

An ergodic theorem for iterated maps

JOHN H ELTON

School of Mathematics, Georgia Institute of Technology, Atlanta, Georgia 30332, USA

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Abstract Consider a Markov process on a locally compact metric space arising from iteratively applying maps chosen randomly from a finite set of Lipschitz maps which, on the average, contract between any two points (no map need be a global contraction) The distribution of the maps is allowed to depend on current position, with mild restrictions Such processes have unique stationary initial distribution [BE], [BDEG]

We show that, starting at any point, time averages along trajectories of the process converge almost surely to a constant independent of the starting point This has applications to computer graphics

1 Introduction

Let (X, d) be a metric space in which sets of finite diameter are relatively compact. Let $w_i, X \rightarrow X$ be Lipschitz maps, with $d(w_i x, w_i y) \leq s_i d(x, y)$ for x, y in X , $i = 1, \dots, N$. A good example is affine maps on \mathbb{R}^n . Let $p_i, X \rightarrow [0, 1]$ such that $p_i(x) \geq 0$ and $\sum_{i=1}^N p_i(x) = 1$, and assume that the p_i 's are continuous. Define a Markov transition probability by

$$p(x, B) = \sum_{i=1}^N p_i(x) 1_B(w_i x)$$

This is the probability of transfer from $x \in X$ into the Borel set B . Intuitively, pick a number i between 1 and N according to the distribution $p_i(x)$ and go to $w_i x$.

Such processes have been discussed in many places under the assumption that the maps are contractions and usually that the p_i 's are constants [BD], [DF], [DS], [H], [K] (Karlín [K] discussed variable p_i 's). It was shown recently [BE], [BDEG] that none of the w_i 's need be contractions, but that if there is contraction 'on the average' between any two points, i.e.

$$\prod_{i=1}^N d(w_i x, w_i y)^{p_i(x)} \leq r d(x, y) \quad \forall x, y, \text{ where } r < 1,$$

and if the p_i 's are bounded away from 0 and have moduli of continuity ϕ_i satisfying Dini's condition (i.e. $\phi_i(t)/t$ is integrable over $(0, \alpha)$ for some $\alpha > 0$), then there is a unique, attractive stationary initial probability distribution μ for the process. This means

$$\int p(x, B) d\mu(x) = \mu(B)$$

for all Borel sets B and, for all initial probability distributions ν , $V^n\nu$ converges in distribution (that is, weakly) to μ , $\int f dV^n\nu \rightarrow \int f d\mu$ for all bounded continuous functions f on X

Our object is to show that starting at *any* $x \in X$, the trajectories (orbits) of the process converge in distribution to μ almost surely. By this we mean that for almost all trajectories x, x_1, x_2, \dots of the process starting at x , the time averages

$$\frac{1}{n+1} \sum_{k=0}^n f(x_k)$$

converge to $\int f d\mu$ for all bounded continuous f , or in yet other terms, the empirical distribution

$$\nu_n = \frac{1}{n+1} \sum_{k=0}^n \delta_{x_k}$$

of the first $n+1$ points along the trajectory converges weakly to μ as $n \rightarrow \infty$

Let us explain why this is important. It follows (see Lemma 1) from the classical pointwise ergodic theorem that for μ -almost all $x \in X$, almost all trajectories starting at x converge in distribution to μ (in the sense just explained). But in applications to computer graphics, for example (see [BD]) we may have no way of choosing the starting x according to the measure μ , in fact, the idea is to start at some x and let a computer-generated realization of the process ‘draw a picture’ of μ .

A special case of this result, when the maps w_i are contractions with a special disjointness condition, and the p_i ’s are constants, was stated already by Diaconis and Shashahani [DS]. Most of the difficulty of our proof arises from having non-contractions and variable p_i ’s.

First we prove a general lemma about Markov processes, and then we state and prove the main theorem, using a martingale argument.

