An Evidential Reasoning Approach for Multiple-Attribute Decision Making with Uncertainty

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Abstract—**A new evidential reasoning based approach is proposed that may be used to deal with uncertain decision knowledge in multiple-attribute decision making (MADM) problems with both quantitative and qualitative attributes. This approach is based on an evaluation analysis model and the evidence combination rule of the Dempster–Shafer theory. It is akin to a preference modeling approach, comprising an evidential reasoning framework for evaluation and quantification of qualitative attributes. Two operational algorithms have been developed within this approach for combining multiple uncertain subjective judgments. Based on this approach and a traditional MADM method, a decision making procedure is proposed to rank alternatives in MADM problems with uncertainty. A numerical example is discussed to demonstrate the implementation of the proposed approach. A multiple-attribute motor cycle evaluation problem is then presented to illustrate the hybrid decision making procedure.**

I. Introduction

MULTIPLE-ATTRIBUTE decision making problems with both quantitative and qualitative attributes are common in practice [6], which we simply call hybrid MADM problems in this paper. At the concept design stage in engineering design, for example, alternative designs for a large engineering product need to be ranked or sorted by taking into account many technical and economical performances which are usually measured or evaluated using either numerical values with certain units or subjective judgments with uncertainty based on *a priori* experience. The aim of design is then to select from the existing alternative designs the best compromise alternative which attains these performances as closely as possible.

To solve a hybrid MADM problem, the first step is to evaluate and quantify the state of a qualitative attribute at each alternative. One of the simplest ways is to define a few evaluation grades for the attribute, which are quantified using a certain scale. The state of the attribute at an alternative may be evaluated to one of the grades. The scale of the confirmed grade may then be used as a numerical value for measuring the sate of the attribute at the alternative [6]. This approach is conceptually clear and easy to understand. However, it may be difficult to apply in practice because of two main reasons.

First of all, in a MADM problem, a qualitative attribute often represents an abstract concept representing an aggregated technical or economical performance comparable with other attributes. Such a qualitative attribute is generally difficult to assess directly, but may be possible to evaluate indirectly through a number of factors, which detail the attribute and are easier to assess directly. Secondly, it is improper to assume that subjective judgments for such evaluations might always be deterministic, even for the assessments of a factor. In other words, the decision maker may not always be 100% sure that the state of a factor is exactly confirmed to one of the evaluation grades. In fact, one or more grades may be confirmed at the same time with total confidence of exact or smaller than 100%. In the new approach to be reported in this paper, uncertain subjective judgments for the evaluation of qualitative attributes through multiple relevant factors will be accommodated within a framework based on the concept of preference degree and an evaluation analysis model [35].

Several tools are available for reasoning with uncertain decision knowledge. The Dempster–Shafer theory (simply D–S theory) is selected for the development of the new approach because of: 1) its powerful evidence combination rule, and 2) its reasonable requirement for the basic probability assignments that given a piece of evidence, the commitment of belief in a hypothesis does not necessarily mean that the remaining belief must be assigned to the complement of the hypothesis, but to the whole sample space [1], [19].

The second advantage of the D–S theory [i.e., 2)] indicates that the theory is well suited for handling incomplete uncertainty. This is particularly important and useful for dealing with uncertain subjective judgments when multiple factors need to be considered simultaneously. This is because even though each uncertain subjective judgment for the evaluation of a single factor provides a complete commitment to the evaluation grades (i.e., with the total confidence of exactly 100%), the total support from the factor for evaluation of its associated attribute may still be incomplete as each factor may have a different relative importance or a different role in evaluation of the attribute, as will be shown in the application examples in Section V. In other words, the total support from a

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single factor could only be 100% if the factor absolutely dominated all other factors. However, this is not always the case. Otherwise, a single-factor analysis should be enough, which actually means that the attribute could be assessed directly. The D–S theory is so great that it can deal with such incomplete uncertainty in a more rational way than other tools in that given a piece of evidence, the unassigned belief in a hypothesis is just supposed to denote the unknown uncertainty, which instead of being necessarily assigned to the complement of the hypothesis, may eventually be assigned to any hyperthesis in the sample space when more evidence is gathered. This is the main reason why we have chosen the D–S theory to handle uncertainty for multiple factor analysis. Some concepts and the evidence combination rule of the D–S theory are introduced to develop evidence combination algorithms for combining uncertain decision knowledge.

In this paper, we will focus on developing the evidential reasoning approach. A decision making procedure for ranking alternatives in a hybrid MADM problem with uncertainty is also proposed, which is composed of the new approach and a traditional MADM method. In Section II, necessary basics are briefly discussed about hybrid MADM with uncertainty, the concept of preference degree, the evaluation analysis model, and the evidence combination rule. The new evidential reasoning approach is then explored in detail. In Section IV, the procedure for alternative ranking is proposed. Section V first presents a numerical example to demonstrate the implementation process of the new evidential reasoning approach. A multiple-attribute motorcycle evaluation problem is then presented to illustrate how to use the approach to deal with a real-world hybrid MADM problem with uncertainty.

II. Basics About Hybrid MADM with Uncertainty

A. Hybrid MADM Problems with Uncertainty

A hybrid MADM problem may be expressed using the following formula (1) or by an extended decision matrix, such as Table I.

$$\underset{a\in\Omega}{\text{optimize}}\; y(a) = [\, y_1(a) \;\cdots\; y_k(a) \;\cdots\; y_{k_1+k_2}(a)]. \tag{1}$$

In (1), Ω is a discrete set of alternatives. In Table I, y_{ij} is a numerical value of y_j at a_i $(i = 1, \cdots, l;\ j = 1, \cdots, k_1)$ and SJ_{ij} are subjective judgments with uncertainty for evaluation of the states of y_{k_1+j} at a_i $(i = 1, \cdots, l;\ j = 1, \cdots, k_2)$. The problem is to rank these alternatives or to select the best compromise alternative, with both quantitative and qualitative attributes being simultaneously satisfied as much as possible.

It is therefore fundamental to evaluate and quantify qualitative attributes so that the extended decision matrix can be transformed into an ordinary decision matrix, and then a traditional MADM method may be used for ranking alternatives. A simple method for the evaluation and quantification is to define a few evaluation grades such that the state of an attribute at an alternative could be evaluated to one of the grades. Then, these grades may be quantified using certain scales [6]. This method may be practical if the decision maker is able to evaluate qualitative attributes synthetically and deterministically using only a few discrete evaluation grades.

In a hybrid MADM problem, however, a qualitative attribute may represent an aggregated technical and economical concept so that it is comparable with other attributes. Such an attribute may only be evaluated through a number of relevant factors which detail the attribute and are easier to evaluate directly. In addition to this, the evaluations of a factor may not always be deterministic. Rather, uncertain subjective judgments may often be provided by the decision maker.

In a problem of evaluating different types of motorcycles, for example, the following type of uncertain subjective judgments was frequently used [7].

Statement i > The *responsiveness* of the *engine* of "Yamaha" is evaluated to be *good* with a confidence degree of 0.3 and to be *excellent* with a confidence degree of 0.6.

In the statement, "Yamaha" is an alternative motor cycle, *engine* a qualitative attribute comparable with other attributes such as *price*, *responsiveness* a factor for the evaluation of *engine*, *good* and *excellent* are evaluation grades representing distinct states of *engine*, and the confidence degrees 0.3 and 0.6 represent the uncertainty in the evaluation. Note that the total confidence degree in Statement i> is 0.9, smaller than one.

To evaluate *engine* of a motorcycle, other factors such as *fuel economy* and *quietness* may need to be considered as well. In this case, similar statements may also be used to evaluate the *fuel economy* and *quietness* of the *engine* of "Yamaha." It is then essential to combine these multiple uncertain judgments to produce an aggregated evaluation for *engine*. The following sections are therefore focused on the development of an approach, comprising multiple-factor analysis and evidential reasoning, so that qualitative attributes may be evaluated and quantified. As a result, a decision making procedure will be proposed for ranking alternatives in a hybrid MADM problem with uncertainty.

B. Evaluation Analysis Model

In [27], [35], an evaluation analysis model was proposed to represent uncertain subjective judgments, such as Statement i>. The model is shown in Fig. 1.

In the attribute level of the model, the state of an attribute (such as *engine*) at each alternative a (such as "Yamaha") is required to be evaluated. In the evaluation grade level, H_n is called an evaluation grade (such as *good*) $(n = 1, \cdots, N)$. A set of evaluation grades for an attribute y_k is denoted by

$$H = \{H_1 \;\; H_2 \;\cdots\; H_n \;\cdots\; H_N\} \tag{2}$$

where N is the number of evaluation grades. H_n represents a grade to which the state of y_k may be evaluated. H_1 and

TABLE I
AN EXTENDED DECISION MATRIX

Alternatives (a_r)	Quantitative Attributes (y_k)				Qualitative Attributes (y_k)			
	y_1	y_2	$\cdots$	y_{k_1}	y_{k_1+1}	y_{k_1+2}	$\cdots$	$y_{k_1+k_2}$
a_1	y_{11}	y_{12}	$\cdots$	y_{1k_1}	SJ_{11}	SJ_{12}	$\cdots$	SJ_{1k_2}
a_2	y_{21}	y_{22}	$\cdots$	y_{2k_1}	SJ_{21}	SJ_{22}	$\cdots$	SJ_{2k_2}
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
a_l	y_{l1}	y_{l2}	$\cdots$	y_{lk_1}	SJ_{l1}	SJ_{l2}	$\cdots$	SJ_{lk_2}

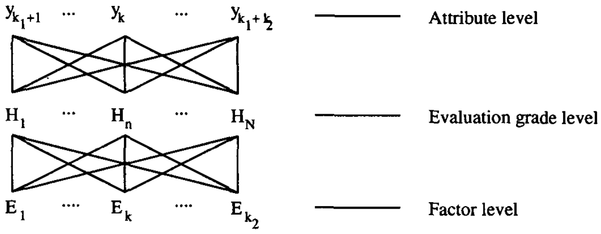

Fig. 1. An evaluation analysis model.

H_N are set to be the worst and the best grades, respectively, and H_{n+1} is supposed to be preferred to H_n. It should be kept in mind that an attribute may have its own set of evaluation grades different from those of other attributes, although Fig. 1 only lists one set and H of (2) is not defined as H^k in order to simplify the following discussion.

In the model, the concept of preference degree was introduced, which may be used to quantify these evaluation grades and eventually to quantify subjective judgments with uncertainty. A preference degree takes values from the close interval $[-1 \quad 1]$, which may be called the preference degree space. The set of evaluation grades may thus be quantified by

$$p\{H\} = [p(H_1) \cdots p(H_n) \cdots p(H_N)]^T \tag{3}$$

where $p(H_n)$ is the scale of H_n and satisfies the following basic conditions:

$$p(H_1) = -1, \quad p(H_N) = 1;$$
$$p(H_{n+1}) > p(H_n), \; n = 1, \cdots, N-1. \tag{4}$$

Besides, $p(H_n)$ $(n = 2, \cdots, N-1)$ should be so assigned that an additional consistence condition, defined by (21) in the next section, can be satisfied.

