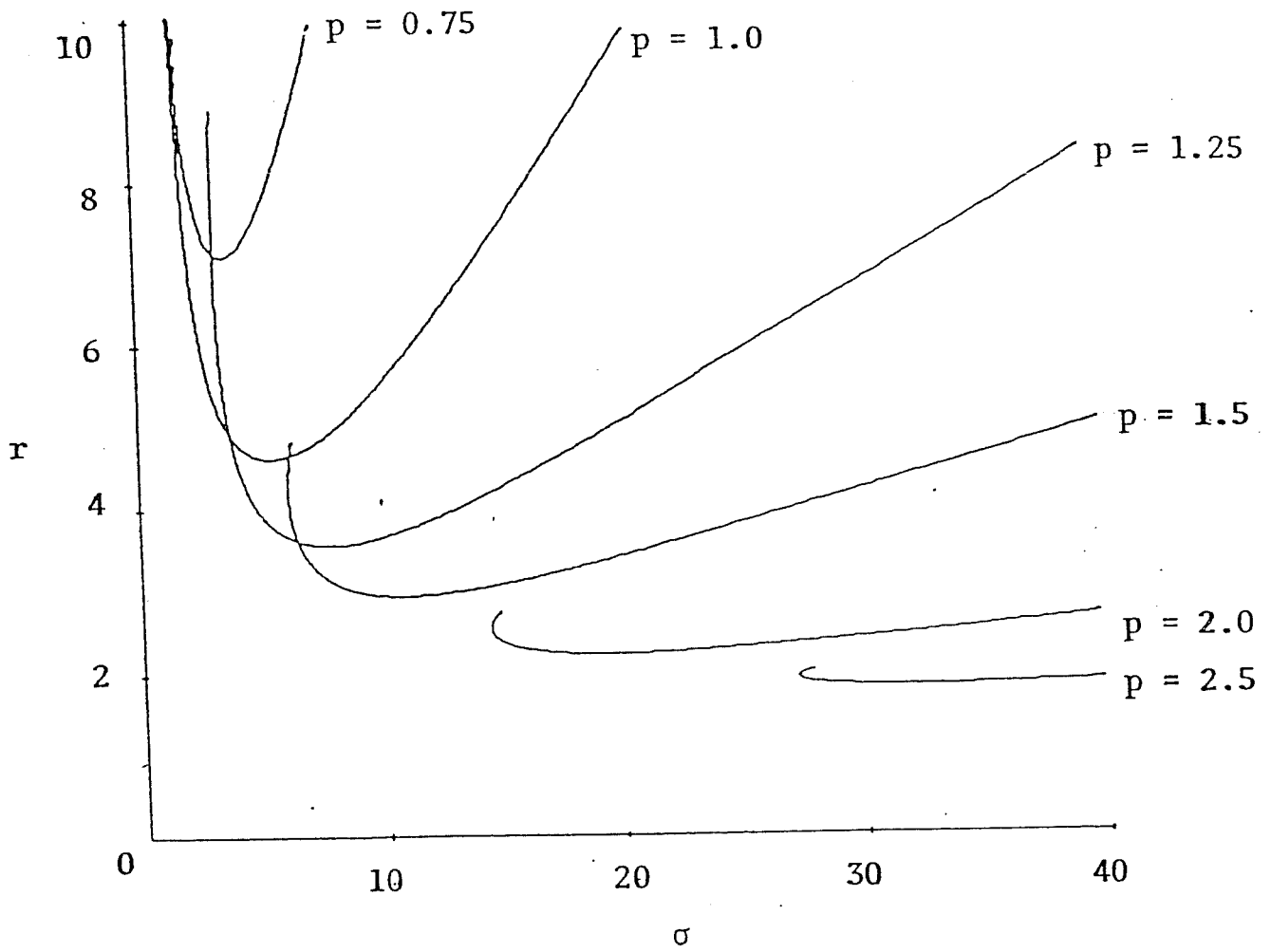


Figure 3(a)

