AN EQUIVALENCE THEOREM FOR EMBEDDINGS OF COMPACT ABSOLUTE NEIGHBORHOOD RETRACTS

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In this paper we wish to prove the following theorem.

THEOREM 1. *Suppose that each of A_0 and A_1 is a compact absolute neighborhood retract (ANR) of dimension k in Euclidean n-space E^n $(2k+2\leqq n, n\geqq 5)$ such that E^n-A_i is 1-ULC (uniformly locally simply connected) for $i=0$, 1, and $f: A_0\to A_1$ is a homeomorphism such that $d(a, f(a))<\epsilon$ for each $a\in A_0$. Then there exists an ϵ-push h of (E^n, A^0) such that $h|A_0=f$.*

In [2] the authors showed that if A is a k-dimensional polyhedron topologically embedded in E^n $(2k+2\leqq n, n\geqq 5)$ such that E^n-A is $1=$ULC, then for each $\epsilon>0$, there is an ϵ-push h of (E^n, A) such that $h|A: A\to E^n$ is piecewise linear. Hence, a well-known theorem of Bing and Kister [1, Theorem 5.5] applies to prove Theorem 1 when A_0 is a polyhedron. In fact, Theorem 5.5 of [1], together with the techniques of Homma [4] and Gluck [3] and the following engulfing theorem proved in [2], make our result possible.

THEOREM 2. *Suppose that A is a k-dimensional compact ANR in E^n $(n-k\geqq 3, n\geqq 5)$ such that E^n-A is 1-ULC and $\epsilon>0$. Then there exists $\delta>0$ such that if $f: A\to E^n$ is a δ-homeomorphism and U is an open subset of E^n containing $f(A)$, then there exists an ϵ-push h of (E^n, A) such that $h(U)\supset A$.*

Following Gluck [3], we define an *ϵ-push h of the pair* (X, A), where X is a metric space and A is a subset of X such that $\overline{A}$ is compact, to be a homeomorphism of X onto itself that is ϵ-isotopic to the identity by an isotopy h_t $(t\in[0, 1])$ of X such that for each $t\in[0,1]$, $h_t|X-N_\epsilon(A)=$identity. Other terminology used here is standard, and we shall assume that it is familiar to the reader.

Actually, the proof of Theorem 1 follows from known results, once we prove a

LEMMA. *Suppose that A is a compact ANR of dimension k in E^n $(2k+2\leqq n, n\geqq 5)$ such that E^n-A is 1-ULC and $f: A\to E^n$ is an embedding such that $d(a, f(a))<\epsilon$ for each $a\in A$. Then for each $\delta>0$ there exists an ϵ-push h of (E^n, A) such that $d(h(a), f(a))<\delta$ for each $a\in A$.*

Received by the editors September 12, 1967.

[1] This research was supported in part by National Science Foundation Grant GP 5458.

Theorem 1 is then proved exactly as is Theorem 4.4 of [3], and we shall not repeat the details of the constructions involved.

PROOF OF THE LEMMA. Given $\delta>0$, there exists $\eta>0$ such that if $a, b\in A$ with $d(a, b)<\eta$, then $d(f(a), f(b))<\frac{1}{2}\delta$. Let N be a polyhedral neighborhood of A in E^n that retracts onto A by a retraction $r\colon N\rightarrow A$ such that $d(x, r(x))<\frac{1}{2}\eta$ and $d(x, fr(x))<\epsilon$ for each $x\in N$. Let T be a triangulation of N with mesh less than $\frac{1}{4}\eta$ and let T^k denote the k-skeleton of T, with $N^k=|T^k|$, the polyhedron of T^k.

The mapping $fr\colon N^k\rightarrow E^n$ has the property that if $x\in N^k$, $a\in A$, and $d(x, a)<\frac{1}{2}\eta$, then $d(fr(x), f(a))<\frac{1}{2}\delta$, since $d(r(x), a)<\eta$.

Let $g'\colon N^k\rightarrow E^n$ be a piecewise linear embedding such that $d(g'(x), fr(x))<\frac{1}{2}\delta$ and $d(x, g'(x))<\epsilon$ for each $x\in N^k$. Since $2k+2\leqq n$, we may apply Theorem 5.5 of [1] to obtain an ϵ-push g of (E^n, N^k) such that $g|N^k=g'$. Notice that if $x\in N^k$, $a\in A$, and $d(x, a)<\frac{1}{2}\eta$, then $d(g(x), f(a))\leqq d(g(x), fr(x))+d(fr(x), f(a))<\delta$. Thus there is an open set U in E^n containing N^k such that the above implication is true for $x\in U$; that is, if $x\in U$, $a\in A$, and $d(x, a)<\frac{1}{2}\eta$, then $d(g(x), f(a))<\delta$. We need one additional fact concerning the open set U.

SUBLEMMA. *There exists a $\frac{1}{2}\eta$-push ϕ of (E^n, A) such that $\phi(A)\subset U$.*

PROOF. Let $\tilde{T}^{n-k-1}$ be the dual $(n-k-1)$-skeleton of T with $\tilde{N}^{n-k-1}=|\tilde{T}^{n-k-1}|$. Choose $\eta'>0$ corresponding to $\frac{1}{4}\eta$ as in Theorem 2. From the construction in the proof of Theorem V5 of [5], we can obtain an embedding ψ of A into E^n such that $d(a, \psi(a))<\eta'$ for each $a\in A$ and $\psi(A)\cap\tilde{N}^{n-k-1}=\varnothing$. Let V be an open subset of E^n containing $\psi(A)$ such that $V\cap\tilde{N}^{n-k-1}=\varnothing$.

By Theorem 2, there exists a $\frac{1}{4}\eta$-push ϕ_1 of (E^n, A) such that $\phi_1(V)\supset A$. Then ϕ_1^{-1} is a $\frac{1}{4}\eta$-push of (E^n, A) and $\phi_1^{-1}(A)\cap\tilde{N}^{n-k-1}=\varnothing$. Since the mesh of T is less than $\frac{1}{4}\eta$, the technique of Stallings [7] may be used to obtain a $\frac{1}{4}\eta$-push ϕ_2 of $(E^n, \phi_1^{-1}(A))$ such that $\phi_2\phi_1^{-1}(A)\subset U$. Then $\phi=\phi_2\phi_1^{-1}$ is the desired $\frac{1}{2}\eta$-push of (E^n, A).

We complete the proof of the Lemma by setting $h=g\phi$. We may assume that η is chosen sufficiently small so that the composition $g\phi$ is an ϵ-push of (E^n, A). Given $a\in A$, we have $d(\phi(a), a)<\frac{1}{2}\eta$ and $\phi(a)\in U$, so that $d(g\phi(A), f(a))=d(h(a), f(a))<\delta$.

The question as to whether Theorem 1 is true when $k=1$ and $n=4$ seems very hard to answer. The method used to prove Theorem 1 involves engulfing techniques that are valid only for $n\geqq 5$. It might be possible, however, to improve the codimension restriction by one if certain other conditions are satisfied. For example, Price has shown [6] that any two piecewise linear embeddings of a k-complex K into E^n $(n=2k+1)$ are equivalent by an isotopy of E^n that is the identity outside a compact set if $H^k(K, Z)=0$. A natural question then is

QUESTION 1. Is Theorem 1 true with $n = 2k+1$ if $H^k(A_0, Z) = 0$? In particular, one might consider a special case.

QUESTION 2. Is Theorem 1 true with $n = 2k+1$ if A_0 is an absolute retract?

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