Suppose $N = 7$, for example, H may be defined as follows:

$$H = \{H_1 \quad H_2 \quad H_3 \quad H_4 \quad H_5 \quad H_6 \quad H_7\}$$
$$= \{\text{the most unsatisfactory, very unsatisfactory, unsatisfactory, indifferent, satisfactory, very satisfactory, the most satisfactory}\}. \tag{5}$$

Without loss of generality, we may scale H_n $(n = 1, \cdots, 7)$ by using real numbers in $[-1 \quad 1]$; for example,

$$p\{H\} = [p(H_1) \quad p(H_2) \quad p(H_3) \quad p(H_4) \quad p(H_5) \quad p(H_6) \quad p(H_7)]^T$$
$$= [-1 \quad -0.8 \quad -0.4 \quad 0 \quad 0.4 \quad 0.8 \quad 1]^T. \tag{6}$$

In the factor level, E_k represents a set of factors which are associated with the evaluation of the attribute $y_k(a)$ and denoted by

$$E_k = \{e_k^1 \quad e_k^2 \cdots e_k^{L_k}\} \qquad k = k_1 + 1, \cdots, k_1 + k_2 \tag{7}$$

where e_k^i $(i = 1, \cdots, L_k)$ are factors (such as *responsiveness*) influencing the evaluation of $y_k(a)$. The state of e_k^i can be directly evaluated at an alternative a, that is, $e_k^i = e_k^i(a)$.

The new approach developed in this paper is then devoted to generating the preference degree for the state of an attribute $y_k(a)$ at each alternative a through the direct evaluations of the relevant factors e_k^i $(i = 1, \cdots, L_k)$. The generated preference degrees for the attribute have to satisfy certain rational assumptions such as the monotonicity of its marginal utilities.

If there is only one factor e_k^1 associated with $y_k(a)$ and its state at a is exactly confirmed to one of the evaluation grades in H, such as H_n, the procedure for evaluation of $y_k(a)$ through e_k^1 may be as simple as to use the scale of the confirmed evaluation grade as the preference degree of $y_k(a)$, denoted by $p(y_k(a))$, or $p(y_k(a)) = p(e_k^1(a)) = p(H_n)$ where $p(e_k^1(a))$ denotes the preference degree of e_k^1 at a. A more general evaluation procedure is to be explored based on the evaluation analysis model and the evidence combination rule of the Dempster–Shafer theory.

C. Evidence Combination Rule

The D–S theory is one of the powerful tools to deal with uncertainty. We do not attempt to discuss all of its details in this paper, but we will only use its evidence combination rule to develop evidence combination algorithms for the new approach.

In the D–S theory, a sample space is called a "frame of discernment," defined as Θ. A basic hyperthesis (singleton) in Θ is denoted by H_s, i.e., $H_s \subseteq \Theta$. In Θ, all basic hypertheses are required to be mutually exclusive and exhaustive. A probability mass to every subset Ψ of

Θ ($\Psi \subseteq \Theta$) can be assigned, denoted by $m(\Psi)$. The probability mass is called the basic probability assignment, which is a number in the interval [0 1] to indicate belief in a hypothesis given a piece of evidence, or the degree to which the evidence supports the hypothesis.

A basic probability assignment satisfies the following condition [19]:

$$\sum_{\Psi \subseteq \Theta} m(\Psi) = 1, \quad m(\varnothing) = 0;$$
$$0 \leq m(\Psi) \leq 1, \text{ for all } \Psi \subseteq \Theta. \tag{8}$$

$m(\Psi)$ indicates that portion of the total belief exactly committed to hypothesis Ψ given a piece of evidence. In other words, $m(\Psi)$ represents the direct support of evidence on Ψ. This portion of belief cannot be further subdivided among the subsets of Ψ, and does not include the portion of belief committed to subsets of Ψ.

The quantity $m(\Theta)$ is a measure of that portion of the total belief that remains unassigned after commitment of belief to all subsets of Θ. If $m(\Psi) = s(\Psi \subseteq \Theta)$ and no belief is assigned to other subsets of Θ, for example, then $m(\Theta) = 1 - s$. Thus, the remaining belief is assigned to Θ, but not to the negation of the hypothesis Ψ (the complement of Ψ).

Suppose there exist two pieces of evidence in Θ, and that they provide two basic probability assignments to a subset Ψ of Θ, i.e., $m_1(\Psi)$ and $m_2(\Psi)$. The problem is to obtain a combined probability assignment $m_{12}(\Psi) = m_1(\Psi) \oplus m_2(\Psi)$. The D-S theory provides an evidence combination rule defined below [19]:

$$m_{12}(\varnothing) = 0, \quad m_{12}(\Psi) = \sum_{A \cap B = \Psi} \frac{m_1(A) m_2(B)}{1 - K} \tag{9}$$

$$K = \sum_{A \cap B = \varnothing} m_1(A) m_2(B). \tag{10}$$

In the rule, $m_{12}(\Psi)$ for hypothesis Ψ ($\subseteq \Theta$) is computed from m_1 and m_2 by adding all products of the form $m_1(A) m_2(B)$ where A and B are selected from the subsets of Θ in all possible ways such that their intersection is Ψ. K reflects the conflicting situations where both $m_1(A)$ and $m_2(B)$ are not zero, but the intersection $A \cap B$ is empty. The commutativity of multiplication in the rule ensures that the rule yields the same value regardless of the order in which the two pieces of evidence are combined.

It is easy to show that the direct use of the combination rule will result in an exponential increase in computational complexity [1]. This is due to the need to enumerate all subsets or supersets of a given subset Ψ of Θ. The following section is therefore intended to develop operational algorithms for evidence combination which reduce the computational complexity to linear time by utilizing the characteristics of the evidence combination process based on the evaluation analysis model.

III. Evidential Reasoning Approach

A. An Evidential Reasoning Framework

In the evaluation analysis mode, an evaluation grade H_n may be considered as a basic hypothesis (singleton) in the D-S theory, a factor e_k^i as a piece of evidence, and a basic probability assignment may be obtained from a confidence degree. To apply the evidence combination rule, however, the mutual exclusiveness and exhaustiveness of all basic hyperthesis have to be satisfied. It is therefore necessary that all the evaluation grades in H be defined as distinct grades. In other words, if one of the evaluation grades is absolutely confirmed, that is, the confidence degree is one, all the other grades must not be confirmed at all; if more than one grade is confirmed simultaneously, the total confidence degree must be one or smaller than one. In addition to this requirement, the evaluation grades defined in H must cover all possible grades the decision maker may use for evaluation of an attribute at all alternatives. Then, the frame of discernment may be defined by

$$\Theta = H = \{H_1 \cdots H_n \cdots H_N\}. \tag{11}$$

Let $m(H_n / e_k^i(a))$ express a basic probability assignment to which e_k^i supports a hypothesis that the state of y_k at an alternative a is confirmed to H_n. Let $\beta_{H_n}(e_k^i(a))$ be a confidence degree to which the decision maker considers that the state of e_k^i at an alternative a is confirmed to H_n. $m(H_n / e_k^i(a))$ may be obtained from $\beta_{H_n}(e_k^i(a))$. For a rational decision maker, we assume that he only provides uncertain subjective judgments satisfying the following rationality assumption.

Rationality Assumption: If a decision maker recognizes that the state of a factor e_k^i has to be confirmed to an evaluation grade H_n to some extent, then he may express his uncertain subjective judgments only in one of the following three manners.

1) e_k^i is only confirmed to H_n to the extent of $\beta_{H_n}(e_k^i(a))$ while $0 < \beta_{H_n}(e_k^i(a)) \leq 1$.

2) e_k^i may be confirmed to H_n and to H_{n+1} at the same time to the extents of $\beta_{H_n}(e_k^i(a))$ and $\beta_{H_{n+1}}(e_k^i(a))$, respectively, while $0 < \beta_{H_n}(e_k^i(a)), \beta_{H_{n+1}}(e_k^i(a)) \leq 1$, and $\beta_{H_n}(e_k^i(a)) + \beta_{H_{n+1}}(e_k^i(a)) \leq 1$.

3) e_k^i may be confirmed to H_{n-1} and to H_n at the same time to the extents of $\beta_{H_{n-1}}(e_k^i(a))$ and $\beta_{H_n}(e_k^i(a))$, respectively, while $0 < \beta_{H_{n-1}}(e_k^i(a)), \beta_{H_n}(e_k^i(a)) \leq 1$, and $\beta_{H_{n-1}}(e_k^i(a)) + \beta_{H_n}(e_k^i(a)) \leq 1$.

This assumption, however, is only made based upon our experience and may not be universally satisfied. In the extensions of the approach reported in [31], [32], this assumption has actually been abandoned, although the newly developed approaches need more computational effort. Obviously, Statement i> satisfies the rationality assumption. From the rationality assumption, we can classify the set of factors $E_k = [e_k^1 \;\; e_k^2 \;\; \cdots \;\; e_k^{L_k}]$ into $N - 1$ subsets S_n, defined by

$$S_n = \{e_{n,n+1}^1, \cdots, e_{n,n+1}^i, \cdots, e_{n,n+1}^{R_n}\}$$
$$n = 1, \cdots, N - 1 \tag{12}$$

where $e_{n,n+1}^i$ is a factor in E_k, the state of which is confirmed to H_n and/or to H_{n+1}, and $L_k = R_1 + R_2 + \cdots + R_{N-1}$. Because the evidence combination rule defined in (9) and (10) is independent of the order in which factors are gathered [19], such classification will be useful to develop operational algorithms for evidence combination.

We are now in a position to summarize an evidential reasoning framework. The basic probability assignments of all hypothesis (subsets of Θ) are first generated from the confidence degrees. Suppose the basic probability assignment of H_n with respect to $e^i_{n,n+1}$ is denoted by $m(H_n/e^i_{n,n+1})$, simply $m^{n,i}_n$, that of H_{n+1} by $m(H_{n+1}/e^i_{n,n+1})$, simply $m^{n,i}_{n+1}$, and that of Θ by $m(\Theta/e^i_{n,n+1})$, simply $m^{n,i}_\Theta$.

Secondly, combine the R_n factors which confirm H_n and/or H_{n+1}. At this step, let

$$e^{I(i)}_{n,n+1} = \{e^1_{n,n+1}, e^2_{n,n+1}, \cdots, e^i_{n,n+1}\} \tag{13}$$

$$m^{I(i)}_n = m(H_n/e^{I(i)}_{n,n+1}), \quad m^{I(i)}_{n+1} = m(H_{n+1}/e^{I(i)}_{n,n+1}),$$

$$m^{I(i)}_\Theta = m(\Theta/e^{I(i)}_{n,n+1}). \tag{14}$$

Therefore, $e^{I(R_n)}_{n,n+1} = \{e^1_{n,n+1}, \cdots, e^{R_n}_{n,n+1}\} = S_n$, and $m^{I(R_n)}_n$, $m^{I(R_n)}_{n+1}$, and $m^{I(R_n)}_\Theta$ $(n = 1, \cdots, N-1)$ are called the local probability assignments partially combined from the R_n factors.

Then, combine all factors in E_k. At this step, define

$$e^{C(j)}_{1,j+1} = \{e^{I(R_1)}_{1,2}, e^{I(R_2)}_{2,3}, \cdots, e^{I(R_j)}_{j,j+1}\} \tag{15}$$

$$b^{C(j)}_n = m(H_n/e^{C(j)}_{1,j+1}), \qquad n = 1, \cdots, N. \tag{16}$$

These symbols defined in (13)–(16) will be used to develop our partial and overall evidence combination algorithms, and they are expected to make it clearer to describe the computational procedures of the algorithms separately.

From (15), it is obvious that $e^{C(N-1)}_{1,N} = \{e^1_k \ e^2_k \ \cdots \ e^{L_k}_k\} = E_k$ and $b^{C(N-1)}_n = m(H_n/E_k)$, where $m(H_n/E_k)$ is the overall probability assignment to which the state of an attribute $y_k(a)$ at an alternative a is confirmed to H_n. Suppose Ψ is a subset of Θ. Then, $m(\Psi/E_k)$ is defined as the overall probability assignment to which the state of $y_k(a)$ at a is confirmed to Ψ. Let $p(\Psi)$ stand for the scale of Ψ, which is defined as the average of the scales of the singletons involved in Ψ.

