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AN ESTIMATE FOR THE LOSS PROBABILITY IN A QUEUEING SYSTEM OF THE $MAP/G/m/0$ TYPE IN THE CASE OF LIGHT TRAFFIC

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D. BAUM AND I. N. KOVALENKO

Abstract. We consider a queueing system with losses and with a general distribution of the service time. It is assumed that the input is of the MAP type and the phase process assumes values in a general measurable space. The asymptotic behavior of the loss probability is studied for the case where the mean service time tends to zero. In particular, we find conditions under which the loss probability is asymptotically invariant with respect to the shape of the service time.

1. Introductory remarks

Algebraic methods have become widely popular in the theory of queueing systems over the last 15–20 years. These methods are useful for the analysis of real systems, mainly for computer networks and telecommunication systems. On the other hand, the implementation of these methods became available due to the growth of possibilities with computers.

The class of models of input random events used in the theory as well as in practice is being extended essentially nowadays (see [1]–[4]). The so-called MAP model (with a Markov input) is the most useful one. The input random events in this model depend on the states of a certain Markov process (called the "phase" process) and on the changes of states of this process. It is worthwhile mentioning that the processes mentioned above as well as many other models are, from a formal point of view, particular cases of processes with homogeneous second component (see [5]).

Models in the case of light traffic are important in many problems in the analysis of queueing systems and the loss probability plays an important rôle for all those models (see [6] and [7]).

Of special interest is the question on the invariance of the loss probability with respect to the distribution of the service time if the mean service time is known. The papers [8]–[12] treat this problem for a wide class of queueing systems and networks. It is proved in [8] that the property of invariance of the loss probability for a $MAP/G/m/0$ queuing system holds only if the input is Poissonian.

It was observed recently that systems that have no invariance of the loss probability in the case of regular traffic may possess this property if the traffic is light (see [6] and the references therein). In this paper, we study this phenomenon for queueing systems of the $MAP/G/m/0$ type.

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2. A queueing system and its basic characteristics

2.1. **Phase process.** Let E be a complete metric space, $\Im$ be a σ-algebra of its subsets, and let $P(u,\cdot)$ be a Markov transition function; that is, $P(u,\cdot)$ is a probability measure for all $u \in E$ and $P(\cdot, A)$ is a Borel function of the first argument for all $A \in \Im$.

We treat the process $(U_n, n \geq 0)$ of phase transitions as a Markov chain for which $P(u,\cdot)$ is its Markov transition function. By $\pi_0 = \pi_0(\cdot)$ we denote the initial distribution of U_0 and let $\pi_n = \pi_n(\cdot)$ be the distribution of U_n; that is,

$$\pi_n = \int_E \pi_0(du) P_n(u,\cdot),$$

where $P_n(u,\cdot)$ is the transition function over n steps. Note that $P_1(u,\cdot) = P(u,\cdot)$ and

$$P_{n+1}(u,\cdot) = \int P(u,dv) P_n(v,\cdot), \qquad n \geq 1.$$

Assumption 1. The sequence (U_n) is stationary. This means that $\pi_n = \pi$, $n \geq 0$, where π is a distribution on $(E, \Im)$.

2.2. **Sojourn times.** Consider the phase process $U(t)$, $t \geq 0$, as a semi-Markov process with the transition times $0 = T_0, T_1, T_2, \dots$. Assume that the transition moments of the process U are determined by (U_n); that is,

$$U(t) = U_n \quad \text{for } T_n \leq t < T_{n+1}$$

and for all $n \geq 0$. Thus $U(t)$ is a right continuous function with probability one. The sojourn times at the states $T_{n+1} - T_n$ are defined as follows. If $U_n = u$, then, independently of the past, $T_{n+1} - T_n$ is an exponentially distributed random variable with parameter $\lambda(u)$ being a Borel function of $u \in E$. The transitions (U_n) are independent of the sojourn times, so that

$$\mathsf{P}\{U_{n+1} \in \cdot \mid \text{the past}; U_n = u\} = P(u,\cdot).$$

2.3. **Input flow.** Assume that

$$P(u,\cdot) = F(u,\cdot) + G(u,\cdot),$$

where F and G are nonnegative measures on $(E, \Im)$ for every fixed u and both F and G are Borel functions of $u \in E$. The transitions of the type F are associated with the arrivals of customers, while customers do not arrive in the case of transitions of the type G. Formally, the input of the queueing system is determined by the following assumptions.