2 Markov processes with unique stationary distribution

Let (X, \mathcal{F}) be an arbitrary measurable space, and let $p(\cdot, \cdot) : X \times \mathcal{F} \rightarrow [0, 1]$ be a transition probability, i.e. $p(x, \cdot)$ is a probability measure for each x , and $p(\cdot, A)$ is a measurable function for all $A \in \mathcal{F}$. A (discrete-time) stochastic process $\{Z_n, n = 0, 1, \dots\}$ with values in X is called a Markov process with transition probability p if

$$P\{Z_{n+1} \in A \mid Z_0 = z_0, \dots, Z_n = z_n\} = p(z_n, A) \quad \text{a.s.}$$

X is called the state space. Define the operator V on finite measures by

$$V\nu(A) = \int p(x, A) d\nu(x), \quad A \in \mathcal{F}$$

A probability measure μ is called a stationary initial distribution if $V\mu = \mu$. If μ is a stationary initial distribution and if Z_0 has distribution μ , then $\{Z_n\}$ will be a stationary stochastic process. Assume for the rest of this paragraph that Z_0 has a stationary initial distribution so that $\{Z_n\}$ is a stationary process. A is called an invariant event if there exists $C \in \mathcal{F}_\infty$ such that $A = \{(Z_k, Z_{k+1}, \dots) \in C\}$ for all $k \geq 0$, where \mathcal{F}_∞ is the σ -algebra in X^∞ generated by measurable cylinders. A is called almost invariant if there is an invariant event B so that $P(A \Delta B) = 0$. Let \mathcal{I} denote

the σ -algebra of almost invariant events. The process $\{Z_n\}$ is called *ergodic* if for every $A \in \mathcal{F}$, $P(A) = 0$ or 1 . A reference for the above definitions is [D].

The next lemma is surely known, but we were unable to find a statement of it for general Markov processes. We did find it stated in [FK] for a special case. In any case, it follows very easily from well-known results.

LEMMA 1 *If μ is the unique stationary initial distribution (or just an extreme point of the set of stationary initial distributions), then the process $\{Z_n\}$ with Z_0 having distribution μ is ergodic.*

Proof If not, there is $A \in \mathcal{F}$ with $0 < P(A) < 1$. Then there exists $C \in \mathcal{F}$ such that $A = \{Z_n \in C\}$ a.e. for all $n \geq 0$, since $\{Z_n\}$ is a stationary Markov process [see S]. Define

$$\nu(B) = \mu(B \cap C) / \mu(C) \quad \text{and} \quad \lambda(B) = \mu(B \cap \sim C) / \mu(\sim C)$$

(note that $\mu(C) = P(A)$ and $\mu(\sim C) = P(\sim A)$). Then $\mu = \mu(C)\nu + \mu(\sim C)\lambda$, so the proof will be completed by showing that ν (and hence λ) is a stationary initial distribution, since clearly $\nu \neq \lambda$.

Now

$$\begin{aligned} \nu(B) &= P(Z_1 \in B \cap C) / \mu(C) \\ &= P((Z_1 \in B) \cap (Z_1 \in C)) / \mu(C) \\ &= P((Z_1 \in B) \cap (Z_0 \in C)) / \mu(C) \\ &= \frac{1}{\mu(C)} \int_C P(Z_1 \in B | Z_0 = z) d\mu(z) \\ &= \frac{1}{\mu(C)} \int_C p(z, B) d\mu(z) \\ &= \int p(z, B) d\nu(z), \end{aligned}$$

since clearly $d\nu/d\mu = (1/\mu(C))1_C$. But this says that ν is a stationary initial distribution. \square

Remark The processes discussed in the introduction and the next section are *not* what is called *indecomposable* in [B] and *Markov ergodic* in [S], as the following simple example shows, so we could not quote the theorems in those references for our application.

Example Let $X = [0, 1]$, $w_1x = \frac{1}{2}x$, $w_2x = \frac{1}{2} + \frac{1}{2}x$, $p_i = 1/2$, $i = 1, 2$. Then all trajectories starting at a rational number in $[0, 1]$ stay in the rationals, and all trajectories starting at an irrational number in $[0, 1]$ stay in the irrationals. Thus the process is *not* indecomposable/Markov ergodic as defined in [B], [S] (some people use the word 'indecomposable' differently). However, the process *is* ergodic, since there is a unique stationary initial distribution.