It is possible that the state of y_k at a_r may be confirmed by the factor set E_k to any subset Ψ of Θ to an extent of $m(\Psi/E_k(a_r))$. The state of $y_k(a_r)$ may therefore be denoted by the following expectation:

$$S(y_k(a_r)) = \{(m(\Psi/E_k(a_r)), \Psi), \quad \text{for all } \Psi \subseteq H\}. \tag{17}$$

In (17), each subset Ψ of H actually represents a possible state (single evaluation grades or their combinations) into which the state of an attribute may possibly fall at a particular alternative and $m(\Psi/E_k(a_r))$ represents the total support by all the factors to the hypothesis that the state of y_k at a_r is confirmed to Ψ. Thus, (17) actually describes the distribution of the state of y_k at a_r among all possible states. If the distribution of the state of y_k at a_r favors good subsets Ψ in H more than the distribution at a_h, $S(y_k(a_r))$ should be better than $S(y_k(a_h))$. As $\Sigma_{\Psi \subseteq H}$ $m(\Psi/E_k(a_r)) = 1$, $m(\Psi/E_k(a_r))$ may thus be explained as a plausible probability that the state of y_k at a_r falls into Ψ. In this context, let us define $p(\Psi)$ as a numerical value to express the relative intensity of the state Ψ compared with all other possible states and let a larger value represent a better state. Then, the intensity of the state of y_k at a_r may be defined as the expected value of the intensities of each state Ψ confirmed at a_r with a plausible probability of $m(\Psi/E_k(a_r))$ for all $\Psi \subseteq H$. Let a preference degree express such an expected value, and the preference degree of $y_k(a_r)$ be denoted by $p_{rk} = p(y_k(a_r))$, quantifying $S(y_k(a_r))$. p_{rk} is then calculated by

$$p_{rk} = \sum_{\Psi \subseteq \Theta} m(\Psi/E_k(a_r))\, p(\Psi). \tag{18}$$

It is therefore rational to state that if at an alternative a_r an attribute y_k has a larger preference degree than at another alternative a_h, then the state of y_k at a_r ought to be better than the state of y_k at a_h. In other words, for two alternatives a_r and a_h, $S(y_k(a_r))$ is preferred to $S(y_k(a_h))$ if and only if $p_{rk} > p_{hk}$. A qualitative attribute y_k can thus be quantified with its marginal utilities being monotonous, which forms a rational basis for further decision analysis.

B. Acquisition and Representation of Uncertain Decision Knowledge

An uncertain subjective judgment may be acquired using a statement such as Statement i>. It is used to evaluate the state of a factor or an attribute at an alternative, indicating to which evaluation grades the state is confirmed and to what extents these evaluation grades are confirmed.

It is assumed that such uncertain subjective judgments satisfy the rationality assumption. Suppose a judgment states that the state of a factor e^i_k at a_r is confirmed to H_n to the extent of $\beta_{H_n}(e^i_k(a_r))$ and to H_{n+1} to the extent of $\beta_{H_{n+1}}(e^i_k(a_r))$. The state of $e^i_k(a_r)$ may be represented by the following expectation:

$$S(e^i_k(a_r)) = \{(\beta_{H_n}(e^i_k(a_r)), H_n); (\beta_{H_{n+1}}(e^i_k(a_r)), H_{n+1})\} \tag{19}$$

where $\beta_{H_n}(e^i_k(a_r)) + \beta_{H_{n+1}}(e^i_k(a_r)) \leq 1$. Compared with (17) and (18), the state of $e^i_k(a_r)$ may then be quantified using the preference degree, defined as the following expected scale:

$$p_{ri} = p(e^i_k(a_r)) = \beta_{H_n}(e^i_k(a_r))\, p(H_n) + \beta_{H_{n+1}}(e^i_k(a_r))\, p(H_{n+1}). \tag{20}$$

The scales of H_n, i.e., $p(H_n)$, $(n = 2, \cdots, N-1)$ therefore need to be defined so that in addition to the basic conditions defined by (4), the following consistence condition is also satisfied, that is, for two alternatives a_r and a_h,

$$S(e^i_k(a_r)) \text{ is preferred to } S(e^i_k(a_h)) \quad \text{if and only if } p_{ri} > p_{hi}. \tag{21}$$

A preference comparison of the state of one factor with that of another, such as $S(e_k^i(a_r))$ with $S(e_k^i(a_h))$, may be provided by the decision maker. If sufficient number of such preference comparisons are obtained, $p(H_n)$ $(n = 1, \cdots, N)$ may then be assessed by satisfying the constraints defined by (4) and (21).

By definition, the confidence degree $\beta_{H_n}(e_k^i(a))$ expresses the intensity to which the state of a single factor e_k^i at a is confirmed to an evaluation grade H_n. On the other hand, the basic probability assignment $m(H_n/e_k^i(a))$ represents the degree to which e_k^i supports a hypothesis that the state of the attribute y_k at a is confirmed to H_n. If there is only one factor e_k^i in E_k, $m(H_n/e_k^i(a))$ should be equal to $\beta_{H_n}(e_k^i(a))$; if there are multiple factors in E_k, however, they may play different roles in evaluation of y_k, depending upon their relative importance. Therefore, the weighted confidence degree may be used as the basic probability assignment. Suppose λ_k^i is the normalized relative weight of e_k^i in E_k and $\lambda_k = [\lambda_k^1 \cdots \lambda_k^{L_k}]^T$. Then, $m(H_n/e_k^i(a))$ may be determined by

$$m(H_n/e_k^i(a)) = \lambda_k^i \beta_{H_n}(e_k^i(a)). \tag{22}$$

λ_k may be obtained as follows. Suppose $\zeta_k = [\zeta_k^1, \cdots \zeta_k^{L_k}]^T$ is a uniform weight vector, where ζ_k^i expresses the relative importance of e_k^i and

$$\sum_{i=1}^{L_k} \zeta_k^i = 1 \qquad 0 \le \zeta_k^i \le 1. \tag{23}$$

Let e_k^l be the most important factor in E_k, called the key factor, that is, $\zeta_k^l = \max_i \{\zeta_k^1, \cdots, \zeta_k^i, \cdots, \zeta_k^{L_k}\}$. Normalize ζ_k as follows:

$$\bar{\zeta}_k^i = \zeta_k^i / \zeta_k^l \qquad i = 1, \cdots, L_k. \tag{24}$$

If for the key factor the following relation is true

$$m(H_n/e_k^l) = \alpha_k \beta_{H_n}(e_k^l) \qquad 0 < \alpha_k \le 1 \tag{25}$$

then λ_k^i $(i = 1, \cdots, L_k)$ may be determined by

$$\lambda_k^i = \alpha_k \bar{\zeta}_k^i \qquad i = 1, \cdots, L_k. \tag{26}$$

α_k may be referred to as a priority coefficient representing the importance of the role the most importance factor plays for evaluation of the attribute y_k.

In this way, the basic probability assignments required in the combination rule can be generated from uncertain subjective judgments. The overall probability assignment can then be obtained by combining all the basic probability assignments using the operational algorithms explored in the following subsections.

C. Partial Combination Algorithm

As mentioned before, the set of factors can be classified into $N - 1$ subsets denoted by S_n $(n = 1, \cdots, N - 1)$. In S_n, there are R_n factors as defined by formula (12), the states of which may be confirmed to H_n and/or to H_{n+1}. In this subsection, an algorithm will be developed to generate the local probability assignments to H_n and H_{n+1} by combining these R_n factors. For computational purposes, the "intersection tableau" [1] with values of probability assignments along the rows and columns, respectively, is adopted to develop the algorithm.

At first, combine two factors, $e_{n,n+1}^1$ and $e_{n,n+1}^2$. Suppose the basic probability assignments $m_n^{n,i}$ and $m_{n+1}^{n,i}$ $(i = 1, \cdots, R_n)$ are obtained using formula [22]; then $m_\Theta^{n,i} = 1 - (m_n^{n,i} + m_{n+1}^{n,i})$ $(i = 1, \cdots, R_n)$. All these basic probability assignments to H_n, H_{n+1}, and Θ with respect to $e_{n,n+1}^i$ $(i = 1, \cdots, R_n; n = 1, \cdots, N - 1)$ may then be expressed by the following basic probability assignment matrices M^n:

$$M^n = \begin{array}{c} \begin{bmatrix} m_n^{n,1} & m_{n+1}^{n,1} & m_\Theta^{n,1} \\ m_n^{n,2} & m_{n+1}^{n,2} & m_\Theta^{n,2} \\ \cdots & \cdots & \cdots \\ m_n^{n,R_n} & m_{n+1}^{n,R_n} & m_\Theta^{n,R_n} \end{bmatrix} \end{array} \begin{array}{l} \{e_{n,n+1}^1\} \\ \{e_{n,n+1}^2\} \\ \cdots \\ \{e_{n,n+1}^{R_n}\} \end{array}$$

$$(n = 1, \cdots, N - 1). \tag{27}$$

If $R_n = 0$, then $m_n^{n,i} = 0$, $m_{n+1}^{n,i} = 0$ and $m_\Theta^{n,i} = 1$. Then, construct intersection tableau 1. (See Table II.)

From the combination rule shown in formulas (9) and (10), we have

$$\{H_n\}: \; m_n^{I(2)} = K^{I(2)}(m_n^{n,1} m_n^{n,2} + m_n^{n,1} m_\Theta^{n,2} + m_\Theta^{n,1} m_n^{n,2})$$

$$\{H_{n+1}\}: \; m_{n+1}^{I(2)} = K^{I(2)}(m_{n+1}^{n,1} m_{n+1}^{n,2} + m_{n+1}^{n,1} m_\Theta^{n,2} + m_\Theta^{n,1} m_{n+1}^{n,2})$$

$$\{\Theta\}: \; m_\Theta^{I(2)} = K^{I(2)} m_\Theta^{n,1} m_\Theta^{n,2}$$

where

$$K^{I(2)} = [1 - (m_n^{n,1} m_{n+1}^{n,2} + m_{n+1}^{n,1} m_n^{n,2})]^{-1}.$$

As to $e_{n,n+1}^{I(2)}$, the partially combined probability assignments to other hypothesis in Θ are all zero.

