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AN EQUATIONAL BASIS IN FOUR VARIABLES FOR THE THREE-ELEMENT TOURNAMENT

BY

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1. Introduction. As in E. Fried [1] and H. L. Skala [3], we can associate with a *tournament* $\langle T; <\rangle$ (T with a binary relation $<$ such that for all $a, b \in T$ exactly one of $a = b$, $a < b$, and $b < a$ holds) an algebra $\langle T; \wedge, \vee\rangle$ by the rule: if $x < y$, then $x = x \wedge y = y \wedge x$ and $y = x \vee y = y \vee x$, and $x = x \wedge x = x \vee x$ for all x.

In this algebra $\langle T; \wedge, \vee\rangle$, neither $\wedge$ nor $\vee$ is associative unless $\langle T; <\rangle$ is a chain, that is, $<$ is transitive. However, the two operations are idempotent, commutative; the absorption identities hold, and a weak form of the associative identities holds. In E. Fried and G. Grätzer [2], such algebras were named "weakly associative lattices."

More formally, following E. Fried [1] and H. L. Skala [3], an algebra $\langle A; \wedge, \vee\rangle$ is called a *weakly associative lattice* (WA-lattice) iff it satisfies the following set of identities:

(1) $x \wedge x = x$,
$x \vee x = x$ (idempotency);

(2) $x \wedge y = y \wedge x$,
$x \vee y = y \vee x$ (commutativity);

(3) $x \wedge (x \vee y) = x$,
$x \vee (x \wedge y) = x$ (absorption identities);

(4) $((x \wedge z) \vee (y \wedge z)) \vee z = z$,
$((x \vee z) \wedge (y \vee z)) \wedge z = z$ (weak associativity).

Define "dual" to mean interchanging $\wedge$ and $\vee$. The dual of a WA-lattice is a WA-lattice. The set of identities (1)–(4) is self-dual (i.e., the dual of

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every identity in the set (1)–(4) is in the set). For a polynomial p, we denote by $\widetilde{p}$ its dual.

The smallest example of a nontransitive tournament is the three-element cycle $\langle\{0,1,2\};<\rangle$ in which $0<1$, $1<2$, and $2<0$. In the corresponding algebra Z, neither $\wedge$ nor $\vee$ is associative.

Z plays the same role for tournaments as the two-element lattice does for distributive lattices. A tournament (algebra) $\langle T;\wedge,\vee\rangle$ is not a chain iff it contains Z as a subalgebra.

Let $\mathbf{Z}$ be the variety generated by Z. Note that $\mathbf{Z}$ is self-dual: if an algebra is in $\mathbf{Z}$, so is its dual.

Let L be a WA-lattice, $a,b\in L$. We denote by $\Theta(a,b)$ the smallest congruence relation in L under which a and b are congruent. The following is a description of $\Theta(a,b)$ in any $L\in\mathbf{Z}$ (E. Fried and G. Grätzer [2, Theorem 1]):

Characterization Theorem of $\Theta(a,b)$ in $\mathbf{Z}$. *Let $L\in\mathbf{Z}$, let $a,b,c,d\in L$, and let $a\le b$, $c\le d$. Then $c\equiv d\ (\Theta(a,b))$ iff the following two equations hold*:

$$a\wedge(c\wedge b)=a\wedge(d\wedge b),\qquad (a\vee c)\vee b=(a\vee d)\vee b\,.$$

One of the main results of E. Fried and G. Grätzer [2] is a characterization of $\mathbf{Z}$ in terms of $\Theta(a,b)$; we will need this in our proof:

Characterization Theorem of $\mathbf{Z}$. *Let $\mathbf{K}$ be a variety of WA-lattices in which for any $A\in\mathbf{K}$, $a,b,c,d\in A$, $a\le b$, $c\le d$, and $c\equiv d\ (\Theta(a,b))$ imply that $a\wedge(c\wedge b)=a\wedge(d\wedge b)$, and $(a\vee c)\vee b=(a\vee d)\vee b$. Then $\mathbf{K}\subseteq\mathbf{Z}$.*

In E. Fried and G. Grätzer [2], a finite set of identities was exhibited that form an equational basis of $\mathbf{Z}$. The identities are in five variables; so from this result we can conclude that if every five-generated subalgebra of an algebra belongs to $\mathbf{Z}$, then so does the algebra. The question was raised whether "five" could be improved to "four." ("Three" is obviously impossible, since every three-variable identity that holds in $\mathbf{Z}$ also holds in any tournament.) In this paper, we answer this question in the affirmative.

2. The identities. We build our identities from the following polynomial:

$$r(x,y,z)=(x\wedge y)\wedge((x\vee y)\wedge z),$$

and its dual. We consider the following identities:

$$r(x,y,z\wedge t)=(r(x,y,z)\wedge\widetilde{r}(x,y,t))\wedge(r(x,y,t)\wedge\widetilde{r}(x,y,z))\,,\tag{5}$$

$$\begin{aligned}r(x,y,z\vee t)=&[(r(x,y,z)\vee\widetilde{r}(x,y,t))\wedge(r(x,y,t)\vee\widetilde{r}(x,y,z))]\\&\wedge(r(x,y,z)\vee r(x,y,t))\,,\end{aligned}\tag{6}$$

and their duals (7) and (8), respectively.

Lemma. *Identities* (5)–(8) *hold in* Z.