3 Main results

Let $\Omega = N^\infty = \{(i_1, i_2, \dots) \mid 1 \leq i_j \leq N \text{ and } i_j \text{ is an integer for each } j\}$. Let \mathcal{A} be the σ -algebra generated by the cylinders in Ω .

Return now to the setup of the introduction For each $x \in X$, let P_x be the probability measure on \mathcal{A} defined on cylinders by

$$P_x((i_1, i_2, \dots, i_n)) = p_{i_1}(x)p_{i_2}(w_{i_1}x)p_{i_3}(w_{i_2}w_{i_1}x) \dots p_{i_n}(w_{i_{n-1}} \dots w_{i_1}x)$$

(we abuse notation by writing $P_x((i_1, i_2, \dots, i_n))$ when we mean $P_x(\{(i_1, i_2, \dots, i_n)\} \times N \times N \times N \times \dots)$) It is clear this is precisely the probability measure for realizations of the Markov process starting at x That is, if we consider a Markov process $\{Z_n, n = 0, 1, \dots\}$ with state space X and transition probability p as given in the introduction, then

$$P((Z_0, Z_1, \dots) \in B | Z_0 = x) = P_x\{(i_1, i_2, \dots) : (x, w_{i_1}x, w_{i_2}w_{i_1}x, \dots) \in B\}$$

for every $B \in \mathcal{F}_\infty$

THEOREM Suppose there exists $r < 1$ such that

$$\prod_{i=1}^N d(w_i x, w_i y)^{p_i(v)} \leq rd(x, y) \quad \forall x, y \text{ in } X$$

Assume there is $\delta > 0$ such that $p_i(x) \geq \delta$ for all x and i , and that the moduli of continuity of the p_i 's satisfy Dini's condition Let μ be the unique stationary initial distribution for the Markov process described above (see [BDEG]) Then for every x in X , there exists $G_x \subset \Omega$ such that $P_x(G_x) = 1$ and for $(i_1, i_2, \dots) \in G_x$, we have

$$\frac{1}{n+1} \sum_{k=0}^n f(w_{i_k} \dots w_{i_1} x) \rightarrow \int f d\mu$$

for all $f \in C(X)$, that is, almost all trajectories $x, w_{i_1}x, w_{i_2}w_{i_1}x, \dots$ starting at x converge in distribution to μ (in the sense explained in the introduction)

COROLLARY 1 Let ν be any probability measure, and let $\{Z_n\}$ be the Markov process with initial distribution ν and transition probability as above Assume the hypotheses of the Theorem Then for all $f \in C(X)$,

$$\frac{1}{n+1} \sum_{k=0}^n f(Z_k) \rightarrow \int f d\mu \quad \text{a.s.}$$

Remark It is shown in [FK] that Corollary 1 holds, in case X is a compact metric space, for a general transition probability for which it is only required that $x \mapsto p(x, \cdot)$ is continuous with the measures being given the w^* -topology, and that there is a unique stationary initial distribution

COROLLARY 2 If $B \subset X$ is such that $\mu(\partial B) = 0$, then for any $x \in X$, if $(i_1, i_2, \dots) \in G_x$, the average amount of time the trajectory spends in B converges to $\mu(B)$, that is,

$$\lim_{k \rightarrow \infty} \frac{\#\{j : 0 \leq j \leq k, w_{i_j} \dots w_{i_1} x \in B\}}{k+1} = \mu(B)$$

This follows from a well-known consequence of weak convergence, and generalizes a statement of [DS]

We prove two lemmas and then the Theorem and Corollary 1 The first lemma uses a martingale argument

LEMMA 2 Let $x, y \in X, x \neq y$ Assume the hypotheses of the theorem Let $r < r_1 < 1$

(i) For all $\epsilon > 0$, there exist n_ϵ and $S \subset \Omega$ with $P_\nu(S) < \epsilon$ such that

$$n \geq n_\epsilon \Rightarrow d(w_{i_n}, w_{i_1}x, w_{i_n}, w_{i_1}y) \leq r_1^n d(x, y)$$

except for (i_1, i_2, \dots) in S ,

$$(ii) \lim_{n \rightarrow \infty} d(w_{i_n}, w_{i_1}x, w_{i_n}, w_{i_1}y) = 0 \text{ a.s. } -P_x$$