Now, let us combine $e_{n,n+1}^{I(3)} = \{e_{n,n+1}^1, e_{n,n+1}^2, e_{n,n+1}^3\}$. Similarly, construct intersection tableau 2 (Table III). From the combination rule, we can obtain

$$\{H_n\}: \; m_n^{I(3)} = K^{I(3)}(m_n^{I(2)} m_n^{n,3} + m_n^{I(2)} m_\Theta^{n,3} + m_\Theta^{I(2)} m_n^{n,3})$$

$$\{H_{n+1}\}: \; m_{n+1}^{I(3)} = K^{I(3)}(m_{n+1}^{I(2)} m_{n+1}^{n,3} + m_{n+1}^{I(2)} m_\Theta^{n,3} + m_\Theta^{I(2)} m_{n+1}^{n,3})$$

$$\{\Theta\}: \; m_\Theta^{I(3)} = K^{I(3)} m_\Theta^{I(2)} m_\Theta^{n,3}$$

where

$$K^{I(3)} = [1 - (m_n^{I(2)} m_{n+1}^{n,3} + m_{n+1}^{I(2)} m_n^{n,3})]^{-1}.$$

Since $m_n^{I(1)} = m_n^{n,1}$, $m_{n+1}^{I(1)} = m_{n+1}^{n,1}$ and $m_\Theta^{I(1)} = m_\Theta^{n,1}$, then it is natural that by combining $e_{n,n+1}^{I(r+1)} = \{e_{n,n+1}^1, \cdots, e_{n,n+1}^{r+1}\}$, we can obtain the following recursive formulas:

$$\{H_n\}: \; m_n^{I(r+1)} = K^{I(r+1)}(m_n^{I(r)} m_n^{n,r+1} + m_n^{I(r)} m_\Theta^{n,r+1} + m_\Theta^{I(r)} m_n^{n,r+1}) \tag{28.a}$$

TABLE II
INTERSECTION TABLEAU 1

$e^{I(2)}_{n,n+1}$		$e^2_{n,n+1}$: $\{H_n\}\,(m_n^{n,2})$	$\{H_{n+1}\}\,(m_{n+1}^{n,2})$	$\{\Theta\}\,(m_\Theta^{n,2})$
$e^1_{n,n+1}$	$\{H_n\}\,(m_n^{n,1})$	$\{H_n\}\,(m_n^{n,1}m_n^{n,2})$	$\{\varnothing\}\,(m_n^{n,1}m_{n+1}^{n,2})$	$\{H_n\}\,(m_n^{n,1}m_\Theta^{n,2})$
	$\{H_{n+1}\}\,(m_{n+1}^{n,1})$	$\{\varnothing\}\,(m_{n+1}^{n,1}m_n^{n,2})$	$\{H_{n+1}\}\,(m_{n+1}^{n,1}m_{n+1}^{n,2})$	$\{H_{n+1}\}\,(m_{n+1}^{n,1}m_\Theta^{n,2})$
	$\{\Theta\}\,(m_\Theta^{n,1})$	$\{H_n\}\,(m_\Theta^{n,1}m_n^{n,2})$	$\{H_{n+1}\}\,(m_\Theta^{n,1}m_{n+1}^{n,2})$	$\{\Theta\}\,(m_\Theta^{n,1}m_\Theta^{n,2})$

TABLE III
INTERSECTION TABLEAU 2

$e^{I(3)}_{n,n+1}$		$e^3_{n,n+1}$: $\{H_n\}\,(m_n^{n,3})$	$\{H_{n+1}\}\,(m_{n+1}^{n,3})$	$\{\Theta\}\,(m_\Theta^{n,3})$
$e^{I(2)}_{n,n+1}$	$\{H_n\}\,(m_n^{I(2)})$	$\{H_n\}\,(m_n^{I(2)}m_n^{n,3})$	$\{\varnothing\}\,(m_n^{I(2)}m_{n+1}^{n,3})$	$\{H_n\}\,(m_n^{I(2)}m_\Theta^{n,3})$
	$\{H_{n+1}\}\,(m_{n+1}^{I(2)})$	$\{\varnothing\}\,(m_{n+1}^{I(2)}m_n^{n,3})$	$\{H_{n+1}\}\,(m_{n+1}^{I(2)}m_{n+1}^{n,3})$	$\{H_{n+1}\}\,(m_{n+1}^{I(2)}m_\Theta^{n,3})$
	$\{\Theta\}\,(m_\Theta^{I(2)})$	$\{H_n\}\,(m_\Theta^{I(2)}m_n^{n,3})$	$\{H_{n+1}\}\,(m_\Theta^{I(2)}m_{n+1}^{n,3})$	$\{\Theta\}\,(m_\Theta^{I(2)}m_\Theta^{n,3})$

$$\{H_{n+1}\}:\ m_{n+1}^{I(r+1)} = K^{I(r+1)}(m_{n+1}^{I(r)}m_{n+1}^{n,r+1} + m_{n+1}^{I(r)}m_\Theta^{n,r+1} + m_\Theta^{I(r)}m_{n+1}^{n,r+1}) \tag{28.b}$$

$$\{\Theta\}:\ m_\Theta^{I(r+1)} = K^{I(r+1)}m_\Theta^{I(r)}m_\Theta^{n,r+1} \tag{28.c}$$

where

$$K^{I(r+1)} = [1 - (m_n^{I(r)}m_{n+1}^{n,r+1} + m_{n+1}^{I(r)}m_n^{n,r+1})]^{-1}$$
$$r = 1, \cdots, R_n - 1; \qquad n = 1, \cdots, N-1. \tag{28.d}$$

The formulas (28) constitute a partial combination algorithm. $m_n^{I(r+1)}$, $m_{n+1}^{I(r+1)}$, and $m_\Theta^{I(r+1)}$ are the partially combined probability assignments to H_n, H_{n+1}, and Θ, respectively, with respect to $e_{n,n+1}^{I(r+1)} = \{e^1_{n,n+1}, \cdots, e^{r+1}_{n,n+1}\}$. The local probability assignments to H_n, H_{n+1}, and Θ with respect to the subset of factors S_n can be represented as $m_n^{I(R_n)}$, $m_{n+1}^{I(R_n)}$, and $m_\Theta^{I(R_n)}$. To represent the results of the partial combination of all subsets of factors, the following matrix is suggested, called the local probability assignment matrix:

$$M = \begin{bmatrix} m_1^{I(R_1)} & m_2^{I(R_1)} & m_\Theta^{I(R_1)} \\ \cdots & \cdots & \cdots \\ m_n^{I(R_n)} & m_{n+1}^{I(R_n)} & m_\Theta^{I(R_n)} \\ \cdots & \cdots & \cdots \\ m_{N-1}^{I(R_{N-1})} & m_N^{I(R_{N-1})} & m_\Theta^{I(R_{N-1})} \end{bmatrix} \begin{matrix} \{e_{1,2}^{I(R_1)}\} \\ \cdots \\ \{e_{n,n+1}^{I(R_n)}\} \\ \cdots \\ \{e_{N-1,N}^{I(R_{N-1})}\} \end{matrix}. \tag{29}$$

D. Overall Combination Algorithm

After the partial combination, the subset of factors S_n may be regarded as an aggregated factor, and $m_n^{I(R_n)}$ as a new basic probability assignment to the hypothesis H_n, confirmed by S_n. The problem is then to combine all these integrated factors in order to obtain the overall probability assignments to all subsets Ψ of Θ, including the singletons H_n ($n = 1, \cdots, N$). We still use the intersection tableau to develop such an overall combination algorithm.

First of all, combine $e_{1,3}^{C(2)} = \{e_{1,2}^{I(R_1)}, e_{2,3}^{I(R_2)}\}$. Construct intersection tableau 3. (See Table IV.)

From the combination rule, we have

$$\{H_1\}:\ b_1^{C(2)} = K^{C(2)}m_1^{I(R_1)}m_\Theta^{I(R_2)}$$

$$\{H_2\}:\ b_2^{C(2)} = K^{C(2)}(m_2^{I(R_1)}m_2^{I(R_2)} + m_2^{I(R_1)}m_\Theta^{I(R_2)} + m_\Theta^{I(R_1)}m_2^{I(R_2)})$$

$$\{H_3\}:\ b_3^{C(2)} = K^{C(2)}m_\Theta^{I(R_1)}m_3^{I(R_2)}$$

$$\{\Theta\}:\ b_\Theta^{C(2)} = K^{C(2)}m_\Theta^{I(R_1)}m_\Theta^{I(R_2)}$$

where

$$K^{C(2)} = [1 - (m_1^{I(R_1)}m_2^{I(R_2)} + m_1^{I(R_1)}m_3^{I(R_2)} + m_2^{I(R_1)}m_3^{I(R_2)})]^{-1}.$$

Then, construct intersection tableau 4 (see Table V) to combine $e_{1,4}^{C(3)} = \{e_{1,3}^{C(2)}, e_{3,4}^{I(R_3)}\} = \{e_{1,2}^{I(R_1)}, e_{2,3}^{I(R_2)}, e_{3,4}^{I(R_3)}\}$. According to the combination rule, we obtain

$$\{H_1\}:\ b_1^{C(3)} = K^{C(3)}b_1^{C(2)}m_\Theta^{I(R_3)}$$

$$\{H_2\}:\ b_2^{C(3)} = K^{C(3)}b_2^{C(2)}m_\Theta^{I(R_3)}$$

$$\{H_3\}:\ b_3^{C(3)} = K^{C(3)}(b_3^{C(2)}m_3^{I(R_3)} + b_3^{C(2)}m_\Theta^{I(R_3)} + b_\Theta^{C(2)}m_3^{I(R_3)})$$

$$\{H_4\}:\ b_4^{C(3)} = K^{C(3)}b_\Theta^{C(2)}m_4^{I(R_3)}$$

$$\{\Theta\}:\ b_\Theta^{C(3)} = K^{C(3)}b_\Theta^{C(2)}m_\Theta^{I(R_3)}$$

where

$$K^{C(3)} = [1 - (b_1^{C(2)}m_3^{I(R_3)} + b_1^{C(2)}m_4^{I(R_3)} + b_2^{C(2)}m_3^{I(R_3)} + b_2^{C(2)}m_4^{I(R_3)} + b_3^{C(2)}m_4^{I(R_3)})]^{-1}$$

TABLE IV
INTERSECTION TABLEAU 3

	$e_{1,3}^{C(2)}$	$e_{2,3}^{I(R_2)}$		
		$\{H_2\}\ (m_2^{I(R_2)})$	$\{H_3\}\ (m_3^{I(R_2)})$	$\{\Theta\}\ (m_\Theta^{I(R_2)})$
$e_{1,2}^{I(R_1)}$	$\{H_1\}\ (m_1^{I(R_1)})$	$\{\varnothing\}\ (m_1^{I(R_1)} m_2^{I(R_2)})$	$\{\varnothing\}\ (m_1^{I(R_1)} m_3^{I(R_2)})$	$\{H_1\}\ (m_1^{I(R_1)} m_\Theta^{I(R_2)})$
	$\{H_2\}\ (m_2^{I(R_1)})$	$\{H_2\}\ (m_2^{I(R_1)} m_2^{I(R_2)})$	$\{\varnothing\}\ (m_2^{I(R_1)} m_3^{I(R_2)})$	$\{H_2\}\ (m_2^{I(R_1)} m_\Theta^{I(R_2)})$
	$\{\Theta\}\ (m_\Theta^{I(R_1)})$	$\{H_2\}\ (m_\Theta^{I(R_1)} m_2^{I(R_2)})$	$\{H_3\}\ (m_\Theta^{I(R_1)} m_3^{I(R_2)})$	$\{\Theta\}\ (m_\Theta^{I(R_1)} m_\Theta^{I(R_2)})$