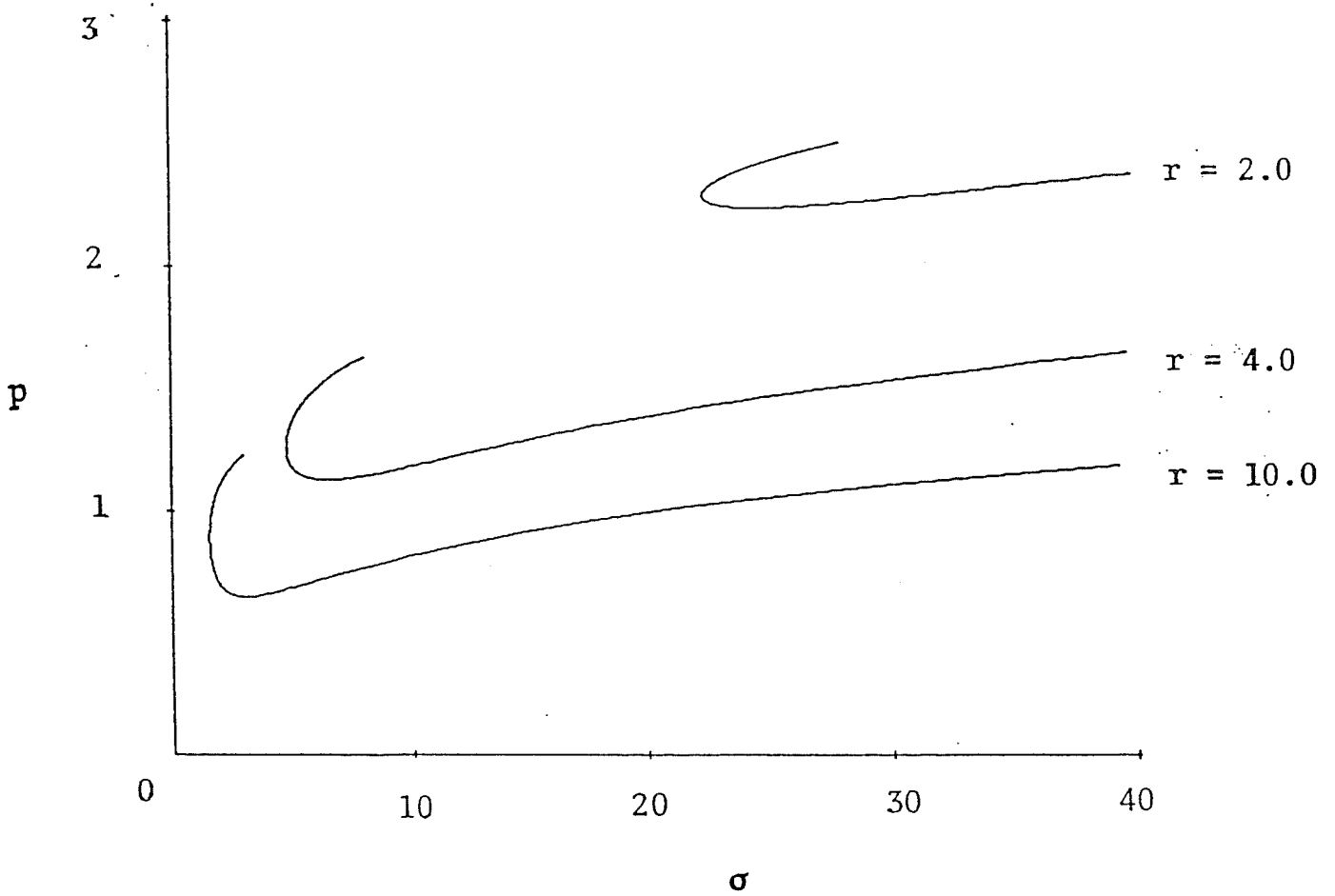


Figure 3(b)