Proof Let $s = \max\{s_i, i = 1, \dots, N\}$ Wlog assume $s \geq 1$ Define random variables X_n on Ω by

$$X_n(i_1, i_2, \dots) = \begin{cases} \left[\log \frac{d(w_{i_n}, w_{i_1}x, w_{i_n}, w_{i_1}y)}{d(w_{i_{n-1}}, w_{i_1}x, w_{i_{n-1}}, w_{i_1}y)} \right] \vee \frac{1}{\delta} \log(r/s) \\ \text{if } d(w_{i_{n-1}}, w_{i_1}x, w_{i_{n-1}}, w_{i_1}y) \neq 0, \\ \log r \text{ otherwise} \end{cases}$$

The purpose of the $(1/\delta) \log(r/s)$ term is to keep X_n bounded below, it is already bounded above by $\log s$

Claim $E(X_n | i_1, \dots, i_{n-1}) \leq \log r$ for all $n \geq 1$ The expectation means with respect to the probability measure P_ν on Ω

Proof Assume $d(w_{i_{n-1}}, w_{i_1}x, w_{i_{n-1}}, w_{i_1}y) \neq 0$ Then

$$E(X_n | i_1, \dots, i_{n-1})$$

$$= \sum_{i_n=1}^N p_{i_n}(w_{i_{n-1}}, w_{i_1}x) \left[\log \frac{d(w_{i_n}, w_{i_1}x, w_{i_n}, w_{i_1}y)}{d(w_{i_{n-1}}, w_{i_1}x, w_{i_{n-1}}, w_{i_1}y)} \right] \vee \frac{1}{\delta} \log(r/s)$$

Assume the expression in brackets is $\geq (1/\delta) \log(r/s)$ for each i_n Then the hypothesis of the Theorem (take logarithms) implies that the above is $\leq \log r$ If for some i_n the expression in brackets is $\leq (1/\delta) \log(r/s)$ (which is negative), the fact that $p_{i_n} \geq \delta$ is easily seen to imply that the above is still $\leq \log r$ The claim is proved

Now let $D_n = X_n - E(X_n | i_1, \dots, i_{n-1})$, so D_n is a martingale difference sequence, and $|D_n| \leq 2|X_n| \leq B$, say

Let $Y_n = \sum_{k=1}^n (1/k)D_k$, so Y_n is a martingale Now $E(Y_n^2) \leq B^2 \sum_{k=1}^n 1/k^2$ since $D_k \perp D_l$ for $k \neq l$ (because they are martingale differences) Thus Y_n is an L^2 -bounded martingale, and so $Y_n \rightarrow a.s.$ Then by Kronecker's lemma,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n D_k = 0 \text{ a.s.}$$

Thus,

$$\overline{\lim} \frac{1}{n} \sum_{k=1}^n \log \frac{d(w_{i_k}, w_{i_1}x, w_{i_k}, w_{i_1}y)}{d(w_{i_{k-1}}, w_{i_1}x, w_{i_{k-1}}, w_{i_1}y)} - \log r \leq 0 \text{ a.s.}$$

This telescopes to

$$\overline{\lim} \log \left(\frac{d(w_{i_n}, w_{i_1}x, w_{i_n}, w_{i_1}y)}{d(x, y)} \right)^{1/n} \leq \log r \text{ a.s.,}$$

that is,

$$\overline{\lim} \left(\frac{d(w_n, w_1, x, w_1, y)}{d(x, y)} \right)^{1/n} \leq r < r_1 \quad \text{as}$$

It is now easy to get from this that (i) of the conclusion holds, (ii) follows immediately from (i) □

LEMMA 3 *Assume the hypotheses of the theorem. Then for all x, y in X , P_x is absolutely continuous with respect to P_x .*

Proof Let $P_x(E) = 0$, and let $\varepsilon > 0$. We shall show $P_y(E) < \varepsilon$. Let r_1 be as in Lemma 2.

Let ϕ_i be the modulus of continuity of p_i , and let $\phi = \phi_1 \vee \phi_2 \vee \dots \vee \phi_N$. Note ϕ is increasing.