TABLE V
INTERSECTION TABLEAU 4

	$e_{1,4}^{C(3)}$	$e_{3,4}^{I(R_3)}$		
		$\{H_3\}\ (m_3^{I(R_3)})$	$\{H_4\}\ (m_4^{I(R_3)})$	$\{\Theta\}\ (m_\Theta^{I(R_3)})$
$e_{1,3}^{C(2)}$	$\{H_1\}\ (b_1^{C(2)})$	$\{\varnothing\}\ (b_1^{C(2)} m_3^{I(R_3)})$	$\{\varnothing\}\ (b_1^{C(2)} m_4^{I(R_3)})$	$\{H_1\}\ (b_1^{C(2)} m_\Theta^{I(R_3)})$
	$\{H_2\}\ (b_2^{C(2)})$	$\{\varnothing\}\ (b_2^{C(2)} m_3^{I(R_3)})$	$\{\varnothing\}\ (b_2^{C(2)} m_4^{I(R_3)})$	$\{H_2\}\ (b_2^{C(2)} m_\Theta^{I(R_3)})$
	$\{H_3\}\ (b_3^{C(2)})$	$\{H_3\}\ (b_3^{C(2)} m_3^{I(R_3)})$	$\{\varnothing\}\ (b_3^{C(2)} m_4^{I(R_3)})$	$\{H_3\}\ (b_3^{C(2)} m_\Theta^{I(R_3)})$
	$\{\Theta\}\ (b_\Theta^{C(2)})$	$\{H_3\}\ (b_\Theta^{C(2)} m_3^{I(R_3)})$	$\{H_4\}\ (b_\Theta^{C(2)} m_4^{I(R_3)})$	$\{\Theta\}\ (b_\Theta^{C(2)} m_\Theta^{I(R_3)})$

$$= \left[1 - \left(\sum_{t=1}^{2} b_t^{C(2)}(m_3^{I(R_3)} + m_4^{I(R_3)}) + b_3^{C(2)} m_4^{I(R_3)}\right)\right]^{-1}.$$

Since $b_1^{C(1)} = m_1^{I(R_1)}$, $b_2^{C(1)} = m_2^{I(R_1)}$, and $b_\Theta^{C(1)} = m_\Theta^{I(R_1)}$, then we can combine $e_{1,j+2}^{C(j+1)} = \{e_{1,2}^{I(R_1)}, \cdots, e_{j+1,j+2}^{I(R_j+1)}\}$ in a similar way and obtain the following recursive algorithm:

$$\{H_1\}:\ b_1^{C(j+1)} = K^{C(j+1)} b_1^{C(j)} m_\Theta^{I(R_j+1)} \tag{30.a}$$

$$\cdots$$

$$\{H_j\}:\ b_j^{C(j+1)} = K^{C(j+1)} b_j^{C(j)} m_\Theta^{I(R_j+1)} \tag{30.b}$$

$$\{H_{j+1}\}:\ b_{j+1}^{C(j+1)} = K^{C(j+1)}(b_{j+1}^{C(j)} m_{j+1}^{I(R_j+1)} + b_{j+1}^{C(j)} m_\Theta^{I(R_j+1)} + b_\Theta^{C(j)} m_{j+1}^{I(R_j+1)}) \tag{30.c}$$

$$\{H_{j+2}\}:\ b_{j+2}^{C(j+1)} = K^{C(j+1)} b_\Theta^{C(j)} m_{j+2}^{I(R_j+1)} \tag{30.d}$$

$$\{\Theta\}:\ b_\Theta^{C(j+1)} = K^{C(j+1)} b_\Theta^{C(j)} m_\Theta^{I(R_j+1)} \tag{30.e}$$

where

$$K^{C(j+1)} = \left[1 - \left(\sum_{t=1}^{j} b_t^{C(j)}(m_{j+1}^{I(R_j+1)} + m_{j+2}^{I(R_j+1)}) + b_{j+1}^{C(j)} m_{j+2}^{I(R_j+1)}\right)\right]^{-1}$$
$$j = 1, \cdots, N-2. \tag{30.f}$$

When $j = N - 2$, the overall probability assignments are generated and can be expressed by the following vector, called the overall probability assignment vector:

$$G = [b_1^{C(N-1)}, \cdots, b_n^{C(N-1)}, \cdots, b_N^{C(N-1)}, b_\Theta^{C(N-1)}]^T. \tag{31}$$

Notice that G is obtained by combining $e_{1,N}^{C(N-1)}$ while

$$\begin{aligned} e_{1,N}^{C(N-1)} &= \{e_{1,2}^{I(R_1)}, \cdots, e_{n,n+1}^{I(R_n)}, \cdots, e_{N-1,N}^{I(R_{N-1})}\} \\ &= \{S_1 \quad S_2 \quad \cdots \quad S_n \quad \cdots \quad S_{N-1}\} \\ &= \{e_k^1 \quad e_k^2 \quad \cdots \quad e_k^{L_k}\} = E_k. \end{aligned} \tag{32}$$

In other words, $b_n^{C(N-1)}$ is the overall probability assignment to which H_n is confirmed by all factors e_k^i ($i = 1, \cdots, L_k$).

From the above discussion, it is obvious that the overall probability assignments are all zero for other hypothesis in Θ except for the singletons H_n ($n = 1, \cdots, N$) and Θ. So, it can be proved that the following equation is true:

$$\sum_{n=1}^{N} b_n^{C(N-1)} + b_\Theta^{C(N-1)} = 1. \tag{33}$$

Since $m(H_n/E_k(a)) = b_{H_n}^{C(N-1)}$, the preference degree p_{rk}, defined by (18), can then be calculated by

$$\begin{aligned} p_{rk} &= \sum_{n=1}^{N} m(H_n/E_k(a_r))\, p(H_n) + m(\Theta/E_k(a_r))\, p(\Theta) \\ &= \sum_{n=1}^{N} b_{H_n}^{C(N-1)} p(H_n) + b_\Theta^{C(N-1)} p(\Theta). \end{aligned} \tag{34}$$

IV. EVALUATION MATRIX AND ALTERNATIVE RANKING

A. Construction of An Evaluation Matrix

The evidential reasoning approach explored above is actually used to transform the uncertain subjective judgments about the state of a qualitative attribute y_k at an alternative a_r into the preference degree $p_{rk} = p(y_k(a_r))$ for all $k = k_1 + 1, \cdots, k_1 + k_2$; $r = 1, \cdots, l$. In this way, all qualitative attributes are evaluated and quantified using the values in the interval $[-1 \quad 1]$.

The values of quantitative attributes which are generally incommensurate may also be transformed into the preference degree space using the following formulas:

$$p_{rk} = p(y_{rk}) = \frac{2(y_{rk} - y_k^{\min})}{y_k^{\max} - y_k^{\min}} - 1, \quad k = 1, \cdots, k_1; r = 1, \cdots, l, \quad \text{for benefit attributes} \tag{35}$$

$$p_{rk} = p(y_{rk}) = \frac{2(y_k^{\max} - y_{rk})}{y_k^{\max} - y_k^{\min}} - 1; \quad k = 1, \cdots, k_1; r = 1, \cdots, l, \quad \text{for cost attributes} \tag{36}$$

$$y_k^{\max} = \max\{y_{1k} \cdots y_{lk}\}, \quad y_k^{\min} = \min\{y_{1k} \cdots y_{lk}\}. \tag{37}$$

The transformed attribute y_k may be denoted by a preference function $p(y_k)$. Thus, the original extended decision matrix defined by Table I is transformed into an evaluation matrix, an ordinary decision matrix defined by Table VI, in which the states of all attributes, either quantitative or qualitative, are represented in the preference degree space. The alternatives may then be ranked based on the evaluation matrix.

B. Alternative Ranking

At this stage, several traditional MADM methods can be selected to rank alternatives on the basis of the evaluation matrix. The CODASID method [21], [30], [33] may be one of them, which is based on a complete concordance and discordance analysis for information aggregation and the decision rule of the TOPSIS method for information synthesis (alternative ranking). The reasons that we have chosen CODASID for alterative ranking are not only that it is appropriate to address a MADM problem represented by an ordinary decision matrix, but also that we have developed software for CODASID so that the application examples can be readily tested [21], [33]. Obviously, the readers are not prohibited from adopting other proper MADM methods they prefer to deal with alternative ranking based on Table VI. The computational steps of CODASID are summarized below.

Step 1: Generate the weighted normalized evaluation matrix Z as follows:

$$Z = (r_{ij})_{l \times (k_1 + k_2)} \times \text{diag}\{\omega_1 \cdots \omega_{k_1 + k_2}\} = (z_{ij})_{l \times (k_1 + k_2)} \tag{38}$$

where ω_k is the relative weight of y_k ($k = 1, \cdots, k_1 + k_2$) and

$$z_{ij} = \omega_j r_{ij}; \quad r_{ij} = \frac{p_{ij} - p_j^{\min}}{p_j^{\max} - p_j^{\min}}, \quad i = 1, \cdots, l; j = 1, \cdots, k_1 + k_2 \tag{39}$$

$$p_j^{\max} = \max\{p_{1j} \cdots p_{lj}\}; \quad p_j^{\min} = \min\{p_{1j} \cdots p_{lj}\}, \quad j = 1, \cdots, k_1 + k_2. \tag{40}$$

TABLE VI
THE EVALUATION MATRIX

Preference Degrees	$p(y_1)$	$\cdots$	$p(y_{k_1})$	$p(y_{k_1+1})$	$\cdots$	$p(y_{k_1+k_2})$
a_1	p_{11}	$\cdots$	p_{1k_1}	p_{1k_1+1}	$\cdots$	$p_{1k_1+k_2}$
a_2	p_{21}	$\cdots$	p_{2k_1}	p_{2k_1+1}	$\cdots$	$p_{2k_1+k_2}$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
a_l	p_{l1}	$\cdots$	p_{lk_1}	p_{lk_1+1}	$\cdots$	$p_{lk_1+k_2}$

Step 2: For each pair of alternatives (a_1, a_j) ($i, j = 1, \cdots, l; i \neq j$), construct the concordance set C_{ij} and the discordance set D_{ij}, based on the evaluation matrix:

$$C_{ij} = \{k \mid p_{ik} \geq p_{jk}, k = 1, \cdots, k_1 + k_2\};$$
$$D_{ij} = \{k \mid {}_{ik} < p_{jk}, k = 1, \cdots, k_1 + k_2\}. \tag{41}$$

Step 3: Calculate the preference-evaluation discordance index d_{ij}, the preference concordance index pc_{ij}, and the evaluation concordance index ec_{ij}:

$$d_{ij} = \frac{\max\limits_{k \in D_{ij}} |z_{ik} - z_{jk}|}{\max\limits_{k \in J} |z_{ik} - z_{jk}|}; \quad pc_{ij} = \frac{\sum\limits_{k \in C_{ij}} \omega_k}{\sum\limits_{k=1}^{k_1+k_2} \omega_k};$$

$$ec_{ij} = \frac{\max\limits_{k \in C_{ij}} |r_{ik} - r_{jk}|}{\max\limits_{k \in J} |r_{ik} - r_{jk}|} \tag{42}$$

where $0 \leq d_{ij}, pc_{ij}, ec_{ij} \leq 1$, and $J = \{1, \cdots, k_1 + k_2\}$ is the index set of attributes.