1. Customers may arrive to the system only at the moments $(T_n, n \geq 1)$.
2. No more than one customer may arrive at every time T_n, $n \geq 1$.
3. If I_n is the indicator of the event {a customer arrives to the system at the moment T_n}, then

$$\mathsf{P}\{I_n = I; U_n \in \cdot \mid U_{n-1} = u; \text{the past}\} = F(u,\cdot);$$
$$\mathsf{P}\{I_n = 0; U_n \in \cdot \mid U_{n-1} = u; \text{the past}\} = G(u,\cdot).$$

If the sequences (U_n) and (T_n) are known, then the input flow is determined by the sequence of independent Bernoulli trials (I_n).

Assumption 2. There exist positive constants $\underline{\lambda}$ and $\overline{\lambda}$ such that

$$\underline{\lambda} \leq \lambda(u) \leq \overline{\lambda}, \qquad u \in E.$$

2.4. **A $MAP/G/m/0$ queueing system.** According to the Kendall classification, the number "0" in this abbreviation means that a customer does not wait for the service in the system if there is no free server, m denotes the number of servers, and G means that the service times are independent random variables Y_n with the distribution function $B_\tau(x) = 1 - B_\tau^c(x)$ whose mean value $\tau > 0$ is finite.

We study such a system in the case of light traffic and for the scheme of series. The main attention is paid to the behavior of the loss probability as $\tau \to 0$. We assume that the input flow is independent of τ. The expression "scheme of series" has its roots in the theory of limit theorems of probability theory, but in our context it means that the dependence of the distribution of $B_\tau(x)$ on τ is rather complicated; the precise statements can be found in Section 4.

Finally, the symbol "MAP" in the abbreviation $MAP/G/m/0$ determines the input flow that satisfies all the assumptions discussed above.

3. The loss probability

Denote by $J_{n\tau}$ the indicator of the event

{a customer arrives at the moment T_n and leaves the system without service}.

Put

$$N_n = I_1 + \cdots + I_n,$$
$$L_{n\tau} = J_{1\tau} + \cdots + J_{n\tau},$$

and

$$Q_{n\tau} = \frac{\mathsf{E}\{L_{n\tau}\}}{\mathsf{E}\{N_n\}}. \tag{1}$$

We study the behavior of $Q_{n\tau}$ for large n and small τ. Since (U_n) is stationary, it follows that

$$\mathsf{E}\{I_n\} = \int_E \pi(du) F(u, E).$$

We avoid the trivial case where this number is zero, so that

$$\mathsf{E}\{N_n\} = n \int_E \pi(du) F(u, E) \tag{2}$$

grows linearly together with n.

Now we consider $J_{n\tau}$. In order that a customer leaves the system without service at a moment T_n, the following event must occur:

> {there are m preceding transition moments, say $T_{n_1}, \dots, T_{n_m}$, and a customer arrives at a moment T_n; moreover, the point T_n belongs to the union of time intervals when the earlier m customers are served}.

Consider sequences of natural numbers

$$\bar{s} = (s_1, s_2, \dots, s_{m+1})$$

such that

$$1 = s_1 < s_2 < \cdots < s_{m+1} = l + 1.$$

Such sequences are called $(l, m+1)$-*chains*. We say that a customer leaves the system at the moment T_n without service due to an $(l, m+1)$-*chain* $\bar{s}$ if $m+1$ customers arrive at the system at the moments $T_{n_1}, \dots, T_{n_m}, T_n$ where $n_1 = n - l$, $n_2 = n - l + s_2 - 1, \dots,$ $n_m = n - l + s_m - 1$, and, moreover, $Y_{n_i} > T_n - T_{n_i}$, $1 \le i \le m$. Denote by r the minimum of the numbers l for which the probability of some $(l, m+1)$-chain is positive.

The latter definition can be explained in a different manner. Assume that a customer arrives at the system at a certain moment T_n. Then the next m customers may arrive at the system after the moment T_{n+r}, that is, after at least r transitions of the phase process. We denote by Γ_0 the set of $(r, m+1)$-chains. Below we list some properties of this set.