Claim $\sum_{k=1}^{\infty} \phi(r_1^k d(x, y)) < \infty$

Proof

$$\begin{aligned} \infty &> \int_0^{d(x,y)} \frac{\phi(t)}{t} dt = \sum_{k=1}^{\infty} \int_{r_1^k d(x,y)}^{r_1^{k-1} d(x,y)} \frac{\phi(t)}{t} dt \\ &\geq \sum_{k=1}^{\infty} (r_1^{k-1} - r_1^k) d(x, y) \frac{\phi(r_1^k d(x, y))}{r_1^{k-1} d(x, y)} \\ &= (1 - r_1) d(x, y) \sum_{k=1}^{\infty} \phi(r_1^k d(x, y)) \end{aligned}$$

which proves the claim

Now choose m so large that $m > n_{\varepsilon/2}$ from (i) of Lemma 2 and also $\sum_{k=m+1}^{\infty} \phi(r_1^k d(x, y)) < \delta/2$

Let \mathcal{A}_n be the cylinder sets in Ω depending only on the first n coordinates. By a standard approximation result, there exist sets $A_n \in \mathcal{A}_n$ such that $E \subset \cup A_n$, the sets A_n are disjoint and $P_x(\cup A_n) < (\varepsilon/4)(\delta/(1-\delta))^{-m}$

Let $Q_n = \{(i_1, i_2, \dots) : d(w_{i_k}, w_1, x, w_{i_k}, w_1, y) \leq r_1^k d(x, y) \text{ for } m \leq k \leq n\}$, $n \geq m$. Let $Q_n = \Omega$ for $n < m$. Thus $Q_n \in \mathcal{A}_n$. Let $Q = \cap_{n \geq 1} Q_n$. By Lemma 2(i), $P_y(\sim Q) < \varepsilon/2$. Let $n \geq m$. Now if $(i_1, i_2, \dots) \in Q_n$,

$$\begin{aligned} &p_{i_1}(y) \quad p_{i_n}(w_{i_{n-1}}, w_1, y) \\ &\leq p_{i_1}(x) \quad p_{i_n}(w_{i_{n-1}}, w_1, x) \left(\frac{1-\delta}{\delta} \right)^m \\ &\quad \times \prod_{k=m+1}^n \left[1 + \frac{p_{i_k}(w_{i_{k-1}}, w_1, y) - p_{i_k}(w_{i_{k-1}}, w_1, x)}{p_{i_k}(w_{i_{k-1}}, w_1, x)} \right] \\ &\leq p_{i_1}(x) \quad p_{i_n}(w_{i_{n-1}}, w_1, x) \left(\frac{1-\delta}{\delta} \right)^m \prod_{k=m+1}^n \left[1 + \frac{\phi(r_1^k d(x, y))}{\delta} \right] \end{aligned}$$

But

$$\prod_{k=m+1}^n \left[1 + \frac{\phi(r_1^k d(x, y))}{\delta} \right] \leq 1 + 2 \sum_{k=m+1}^{\infty} \frac{\phi(r_1^k d(x, y))}{\delta} \leq 2,$$

so

$$p_{i_1}(y) p_{i_n}(w_{i_{n-1}} \dots w_{i_1}y) \leq 2 \left(\frac{1-\delta}{\delta}\right)^m p_{i_1}(x) p_{i_n}(w_{i_{n-1}} \dots w_{i_1}x)$$

When $n < m$, this holds trivially for any (i_1, i_2, \dots)

Thus,

$$\begin{aligned} P_y(Q \cap A_n) &\leq P_x(Q_n \cap A_n) \\ &= \sum_{(i_1, \dots, i_n)} \sum_{(i_1, i_2, \dots) \in Q_n \cap A_n} p_{i_1}(y) p_{i_n}(w_{i_{n-1}} \dots w_{i_1}y) \\ &\leq \sum_{(i_1, \dots, i_n)} \sum_{(i_1, i_2, \dots) \in Q_n \cap A_n} 2 \left(\frac{1-\delta}{\delta}\right)^m p_{i_1}(x) p_{i_n}(w_{i_{n-1}} \dots w_{i_1}x) \\ &\leq 2 \left(\frac{1-\delta}{\delta}\right)^m P_x(A_n) \end{aligned}$$

So $P_y(\cup (Q \cap A_n)) \leq 2((1-\delta)/\delta)^m P_x(\cup A_n)$ since the A_n 's are disjoint. Now the right side is $< \epsilon/2$ by construction.