Step 4: For all alternatives, calculate the net preference concordance dominance index $pc(a_i)$, the net evaluation concordance dominance index $ec(a_i)$, and the net preference-evaluation discordance dominance index $d(a_i)$. Then, construct the Judgment-Evaluation (J-E) matrix (Table VII):

$$pc(a_i) = \sum_{\substack{j=1 \\ j \neq i}}^{l} pc_{ij} - \sum_{\substack{j=1 \\ j \neq i}}^{l} pc_{ji};$$

$$ec(a_i) = \sum_{\substack{j=1 \\ j \neq i}}^{l} ec_{ij} - \sum_{\substack{j=1 \\ j \neq i}}^{l} ec_{ji};$$

$$d(a_i) = \sum_{\substack{j=1 \\ j \neq i}}^{l} d_{ij} - \sum_{\substack{j=1 \\ j \neq i}}^{l} d_{ji}, \quad i = 1, \cdots, l. \tag{43}$$

Step 5: $pc(a)$, $ec(a)$, and $d(a)$ in Table VII are regarded as new composite attributes. $pc(a)$ and $ec(a)$ are for maximization and $d(a)$ for minimization. Suppose ρ_1, ρ_2, and ρ_3 are tradeoff weights representing the relative importance of $pc(a)$, $ec(a)$, and $d(a)$. ρ_i may be determined as follows [30]:

$$\rho_1 = \rho_2 = 0.25; \quad \rho_3 = 0.5. \tag{44}$$

Step 6: Normalize the three indexes $pc(a)$, $ec(a)$, and

TABLE VII
THE J-E MATRIX

a	$pc(a)$	$ec(a)$	$d(a)$
a_1	pc_1	ec_1	d_1
a_2	pc_2	ec_2	d_2
$\cdots$	$\cdots$	$\cdots$	$\cdots$
a_l	pc_l	ec_l	d_l

$d(a)$ as follows:

$$\overline{pc}(a_i) = \frac{pc(a_i)}{\sqrt{\sum_{j=1}^{l} pc^2(a_j)}}; \quad \overline{ec}(a_i) = \frac{ec(a_i)}{\sqrt{\sum_{j=1}^{l} ec^2(a_j)}};$$

$$\bar{d}(a_i) = \frac{d(a_i)}{\sqrt{\sum_{j=1}^{l} d^2(a_j)}}, \quad i = 1, \cdots, l \tag{45}$$

yielding the normalized J-E matrix $\overline{JE}$:

$$\overline{JE} = \begin{bmatrix} \overline{pc}(a_1) & \overline{ec}(a_1) & \bar{d}(a_1) \\ \overline{pc}(a_2) & \overline{ec}(a_2) & \bar{d}(a_2) \\ \cdots & \cdots & \cdots \\ \overline{pc}(a_l) & \overline{ec}(a_l) & \bar{d}(a_l) \end{bmatrix}. \tag{46}$$

Step 7: $\overline{JE}$ is weighted by the tradeoff weight vector $\rho = [\rho_1 \ \ \rho_2 \ \ \rho_3]^T$, resulting in the weighted normalized J-E matrix JE:

$$JE = \begin{bmatrix} \widetilde{pc}(a_1) & \widetilde{ec}(a_1) & \tilde{d}(a_1) \\ \widetilde{pc}(a_2) & \widetilde{ec}(a_2) & \tilde{d}(a_2) \\ \cdots & \cdots & \cdots \\ \widetilde{pc}(a_l) & \widetilde{ec}(a_l) & \tilde{d}(a_l) \end{bmatrix} \tag{47}$$

where $\widetilde{pc}(a_i) = \rho_1 \overline{pc}(a_i)$, $\widetilde{ec}(a_i) = \rho_2 \overline{ec}(a_i)$, and $\tilde{d}(a_i) = \rho_3 \bar{d}(a_i)$, $i = 1, \cdots, l$.

Step 8: An ideal point a^* and a negative ideal point a^- in the J-E space can then be defined as follows:

$$\begin{aligned} \widetilde{pc}(a^*) &= \max\{\widetilde{pc}(a_1) \cdots \widetilde{pc}(a_l)\}, \\ \widetilde{ec}(a^*) &= \max\{\widetilde{ec}(a_1) \cdots \widetilde{ec}(a_l)\}, \\ \tilde{d}(a^*) &= \min\{\tilde{d}(a_1) \cdots \tilde{d}(a_l)\} \end{aligned} \tag{48}$$

and

$$\begin{aligned} \widetilde{pc}(a^-) &= \min\{\widetilde{pc}(a_1) \cdots \widetilde{pc}(a_l)\}, \\ \widetilde{ec}(a^-) &= \min\{\widetilde{ec}(a_1) \cdots \widetilde{ec}(a_l)\}, \\ \tilde{d}(a^-) &= \max\{\tilde{d}(a_1) \cdots \tilde{d}(a_l)\}. \end{aligned} \tag{49}$$

Step 9: The distance s_i^* between an alternative a_i and the ideal point a^* and the distance s_i^- between a_i and the negative ideal point a^- are defined as

$$s_i^* = [(\widetilde{pc}(a_i) - \widetilde{pc}(a^*))^2 + (\widetilde{ec}(a_i) - \widetilde{ec}(a^*))^2 + (\tilde{d}(a_i) - \tilde{d}(a^*))^2]^{1/2} \quad i = 1, \cdots, l \tag{50}$$

$$s_i^- = [(\widetilde{pc}(a_i) - \widetilde{pc}(a^-))^2 + (\widetilde{ec}(a_i) - \widetilde{ec}(a^-))^2 + (\tilde{d}(a_i) - \tilde{d}(a^-))^2]^{1/2} \quad i = 1, \cdots, l. \tag{51}$$

Step 10: The relative closeness index of a_i to the ideal point is finally defined as

$$u(a_i) = \frac{s_i^-}{s_i^- + s_i^*}, \quad 0 \le u(a_i) \le 1, i = 1, \cdots, l;$$
$$u(a^-) = 0, u(a^*) = 1. \tag{52}$$

A large value of $u(a_i)$ indicates that a_i is more favorable since it is simultaneously closer to the ideal point and further from the negative ideal point.

C. A Procedure for Hybrid MADM with Uncertainty

As a result of the discussion in the previous subsections, we are now in a position to formulate a procedure for dealing with a hybrid multiple-attribute decision making problem with uncertainty. This procedure is composed of the transformation, aggregation, and synthesis of information contained in the problem. The procedure may be summarized as the following steps.

Step 1: Define a hybrid MADM problem using the extended decision matrix as defined by Table I, where uncertain subjective judgments for evaluation of a qualitative attribute may be acquired using statements similar to Statement i> and represented by the evaluation analysis model.

Step 2: Transform the numerical value with a certain unit of a quantitative attribute at each alternative into the preference degree space using (35) or (36).

Step 3: Quantify the state of a qualitative attribute y_k at each alternative a_r using the evidential reasoning approach in order to obtain the preference degree $p_{rk} = p(y_k(a_r))$. Let $k = 1$, $r = 1$.

Step 4: First, calculate the basic probability assignments from the confidence degrees given in the uncertain subjective judgments by using (22), resulting in the basic probability assignment matrices $M^n(y_k(a_r))$ $(n = 1, \cdots, N - 1)$ defined by (27).

Step 5: Then, conduct partial combinations for the subsets of factors S_n $(n = 1, \cdots, N - 1)$ in E_k using the algorithm shown in formulas (28), resulting in the local probability assignment matrix $M(y_k(a_r))$ defined by (29).

Step 6: Conduct overall combination for all factors in E_k for y_k using the algorithm listed in formulas (30), yielding the overall probability assignment vector $G(y_k(a_r))$ defined by (31).

Step 7: Using (34), calculate the preference degree of $y_k(a_r)$, i.e., p_{rk}. If $k \ge k_1 + k_2$ and $r \ge l$, continue. If $k \le k_1 + k_2$ and $r < l$, let $r = r + 1$ and then go to Step 4. If $k < k_1 + k_2$ and $r \ge l$, let $k = k + 1$, $r = 1$, and then go to Step 4.

Step 8: Construct the evaluation matrix as shown in Table VI.

Step 9: Aggregate the information contained in the evaluation matrix using formulas (38)–(43), resulting in the J–E matrix defined in Table VII.

Step 10: Synthesize the information contained in the J–E matrix by formulas (44)–(52), generating the relative closeness indexes of all alternatives to the ideal point, that is, $u(a_r)$, $r = 1, \cdots, l$.

Step 11: Rank a_r based on $u(a_r)$, $r = 1, \cdots, l$. If $u(a_{r_1}) \geq u(a_{r_2})$, then a_{r_1} is preferred to a_{r_2}, $r_1, r_2 = 1, \cdots, l$; $r_1 \neq r_2$.

V. Examples

A. A Numerical Example

A numerical example is discussed in this subsection to show how to implement the new evidential reasoning approach. The problem is to evaluate and quantify the state of an attribute y_k at an alternative a_r within an evidential reasoning framework.

First, define the set of evaluation grades for y_k as

$$H = \{H_1 \quad H_2 \quad H_3 \quad H_4 \quad H_5 \quad H_6 \quad H_7\} \tag{53}$$

which may be interpreted as in formula (5) and scaled as in (6) where $p(H) = \sum_{n=1}^{7} p(H_n)/7 = 0$. In (53), H_n ($n = 1, \cdots, 7$) are supposed to be distinct grades. Suppose there are ten factors influencing the evaluation of the attribute y_k, denoted by

$$E_k = \{e_1 \quad e_2 \quad e_3 \quad e_4 \quad e_5 \quad e_6 \quad e_7 \quad e_8 \quad e_9 \quad e_{10}\}. \tag{54}$$

The evaluation analysis model can then be depicted as in Fig. 2.

The uniform weights representing the relative importance of these factors are given by

$$\begin{aligned}\zeta_k &= [\zeta_k^1 \quad \zeta_k^2 \quad \zeta_k^3 \quad \zeta_k^4 \quad \zeta_k^5 \quad \zeta_k^6 \quad \zeta_k^7 \quad \zeta_k^8 \quad \zeta_k^9 \quad \zeta_k^{10}]^T \\ &= [0.12, 0.085, 0.095, 0.09, 0.1, 0.14, 0.08, \\ &\qquad 0.07, 0.13, 0.09]^T.\end{aligned} \tag{55}$$

Normalize ζ_k, resulting in $\bar{\zeta}_k$, where

$$\begin{aligned}\bar{\zeta}_k = \zeta_k/\zeta_k^6 &= [0.86, 0.61, 0.68, 0.64, 0.71, 1.0, 0.57, \\ &\qquad 0.5, 0.93, 0.64]^T.\end{aligned} \tag{56}$$

Note that e_6 is the key factor. Suppose the decision maker considers that e_6 has the absolute priority in evaluation of $y_k(a)$, that is, $\alpha_k = 1$. So, $\lambda_k = \alpha_k \bar{\zeta}_k$.

The uncertain subjective judgments for evaluation of the state of y_k at a_r are acquired and listed in Table VIII. These judgments may also be described using statements. For instance, it is stated that the state of the factor e_6 at a_r is evaluated to be *indifferent* (H_4) with a confidence degree of 0.7 and to be *satisfactory* (H_5) with 0.2.

From Fig. 2 and Table VIII, we can then obtain the following notation:

$$R_1 = 0;$$

$$e_{2,3}^{I(R_2)} = \{e_{2,3}^1 = e_1\}, \qquad R_1 = 1;$$

$$e_{3,4}^{I(R_3)} = \{e_{3,4}^1 = e_2, e_{3,4}^2 = e_3, e_{3,4}^3 = e_4, e_{3,4}^4 = e_5,\}$$

$$R_3 = 4;$$

$$e_{4,5}^{I(R_4)} = \{e_{4,5}^1 = e_6, e_{4,5}^2 = e_7, e_{4,5}^3 = e_8, e_{4,5}^4 = e_9\}$$

$$R_4 = 4;$$

$$e_{5,6}^{I(R_5)} = \{e_{5,6}^1 = e_{10}\}, \qquad R_5 = 1;$$

$$R_6 = 0.$$

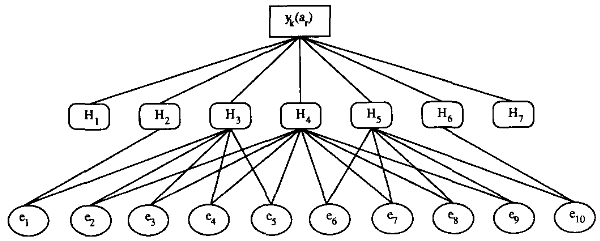

Fig. 2. The evaluation analysis submodel for the numerical example.