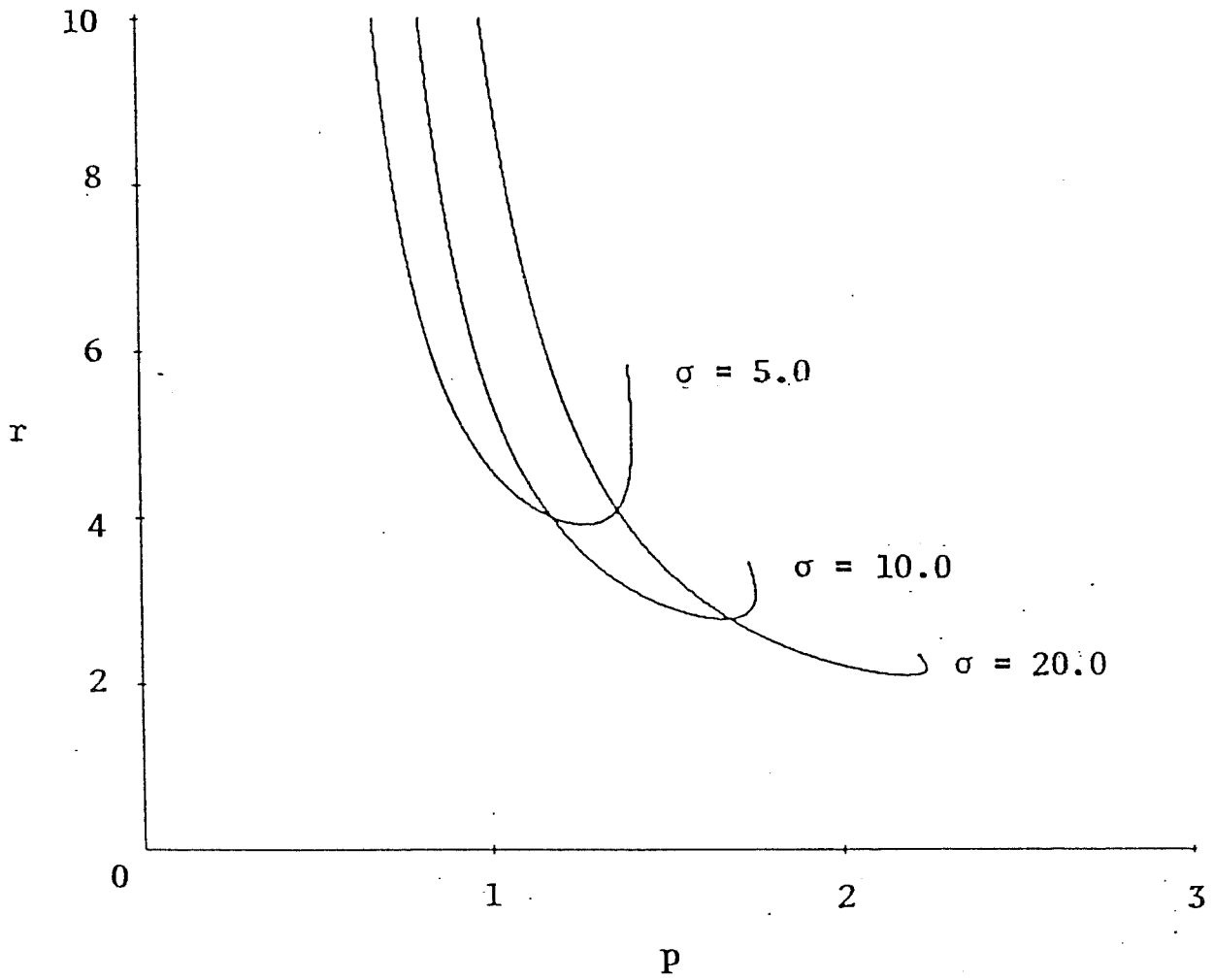


Figure 3(c)