P r o o f. These identities were checked with a computer program. For the reader's convenience, we show a quick way to check them by hand. Let a, b, and c be three distinct elements of Z. It is easily verified that

$$a \wedge (b \wedge c) = a \vee (b \vee c) = a\,.$$

Therefore, if $a \neq b$ in Z, then $a \wedge b < a \vee b$, $r(a,b,z) = a \wedge b$, and $\widetilde{r}(a,b,z) = a \vee b$, for all values of z in Z. This reduces the identities to relations involving only the two elements $x = a$ and $y = b$, which are easily verified.

Otherwise, $x = y$. If $\{x,z,t\}$ is contained in a two-element subset of Z, then the result follows easily since we work in a distributive lattice. Let $\{x,z,t\} = \{0,1,2\}$. Since the 3-cycle $0 \to 1 \to 2 \to 0$ is an automorphism of Z, we can assume that $x = 0$. Each identity is symmetric in z and t; therefore, it suffices to take $z = 1$ and $t = 2$. A check of this single case completes the proof.

Let $\mathbf{K}$ be the class of weakly associative lattices satisfying the identities (5)–(8). Then the Lemma can be restated as follows: $\mathbf{Z} \subseteq \mathbf{K}$.

3. The Theorem. Our main result is the following:

Theorem. *Identities* (1)–(8) *define* $\mathbf{Z}$.

P r o o f. Let $A \in \mathbf{K}$ and $a, b \in A$. Consider a binary relation $\sim$ on A defined as follows: for $c, d \in A$, let $c \sim d$ iff

$$(*) \qquad r(a,b,c) = r(a,b,d) \quad \text{and} \quad \widetilde{r}(a,b,c) = \widetilde{r}(a,b,d).$$

This is clearly an equivalence relation, and $a \sim b$ holds. We show that $\sim$ has the Substitution Property. Indeed, if $c \sim d$, then for all $e \in A$,

$$\begin{aligned} r(a,b,c \wedge e) &= r(a,b,d \wedge e)\,, & \widetilde{r}(a,b,c \wedge e) &= \widetilde{r}(a,b,d \wedge e)\,,\\ r(a,b,c \vee e) &= r(a,b,d \vee e)\,, & \widetilde{r}(a,b,c \vee e) &= \widetilde{r}(a,b,d \vee e)\,. \end{aligned}$$

To prove the first equation, compute:

$$\begin{aligned} &r(a,b,c \wedge e) = && \text{(by (5))}\\ &(r(a,b,c) \wedge \widetilde{r}(a,b,e)) \wedge (r(a,b,e) \wedge \widetilde{r}(a,b,c)) = && \text{(by } (*))\\ &(r(a,b,d) \wedge \widetilde{r}(a,b,e)) \wedge (r(a,b,e) \wedge \widetilde{r}(a,b,d)) = && \text{(by (5))}\\ &r(a,b,d \wedge e)\,. \end{aligned}$$

The other three proofs are similar. Thus, $\sim$ is a congruence relation on A.

Now let $c \equiv d$ $(\Theta(a,b))$. Since $a \sim b$ and $\sim$ is a congruence relation, it follows that $c \sim d$; therefore, $(*)$ holds. If, in addition, $a \leq b$ and $c \leq d$, then $(*)$ simplifies to $a \wedge (b \wedge c) = a \wedge (b \wedge d)$ and $b \vee (a \vee c) = b \vee (a \vee d)$.

Thus, in view of the Characterization Theorem of $\mathbf{Z}$, quoted in §2, this proves that $\mathbf{K} \subseteq \mathbf{Z}$; since by the Lemma, $\mathbf{K} \supseteq \mathbf{Z}$, this completes the proof of the Theorem.

COROLLARY 1. *Let A be an algebra. If every four-generated subalgebra of A belongs to $\mathbf{Z}$, then so does A.*

The identities (1)–(8) correspond closely to the identities defining distributive lattices. The identities (1)–(4) define WA-lattices, and (5)–(8) are the distributive identities. One difference shows up in (1)–(4): for lattices, we have three identites for $\vee$, the three dual ones for $\wedge$, and these two sets of identities are connected by the two absorption identities. For WA-lattices, weak associativity involves both operations. We can remedy this situation for $\mathbf{Z}$.

Consider the identities:

$$\begin{aligned} (x \wedge z) \wedge (x \wedge (y \wedge z)) &= (x \wedge z) \wedge ((x \wedge y) \wedge z)\,, \\ (x \vee z) \vee (x \vee (y \vee z)) &= (x \vee z) \vee ((x \vee y) \vee z)\,. \end{aligned} \tag{4'}$$

It is easy to see that $(4')$ holds in Z, and therefore in $\mathbf{Z}$. The role of (4) is to ensure that $a \vee b$ is the least upper bound of a and b (in the sense that $a \leq a \vee b$, $b \leq a \vee b$, and if $a \leq d$ and $b \leq d$, then $a \vee b \leq d$); and dually. This readily follows also from $(4')$.

COROLLARY 2. *The identities* (1)–(3), $(4')$, (5)–(8) *define* $\mathbf{Z}$.

Finally, we would like to point out a curiosity. The set of identities in [2] characterizing $\mathbf{Z}$ is equivalent to the identities (1)–(8) in this paper. The proof of the equivalence uses the Characterization Theorem of $\mathbf{Z}$ from [2], a result that cannot be proved without some form of the Axiom of Choice. It would be interesting to find a direct equational theoretic proof of the equivalence. The existence of such a proof is known.

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