TABLE VIII
UNCERTAIN SUBJECTIVE JUDGMENTS FOR EVALUATION OF $y_k(a_r)$

Confidence Degrees (β)		Evaluation Grades						
		H_1	H_2	H_3	H_4	H_5	H_6	H_7
Factors	e_1		0.4	0.2				
	e_2			0.5	0.4			
	e_3			0.4	0.5			
	e_4			0.3	0.6			
	e_5			0.8				
	e_6				0.7	0.2		
	e_7				0.8			
	e_8				0.5	0.5		
	e_9				0.6	0.25		
	e_{10}					0.5	0.5	

The basic probability assignments can be calculated from the given confidence degrees by using formula (22). For instance,

$$\begin{aligned}
m_2^{2,1} &= m(H_2/e_{2,3}^1) = m(H_2/e_1) = 0.4 \times \lambda_k^1 \\
&= 0.4 \times 0.86 \approx 0.344; \\
m_3^{2,1} &= m(H_3/e_{2,3}^1) = m(H_3/e_1) = 0.2 \times \lambda_k^1 = 0.172; \\
m_\Theta^{2,1} &= 1 - (m_2^{2,1} + m_3^{2,1}) = 0.484; \\
m_3^{3,1} &= m(H_3/e_{3,4}^1) = m(H_3/e_2) = 0.5 \times \lambda_k^2 \\
&= 0.5 \times 0.61 = 0.305; \\
m_3^{3,2} &= m(H_3/e_{3,4}^2) = m(H_3/e_3) = 0.4 \times \lambda_k^3 \\
&= 0.4 \times 0.68 \approx 0.272; \\
m_3^{3,3} &= m(H_3/e_{3,4}^3) = m(H_3/e_4) = 0.3 \times \lambda_k^4 \\
&= 0.3 \times 0.64 = 0.192; \\
m_3^{3,4} &= m(H_3/e_{3,4}^4) = m(H_3/e_5) = 0.8 \times \lambda_k^5 \\
&= 0.8 \times 0.71 = 0.568.
\end{aligned}$$

On the whole, the following basic probability assignment matrices are obtained:

$$M^1 = [0 \quad 0 \quad 1] \qquad R_1 = 0;$$

$$M^2 = [0.344 \quad 0.172 \quad 0.484] \qquad R_2 = 1;$$

$$M^3 = \begin{bmatrix} 0.305 & 0.244 & 0.451 \\ 0.272 & 0.34 & 0.388 \\ 0.192 & 0.384 & 0.424 \\ 0.568 & 0.0 & 0.432 \end{bmatrix} \qquad R_3 = 4;$$

$$M^4 = \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.456 & 0.0 & 0.544 \\ 0.25 & 0.25 & 0.5 \\ 0.558 & 0.233 & 0.209 \end{bmatrix} \qquad R_4 = 4;$$

$$M^5 = [0.32 \quad 0.32 \quad 0.36] \qquad R_5 = 1;$$

$$M^6 = [0 \quad 0 \quad 1] \qquad R_6 = 0.$$

These ten factors are then combined using the partial combination algorithm (28) and the overall combination algorithm (30). As a result of the partial combination, the following local probability assignment matrix can be produced:

$$M = \begin{bmatrix} 0.0 & 0.0 & 1.0 \\ 0.344 & 0.172 & 0.484 \\ 0.609 & 0.32 & 0.071 \\ 0.894 & 0.095 & 0.011 \\ 0.32 & 0.32 & 0.36 \\ 0.0 & 0.0 & 1.0 \end{bmatrix}.$$

As a result of the overall combination, the following overall probability assignment vector can be obtained:

$$G = [b_1^{C(6)} \quad b_2^{C(6)} \quad b_3^{C(6)} \quad b_4^{C(6)} \quad b_5^{C(6)} \quad b_6^{C(6)} \quad b_7^{C(6)} \quad b_H^{C(6)}]^T$$
$$= [0.0 \quad 0.001 \quad 0.025 \quad 0.934 \quad 0.036 \quad 0.002 \quad 0.0 \quad 0.002]^T.$$

From the above distribution of the overall probability assignments, it is obvious that the state of y_k at a_r is confirmed by the whole set of factors to the grade H_4 to a very high extent of 0.934, although the confidence degrees are almost uniformly distributed among the grades H_2, H_3, H_4, H_5, and H_6. Such a result is quite reasonable because the states of eight factors at a_r are confirmed to H_4 to different extents, including those of the two most important factors e_6 and e_9 at a_r. This result may demonstrate the property of the D–S theory that it can model the narrowing of the hypothesis set with the accumulation of evidence.

Finally, the preference degree of $y_k(a_r)$, i.e., $p(y_k(a_r))$, can be generated by formula (34):

$$p(y_k(a_r)) = \sum_{n=1}^{7} b_{H_n}^{C(6)} p(H_n) + b_H^{C(6)} p(H) = 0.0052 \tag{57}$$

which means that it is almost certain that the state of y_k at a_r is *indifferent*.

B. A Motorcycle Evaluation Problem

A customer intends to buy a motorcycle. Four types of motorcycle are available for selection, that is, "Kawasaki," "Yamaha," "Honda," and "BMW." The technical and economical performances of the four types of motorcycle are also available [7]. These performances are represented by either numerical values with units or subjective judgments with uncertainty. The customer, however, only takes into account six of the performances (attributes), including both qualitative and quantitative attributes. These six attributes are described in Table IX, in which the numerical values of the quantitative attributes and the uncertain subjective judgments for evaluation of the qualitative attributes are discussed in depth in [7].

The uncertain subjective judgments listed in Table IX are represented in a compact form. In Table X, the uncertain subjective judgments for evaluation of the *engine* of "Kawasaki" are demonstrated. These judgments can be described using the following statements.

1) The *responsiveness* of the *engine* of "Kawasaki" is *excellent* with a confidence degree of 0.8,

2) the *fuel economy* of the *engine* of "Kawasaki" is absolutely *average*, and

3) The *quietness* of the *engine* of "Kawasaki" might be half *indifferent* and half *average*.

Since no single motorcycle type dominates or is dominated by the other types from Table IX, the customer has to provide his preference information about the relative importance of the six attributes. He uses a ten-point scale to estimate the relative importance. The relative weights of the six attributes are thus estimated as follows:

$$\hat{W} = [\hat{\omega}_1 \quad \hat{\omega}_2 \quad \hat{\omega}_3 \quad \hat{\omega}_4 \quad \hat{\omega}_5 \quad \hat{\omega}_6]^T = [9 \quad 5 \quad 7 \quad 7 \quad 7 \quad 4]^T. \tag{58}$$

$\hat{W}$ is then normalized by

$$W = \hat{W}/39 = [\omega_1 \quad \omega_2 \quad \cdots \quad \omega_6]^T = [0.23 \quad 0.127 \quad 0.18 \quad 0.18 \quad 0.18 \quad 0.103]^T. \tag{59}$$

Three sets of factors for evaluation of the three qualitative attributes are defined by

$$E_4 = \{e_4^1 \quad e_4^2 \quad e_4^3\} = \{\text{responsiveness} \quad \text{fuel economy} \quad \text{quietness}\} \tag{60}$$

TABLE IX
AN EXTENDED DECISION MATRIX FOR EVALUATION OF FOUR TYPES OF MOTORCYCLE

Types of Attributes	Definition of Attributes	Units or Factors	Types of Motorcycle (Alternatives) Kawasaki (a_1)	Yamaha (a_2)	Honda (a_3)	BMW (a_4)
Quantitative	Price (y_1)	lb	6499	5199	6199	8220
	Displacement (y_2)	cm^3	1052	1188	998	987
	Top Speed (y_3)	mi/h	160	155	160	145
Qualitative	Engine (y_4)	Responsiveness (e_4^1)	$E(0.8)$	$G(0.3)$ $E(0.6)$	$G(1.0)$	$I(1.0)$
		Fuel Economy (e_4^2)	$A(1.0)$	$I(1.0)$	$I(0.5)$ $A(0.5)$	$E(1.0)$
		Quietness (e_4^3)	$I(0.5)$ $A(0.5)$	$A(1.0)$	$G(0.5)$ $E(0.3)$	$E(1.0)$
	Handling (y_5)	Steering (e_5^1)	$E(0.9)$	$G(1.0)$	$A(1.0)$	$A(0.6)$
		Bumping Bends (e_5^2)	$A(0.5)$ $G(0.5)$	$G(1.0)$	$G(0.8)$ $E(0.1)$	$P(0.5)$ $I(0.5)$
		Maneuverability (e_5^3)	$A(1.0)$	$E(0.9)$	$I(1.0)$	$P(1.0)$
		Top Speed Stability (e_5^4)	$E(1.0)$	$G(1.0)$	$G(1.0)$	$G(0.6)$ $E(0.4)$
	General (y_6)	Quality of Finish (e_6^1)	$P(0.5)$ $I(0.5)$	$G(1.0)$	$E(1.0)$	$G(0.5)$ $E(0.5)$
		Seat Comfort (e_6^2)	$G(1.0)$	$G(0.5)$ $E(0.5)$	$G(0.6)$	$E(1.0)$
		Headlight (e_6^3)	$G(1.0)$	$A(1.0)$	$E(1.0)$	$G(0.5)$ $E(0.5)$

The evaluation grades for the qualitative attributes are defined as $P(\beta)$—*poor*, $I(\beta)$—*indifferent*, $A(\beta)$—*average*, $G(\beta)$—*good*, and $E(\beta)$—*excellent*, where β represents confidence degree [7].

TABLE X
UNCERTAIN SUBJECTIVE JUDGMENTS FOR $y_4(a_1)$

Confidence Degrees (β)		Evaluation Grades Poor	Indifferent	Average	Good	Excellent
Factors	Responsiveness					0.8
	Fuel economy			1.0		
	Quietness		0.5	0.5		

$$E_5 = \{e_5^1 \quad e_5^2 \quad e_5^3 \quad e_5^4\} = \{\text{steering} \quad \text{bumpy bends} \quad \text{maneuverability} \quad \text{top speed stability}\} \tag{61}$$

$$E_6 = \{e_6^1 \quad e_6^2 \quad e_6^3\} = \{\text{quality of finish} \quad \text{seat comfort} \quad \text{headlight}\}. \tag{62}$$

The customer realizes that the factors in the same factor set have equal relative importance for evaluation of the corresponding attribute, and that the priority coefficients α_4, α_5, α_6 are all equal to 0.9. This means that the state of the corresponding attribute is only regarded to be confirmed by 90% to the same evaluation grade as that confirmed absolutely by a key factor's state. Thus, the following weights for the factors are obtained:

$$\lambda_4 = [\lambda_4^1 \quad \lambda_4^2 \quad \lambda_4^3]^T = [0.9 \quad 0.9 \quad 0.9]^T \tag{63}$$

$$\lambda_5 = [\lambda_5^1 \quad \lambda_5^2 \quad \lambda_5^3 \quad \lambda_5^4]^T = [0.9 \quad 0.9 \quad 0.9 \quad 0.9]^T \tag{64}$$

$$\lambda_6 = [\lambda_6^1 \quad \lambda_6^2 \quad \lambda_6^3]^T = [0.9 \quad 0.9 \quad 0.9]^T. \tag{65}$$

In [7], the same set of evaluation grades is used for the three qualitative attributes, which includes five distinct evaluation grades and is defined by

$$H = \{H_1 \quad H_2 \quad H_3 \quad H_4 \quad H_5\} = \{\text{poor} \quad \text{indifferent} \quad \text{average} \quad \text{good} \quad \text{excellent}\}. \tag{66}$$

H is transformed into the preference degree space using the following scale:

$$p\{H\} = [p(H_1) \quad p(H_2) \quad p(H_3) \quad p(H_4) \quad p(H_5)]^T = [-1 \quad -0.4 \quad 0 \quad 0.4 \quad 1]^T \tag{67}$$

where $p(H_n)$ ($n = 2, 3, 4$) are assigned by the customer so that they satisfy the basic conditions (4) and the consistence condition (21), and $p(H) = \Sigma_{n=1}^{5} p(H_n)/5 = 0$.