1. If $A_{(r,m+1)}$ denotes the event that the customer n leaves the system without service due to an $(r, m+1)$-chain, then these events are disjoint for different chains.
2. There is no customer arriving at the system between the moments
$$T_{n_1}, \quad \dots, \quad T_{n_m}, \quad T_n.$$

Now we define the *weight* $w(\bar{s})$ of an $(r, m+1)$-chain $\bar{s}$ as follows:

(3)
$$w(\bar{s}) = \frac{1}{(s_2 - s_1 - 1)! \cdots (s_{m+1} - s_m - 1)!} \int \cdots \int_{E^{r+1}} \int \cdots \int \lambda(u_1) \cdots \lambda(u_r)\, \pi(du_0) \times \left(\prod_{j=1}^{m+1} F\left(u_{s_j - 1}, du_{s_j}\right) \right) \left(\prod_{i \notin \{s_1, \dots, s_{m+1}\}, 2 \le i \le r} G(u_{i-1}, du_i) \right).$$

According to Assumption 2, we have

$$w(\bar{s}) \ge c(\bar{s}) \underline{\lambda}^r, \qquad \bar{s} \in \Gamma_0, \tag{4}$$

for some constants $c(\bar{s}) > 0$.

Taking into account the weight of a chain $\bar{s}$, the joint probability density of increments $T_{n_2} - T_{n_1}, \dots, T_n - T_{n_m}$ is estimated from above as follows:

$$w(\bar{s}) t_1^{s_2 - s_1 - 1} (t_2 - t_1)^{s_3 - s_2 - 1} \cdots (t_m - t_{m-1})^{s_{m+1} - s_m - 1}.$$

Thus the loss probability at the moment T_n due to the $(r, m+1)$-chain $\bar{s}$ is bounded from above by

(5)
$$\begin{aligned} &\mathsf{E}\{J_{n\tau}; \bar{s}\} \\ &\quad \le w(\bar{s}) \int \cdots \int_{0 < t_1 < \cdots < t_m} \int \cdots \int t_1^{s_2 - s_1 - 1} (t_2 - t_1)^{s_3 - s_2 - 1} \cdots (t_m - t_{m-1})^{s_{m+1} - s_m - 1} \\ &\qquad \times B_\tau^c(t_m) B_\tau^c(t_m - t_1) B_\tau^c(t_m - t_2) \cdots B_\tau^c(t_m - t_{m-1})\, dt_1 \cdots dt_m \\ &\quad \stackrel{\text{def}}{=} \bar{J}_{0\tau}(\bar{s}). \end{aligned}$$

Put

$$\alpha_{k\tau} = \int_0^\infty x^k \, dB_\tau(x).$$

Theorem 1. *If*

$$\alpha_{r-m+2} = O\left(\tau^{r-m+2}\right), \tag{6}$$

then

$$\mathsf{E}\{L_n\} \sim n \sum_{s \bar{\in} \Gamma_0} \bar{J}_{0\tau}(\bar{s}) \tag{7}$$

as $n \to \infty$ *and* $\tau \to 0$. *(The terms in this sum are defined by the right-hand side of inequality* (5).*)*

Proof. Let Γ_1 be the set of all $(l, m+1)$-chains such that $l > r$. Then $\mathsf{E}\{J_{n\tau}; \Gamma_1\}$ does not exceed the probability that some customers arrive at moments $T_{n_1}, \dots, T_{n_m}$ and T_n where $n_1 < \dots < n_m < n$, $n - n_1 > r$, and $Y_{n_i} > T_n - T_{n_i}$, $1 \le i \le m$, where the Y_{n_i} are corresponding service times. The latter probability is bounded from above by

$$\mathsf{P}\{Z_1 + \dots + Z_m > r; Z_i > 0, 1 \le i \le m\},$$

where the Z_i are independent random variables with the distribution

$$\mathsf{P}\{Z_1 = k\} = \int_0^\infty e^{-\bar{\lambda} x} \frac{(\bar{\lambda} x)^k}{k!}\, dB_\tau(x).$$

Thus

$$\begin{aligned}
&\mathsf{P}\{Z_1 + \dots + Z_m > r; Z_i > 0, 1 \le i \le m\} \\
&\quad \le m\mathsf{P}\{Z_1 > z - m + 1; Z_2 > 0, \dots, Z_m > 0\} \\
&\qquad + \sum_{\substack{1 \le i_1, \dots, i_m \le r-m+1, \\ i_1 + \dots + i_m > r}} \mathsf{P}\{Z_1 = i_1, \dots, Z_m = i_m\} \\
&\quad < m\tau^{m-1} \frac{\alpha_{r-m+2}}{(r-m+2)!} + \sum_{\substack{1 \le i_1, \dots, i_m \le r-m+1, \\ i_1 + \dots + i_m > r}} \frac{\alpha_{i_1}}{i_1!} \cdots \frac{\alpha_{i_m}}{i_m!} \\
&\quad = O\left(\alpha_{r-m+2}^{(r+1)/(r-m+2)}\right),
\end{aligned} \tag{8}$$

whence

$$\mathsf{E}\{J_{n\tau}; \Gamma_1\} = O\left(\alpha_{r-m+2}^{(r+1)/(r-m+2)}\right). \tag{9}$$