Also $P_y(\cup (\sim Q \cap A_n)) \leq P_y(\sim Q) < \epsilon/2$, so we have then $P_y(E) \leq P_y(\cup A_n) < \epsilon$. □

Proof of the Theorem Let $\{Z_n\}$ be the Markov process with transition probability p as given in the introduction and such that Z_0 has distribution μ . Then the process is stationary since μ is a stationary initial distribution, and is ergodic by Lemma 1 since μ is unique. Let $f \in C_c(X)$, the continuous functions with compact support. Then $\{f(Z_n), n = 0, 1, \dots\}$ is also stationary and ergodic [B, p 119]. Let

$$B = \left\{ (x_0, x_1, \dots) \in X^\infty : \frac{1}{n+1} \sum_{k=0}^n f(x_k) \rightarrow \int f d\mu \right\}$$

By the classical pointwise ergodic theorem, $P((Z_0, Z_1, \dots) \in B) = 1$.
But

$$\begin{aligned} P((Z_0, Z_1, \dots) \in B) &= \int P((Z_0, Z_1, \dots) \in B | Z_0 = x) d\mu(x) \\ &= \int P_x((i_1, i_2, \dots) (x, w_{i_1}x, w_{i_2}w_{i_1}x, \dots) \in B) d\mu(x) \end{aligned}$$

Thus, for some $x_0 \in X$,

$$P_{x_0}((i_1, i_2, \dots) (x_0, w_{i_1}x_0, w_{i_2}w_{i_1}x_0, \dots) \in B) = 1$$

Let $G = \{(i_1, i_2, \dots) (x_0, w_{i_1}x_0, w_{i_2}w_{i_1}x_0, \dots) \in B\}$. Thus $P_{x_0}(G) = 1$ and for $(i_1, i_2, \dots) \in G$,

$$\frac{1}{n+1} \sum_{k=0}^n f(w_{i_k} \dots w_{i_1}x_0) \rightarrow \int f d\mu$$

By Lemma 3, $P_y(G) = 1$ for every $y \in X$. By Lemma 2(ii), for every $y \in X$, there exists H_y with $P_y(H_y) = 1$ and for $(i_1, i_2, \dots) \in H_y$,

$$\frac{1}{n+1} \sum_{k=0}^n f(w_{i_k} \dots w_{i_1}y) - f(w_{i_k} \dots w_{i_1}x_0) \rightarrow 0$$

(note f is uniformly continuous) Thus for $(t_1, t_2, \dots) \in G \cap H_y$,

$$\frac{1}{n+1} \sum_{k=0}^n f(w_{t_k}, w_{t_k}, y) \rightarrow \int f d\mu,$$

and $P_y(G \cap H_y) = 1$

In the above, G and H_y depended on f . But since $C_c(X)$ is separable (since X is σ -compact), we obtain that for each $y \in X$, there exists G_y with $P_y(G_y) = 1$ such that

$$\frac{1}{n+1} \sum_{k=0}^n f(w_{t_k}, w_{t_k}, y) \rightarrow \int f d\mu$$

for each f in a countable dense subset of $C_c(X)$, and then a 3ϵ argument gives this for each $f \in C_c(X)$. Finally, since μ is a probability measure, it is easy to see (by Urysohn's lemma) that this holds for all $f \in C(X)$ □

Proof of Corollary 1 As in the proof of the theorem, let

$$B = \left\{ (x_0, x_1, \dots) \in X^\infty \mid \frac{1}{n+1} \sum_{k=0}^n f(x_k) \rightarrow \int f d\mu \right\}$$

Then

$$\begin{aligned} P((Z_0, Z_1, \dots) \in B) &= \int P((Z_0, Z_1, \dots) \in B \mid Z_0 = x) d\nu(x) \\ &= \int P_x((t_1, t_2, \dots) (x, w_{t_1}x, w_{t_2}w_{t_1}x, \dots) \in B) d\nu(x) \end{aligned}$$

But the integrand is 1 for each x by the theorem □

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