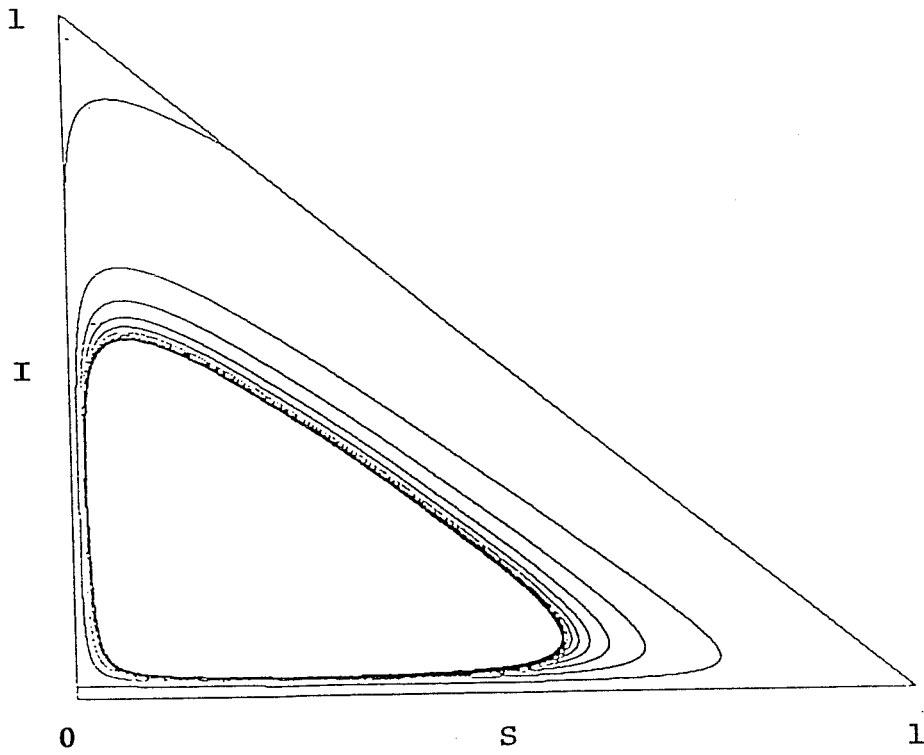


Figure 4(a)

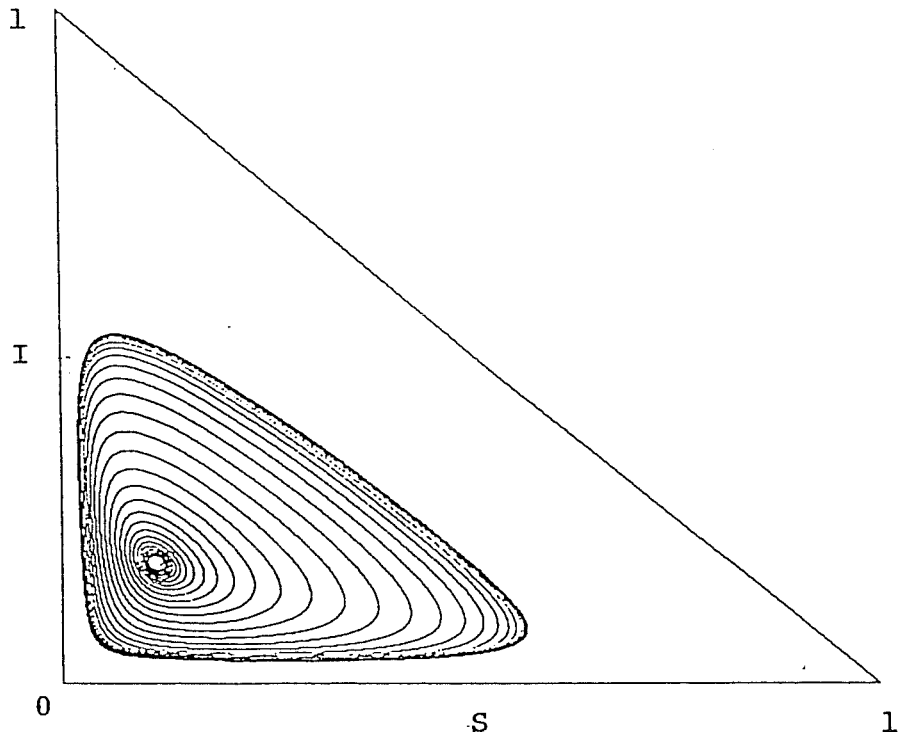


Figure 4(b)