Now we show that $\mathsf{E}\{J_{n\tau}; \Gamma_1\}$ is small compared to $\sum_{\bar{s} \in \Gamma_0} \bar{J}_m(\bar{s})$. First we establish a lower bound for $\bar{J}_m(\bar{s})$. For all $a > \tau/3$,

$$\tau = \left(\int_0^{\tau/3} + \int_{\tau/3}^a + \int_a^\infty\right) B_\tau^c(x)\, dx \le \frac{\tau}{3} + \left(a - \frac{\tau}{3}\right) B_\tau^c\left(\frac{\tau}{3}\right) + \frac{\alpha_{r-m+2}}{(r-m+2)a^{r-m+1}}.$$

Equating the third term of the latter expression to $\tau/3$, we obtain

$$B^c\left(\frac{\tau}{3}\right) \ge \left(3\left(\frac{3}{(r-m+1)} \frac{\alpha_{r-m+2}}{\tau^{r-m+2}}\right)^{1/r-m+1} - 1\right)^{-1} \ge c_0 > 0. \tag{10}$$

Substituting this inequality in (5), we get the lower estimate

$$\begin{aligned}
\bar{J}_{0\tau}(\bar{s}) &\ge w(\bar{s}) c_0^m \\
&\quad \times \int \cdots \int_{0 < t_1 < \dots < t_m < \tau/3} \int \cdots \int t_1^{s_2 - s_1 - 1} \cdots (t_m - t_{m-1})^{s_m - s_{m-1} - 1}\, dt_1 \cdots dt_m \\
&\ge c_1 \tau^r,
\end{aligned} \tag{11}$$

where $c_1 > 0$. Comparing the latter estimate with (9), we prove that

$$\mathsf{E}\{J_{n\tau}; \Gamma_1\} = o\biggl(\sum_{\bar{s} \in \Gamma_0} \bar{J}_{0\tau}(\bar{s})\biggr) \tag{12}$$

as $\tau \to 0$ provided relation (6) holds.

To derive the lower estimate for $\mathsf{E}\{J_\tau\}$, note that

$$\mathsf{E}\{J_\tau\} \ge \sum_{\bar{s} \in \Gamma_0} \underline{J}_{0\tau}(\bar{s}), \tag{13}$$

where $\underline{J}_{0\tau}(\bar{s})$ is defined via the same integral as in the case of $\bar{J}_{0\tau}(\bar{s})$ (see (5)), but with the factor $e^{-\bar{\lambda} t_m}$ under the integral sign and with the factor $(1-\bar{\lambda}\tau)$ before the integral sign. It is easy to prove that

$$(14) \qquad \sum_{\bar{s}\in\Gamma_0} \bar{J}_{0\tau}(\bar{s}) - \sum_{\bar{s}\in\Gamma_0} \underline{J}_{0\tau}(\bar{s}) = O\left(a_{r-m+2}^{(r+1)/(r-m+2)}\right) = O\left(\tau^{r+1}\right) \quad \text{as } \tau \to 0.$$

Now (7) follows from (12), (13), and (14). □

4. The invariance of the loss probability in the case of light traffic

The property of the invariance of the loss probability can be described in this case as follows. Let $B_\tau^1(x)$ and $B_\tau^2(x)$ be two arbitrary parametric sets of distribution functions of the service times such that

$$\int_0^\infty B_\tau^{1c}(x)\,dx = \int_0^\infty B_\tau^{2c}(x)\,dx = \tau$$

and

$$\int_0^\infty x^{r-m+2}\,dB_\tau^i(x) = O\left(\tau^{r-m+2}\right), \qquad i = 1,2,$$

as $\tau \to 0$ where r is defined in Section 3. Then

$$(15) \qquad \frac{Q_{n\tau}^1}{Q_{n\tau}^2} \to 1 \quad \text{as } n \to \infty \text{ and } \tau \to 0,$$

where $Q_{n\tau}^1$ and $Q_{n\tau}^2$ are defined similarly to $Q_{n\tau}$ in equality (1) with $B_\tau = B_\tau^1$ and $B_\tau = B_\tau^2$, respectively.

Theorem 2. *Assume that all the assumptions of Theorem 1 hold. Then $Q_{n\tau}$ is invariant in the case of light traffic with respect to $B_\tau(x)$ if and only if $r = m$.*

Proof. 1. *Sufficiency.* If $r = m$, then there exists a unique $(r, m+1)$-chain

$$\bar{s} = (1, 2, \ldots, m+1).$$

It follows from (5) that

$$\begin{aligned} &w^{-1}(\bar{s})\bar{J}_{0\tau}(\bar{s}) \\ &\quad = \int\cdots\int_{0<t_1<\cdots<t_m}\int\cdots\int B_\tau^c(t_m)B_\tau^c(t_m-t_1)\cdots B_\tau^c(t_m-t_{m-1})\,dt_1\cdots dt_m \\ &\quad = \frac{1}{m!}\left(\int_0^\infty B_\tau^c(x)\,dx\right)^m = \frac{\tau^m}{m!}. \end{aligned}$$

Thus the loss probability is invariant.

2. *Necessity.* Assume that $r > m$. Put

$$B_\tau^c(x) = \theta e^{-\theta x/\tau},$$

where $\theta \in (0,1]$ is a parameter. Then we have

$$\begin{aligned} &w^{-1}(\bar{s})\bar{J}_{0\tau}(\bar{s}) \\ &\quad = \theta^m \int\cdots\int_{0<t_1<\cdots<t_m}\int\cdots\int t_1^{s_2-s_1-1}\cdots t_m^{r-s_m-1}e^{-\theta R(t_1,\ldots,t_m)/\tau}\,dt_1\cdots dt_m \end{aligned}$$

for all chains $\bar{s} \in \Gamma_0$ where $R(t_1,\ldots,t_m)$ is a linear function. Changing the variables $t_i = \tau y_i/\theta$ we prove that the latter integral is equal to

$$\frac{\tau^r}{\theta^{r-m}} \cdot \text{const}.$$

This means that the loss probability is not invariant in the case of light traffic. □

5. Addendum

5.1. **Nonstationary phase process.** Let all the above assumptions be satisfied except for the assumption that the sequence (U_n) is stationary. Instead, we assume that this sequence has a stationary distribution π such that the distribution $\pi_n = (\mathsf{P}\{U_n \in \cdot\})$ converges to π as $n \to \infty$. We consider two types of convergence, namely

(A) $\|\pi_n - \pi\| \to 0$ as $n \to \infty$, that is the total variation of the difference of distributions tends to zero;
(B) $\pi_n \xrightarrow{w} \pi$ as $n \to \infty$ where the symbol $\xrightarrow{w}$ stands for the weak convergence and where the functions $F(u,\cdot)$, $G(u,\cdot)$, and $\lambda(u)$ are continuous with respect to u.

In both cases, (A) and (B), it follows that

$$Q_{n\tau} \sim \frac{\sum_{\bar{s}\in\Gamma_0} \bar{J}_{0\tau}(\bar{s})}{\int_E \pi(du)F(u,E)} \quad \text{as } \tau \to 0, \tag{16}$$

for $n > n(\tau)$ where an integer number $n(\tau)$ depends on the mean service time τ. Note that the uniform convergence

$$\frac{Q_{n\tau}}{\text{right-hand side of (16)}} \to 1$$

as $n \to \infty$ and $\tau \to 0$ does not hold in the general case if the sequence (U_n) is nonstationary.

5.2. **Frequency approach.** One can use another definition of the loss probability, namely

$$Q_\tau(t) = \frac{\mathsf{E}\{L_\tau(t)\}}{\mathsf{E}\{N(t)\}} \tag{17}$$

instead of equality (1) related to the number n of transitions of the phase process. Definition (17) involves $N(t)$, the number of customers arriving during the time period $(0,t)$, and $L_\tau(t)$, the number of customers leaving the system without service during the same time period. The results in this case are similar to those we obtained above for $Q_{n\tau}$; in any case, we have for all $\tau > 0$ that

$$\frac{Q_\tau(t)}{Q_{n\tau}} \to 1 \quad \text{as } t \to \infty \text{ and } n \to \infty.$$

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LG Stochastische Modellierung und Rechnernetze, FB4-Abteilung Informatik, Universität Trier, D-54286 Trier, Germany
E-mail address: baum@info04.uni-trier.de

Glushkov Institute for Cybernetics, Kyiv, Ukraine
E-mail address: kovigo@yandex.ru

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