The evaluation analysis model for evaluation of the three qualitative attributes may then be depicted as in Fig. 3.

Each of the preference degrees for quantifying the states of the qualitative attributes at all alternative motorcycles, i.e., $p_{rk} = p(y_k(a_r))$ ($k = 4, 5, 6$: $r = 1, \cdots, 4$), is generated following the same process demonstrated in the last subsection. The basic probability assignments are obtained from the confidence degrees given in Table IX and the relative weights λ_k ($k = 4, 5, 6$) assigned by the customer. The overall probability assignments are generated using the evidence combination algorithms. The results are listed in Tables XI–XXII.

Then, the preference degrees of the three qualitative attributes at the four alternative motorcycle types are calculated using (34). For instance, from (67) and Table XI,

$$\begin{aligned} p_{14} &= b_{H_1}^{C(4)} p(H_1) + b_{H_2}^{C(4)} p(H_2) + b_{H_3}^{C(4)} p(H_3) \\ &\quad + b_{H_4}^{C(4)} p(H_4) + b_{H_5}^{C(4)} p(H_5) + b_{H}^{C(4)} p(H) \\ &= 0.0 \times (-1) + 0.072 \times (-0.4) + 0.87 \times 0 \\ &\quad + 0.0 \times 0.4 + 0.041 \times 1 + 0.017 \times 0 \\ &= 0.012. \end{aligned} \tag{68}$$

The values of the three quantitative attributes at each alternative are normalized using (35) for y_2 and y_3 and using (36) for y_1. Table XXIII shows the obtained evaluation matrix for the motorcycle evaluation problem.

The CODASID method is then used to rank the four motorcycle types, based on Table XXIII. Using formulas (38)–(43), we can obtain the following judgment and evaluation matrix (Table XXIV).

Finally, the relative closeness indexes of the four motorcycle types are generated by formulas (44)–(52), that is,

$$[u(a_1) \quad u(a_2) \quad u(a_3) \quad u(a_4)]^T = [0.281 \quad 0.895 \quad 0.942 \quad 0.205]^T. \tag{69}$$

So the preference order is

$$a_3 \succ a_2 \succ a_1 \succ a_4. \tag{70}$$

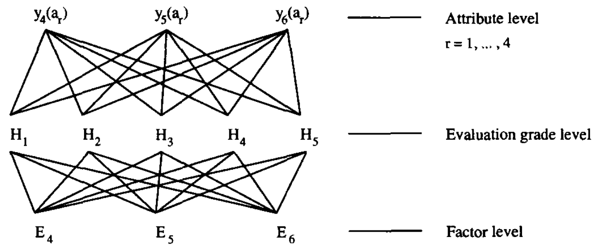

Fig. 3. The evaluation analysis model for the motorcycle evaluation problem.

TABLE XI
PROBABILITY ASSIGNMENTS FOR $y_4(a_1)$

Basic Probability Assignments ($\beta \times \lambda_4$)		Evaluation Grades				
		$P(\beta)$	$I(\beta)$	$A(\beta)$	$G(\beta)$	$E(\beta)$
Factors	e_4^1					0.72
	e_4^2			0.9		
	e_4^3		0.45	0.45		
Overall Probability Assignments $b_{H_n}^{C(4)}$		0.000	0.072	0.870	0.000	0.041

TABLE XII
PROBABILITY ASSIGNMENTS FOR $y_4(a_2)$

Basic Probability Assignments ($\beta \times \lambda_4$)		Evaluation Grades				
		$P(\beta)$	$I(\beta)$	$A(\beta)$	$G(\beta)$	$E(\beta)$
Factors	e_4^1				0.27	0.54
	e_4^2		0.9			
	e_4^3			0.9		
Overall Probability Assignments $b_{H_n}^{C(4)}$		0.000	0.387	0.387	0.061	0.122

TABLE XIII
PROBABILITY ASSIGNMENTS FOR $y_4(a_3)$

Basic Probability Assignments ($\beta \times \lambda_4$)		Evaluation Grades				
		$P(\beta)$	$I(\beta)$	$A(\beta)$	$G(\beta)$	$E(\beta)$
Factors	e_4^1				0.9	
	e_4^2		0.45	0.45		
	e_4^3				0.45	0.27
Overall Probability Assignments $b_{H_n}^{C(4)}$		0.000	0.125	0.125	0.696	0.027

Hence, "Honda" is regarded as the best compromise choice, "Yamaha" is quite competitive, but its *top speed* and *engine* are not as good as those of "Honda," which are supposed to be very important.

It may be noted that the preference order (70) partially depends on the customer's preference, that is, the relative

TABLE XIV
PROBABILITY ASSIGNMENTS FOR $y_4(a_4)$

Basic Probability Assignments ($\beta \times \lambda_4$)		Evaluation Grades				
		$P(\beta)$	$I(\beta)$	$A(\beta)$	$G(\beta)$	$E(\beta)$
Factors	e_4^1		0.9			
	e_4^2					0.9
	e_4^3					0.9
Overall Probability Assignments $b_{H_n}^{C(4)}$		0.000	0.083	0.000	0.000	0.908

TABLE XV
PROBABILITY ASSIGNMENTS FOR $y_5(a_1)$

Basic Probability Assignments ($\beta \times \lambda_5$)		Evaluation Grades				
		$P(\beta)$	$I(\beta)$	$A(\beta)$	$G(\beta)$	$E(\beta)$
Factors	e_5^1					0.81
	e_5^2			0.45	0.45	
	e_5^3			0.9		
	e_5^4					0.9
Overall Probability Assignments $b_{H_n}^{C(4)}$		0.000	0.000	0.486	0.04	0.465

TABLE XVI
PROBABILITY ASSIGNMENTS FOR $y_5(a_2)$

Basic Probability Assignments ($\beta \times \lambda_5$)		Evaluation Grades				
		$P(\beta)$	$I(\beta)$	$A(\beta)$	$G(\beta)$	$E(\beta)$
Factors	e_5^1				0.9	
	e_5^2				0.9	
	e_5^3					0.81
	e_5^4				0.9	
Overall Probability Assignments $b_{H_n}^{C(4)}$		0.000	0.000	0.000	0.995	0.004

TABLE XVII
PROBABILITY ASSIGNMENTS FOR $y_5(a_3)$

Basic Probability Assignments ($\beta \times \lambda_5$)		Evaluation Grades				
		$P(\beta)$	$I(\beta)$	$A(\beta)$	$G(\beta)$	$E(\beta)$
Factors	e_5^1			0.9		
	e_5^2				0.72	0.09
	e_5^3		0.9			
	e_5^4				0.9	
Overall Probability Assignments $b_{H_n}^{C(4)}$		0.000	0.136	0.136	0.707	0.007

TABLE XVIII
PROBABILITY ASSIGNMENTS FOR $y_5(a_4)$

Basic Probability Assignments ($\beta \times \lambda_5$)		Evaluation Grades				
		$P(\beta)$	$I(\beta)$	$A(\beta)$	$G(\beta)$	$E(\beta)$
Factors	e_5^1			0.54		
	e_5^2	0.45	0.45			
	e_5^3	0.9				
	e_5^4				0.54	0.36
Overall Probability Assignments $b_{H_n}^{C(4)}$		0.775	0.065	0.017	0.078	0.052

TABLE XIX
PROBABILITY ASSIGNMENTS FOR $y_6(a_1)$

Basic Probability Assignments ($\beta \times \lambda_6$)		Evaluation Grades				
		$P(\beta)$	$I(\beta)$	$A(\beta)$	$G(\beta)$	$E(\beta)$
Factors	e_6^1	0.45	0.45			
	e_6^2				0.9	
	e_6^3				0.9	
Overall Probability Assignments $b_{H_n}^{C(4)}$		0.041	0.041	0.000	0.908	0.000

TABLE XX
PROBABILITY ASSIGNMENTS FOR $y_6(a_2)$

Basic Probability Assignments ($\beta \times \lambda_6$)		Evaluation Grades				
		$P(\beta)$	$I(\beta)$	$A(\beta)$	$G(\beta)$	$E(\beta)$
Factors	e_6^1				0.9	
	e_6^2				0.45	0.45
	e_6^3			0.9		
Overall Probability Assignments $b_{H_n}^{C(4)}$		0.000	0.000	0.131	0.788	0.066

TABLE XXI
PROBABILITY ASSIGNMENTS FOR $y_6(a_3)$

Basic Probability Assignments ($\beta \times \lambda_6$)		Evaluation Grades				
		$P(\beta)$	$I(\beta)$	$A(\beta)$	$G(\beta)$	$E(\beta)$
Factors	e_6^1					0.9
	e_6^2				0.54	
	e_6^3					0.9
Overall Probability Assignments $b_{H_n}^{C(4)}$		0.000	0.000	0.000	0.012	0.979

weights of the six attributes and those of the factors. If he assigns different weights to the attributes or to the factors, different preference orders may be generated. For instance, if he recognizes that *top speed* (y_3) and *engine* (y_4) are not as important as suggested by (58) and adopts the following devaluated weights for y_3 and y_4,

TABLE XXII
PROBABILITY ASSIGNMENTS FOR $y_6(a_4)$

Basic Probability Assignments ($\beta \times \lambda_6$)		Evaluation Grades				
		$P(\beta)$	$I(\beta)$	$A(\beta)$	$G(\beta)$	$E(\beta)$
Factors	e_6^1				0.45	0.45
	e_6^2					0.9
	e_6^3				0.45	0.45
Overall Probability Assignments $b_{H_n}^{C(4)}$		0.000	0.000	0.000	0.088	0.909

TABLE XXIII
THE EVALUATION MATRIX

	$p(y_1)$	$p(y_2)$	$p(y_3)$	$p(y_4)$	$p(y_5)$	$p(y_6)$
a_1	0.139	−0.353	1.000	0.012	0.481	0.306
a_2	1.000	1.000	0.333	−0.057	0.402	0.381
a_3	0.338	−0.891	1.000	0.255	0.235	0.984
a_4	−1.000	−1.000	−1.000	0.875	−0.718	0.944

TABLE XXIV
THE J-E MATRIX

	$pc(a)$	$ec(a)$	$d(a)$
a_1	0.308	−1.179	−0.964
a_2	0.436	0.566	0.867
a_3	0.769	1.006	0.728
a_4	−1.513	−0.394	−0.631

$$\hat{W} = [9 \quad 5 \quad 5 \quad 5 \quad 7 \quad 4]^T \tag{71}$$

then the four motorcycle types will be ranked as follows:

$$[u(a_1) \quad u(a_2) \quad u(a_3) \quad u(a_4)]^T = [0.306 \quad 0.916 \quad 0.869 \quad 0.144]^T \tag{72}$$

$$a_2 \succ a_3 \succ a_1 \succ a_4. \tag{73}$$

In this case, the cheapest "Yamaha" is ranked to be the best compromise choice.

VI. CONCLUSION

The evidential reasoning approach proposed in this paper provides an alternative way to treat uncertain decision knowledge. The presented decision making procedure composed of this approach and the CODASID method can be used to deal with hybrid multiple-attribute decision making problems with uncertainty. The evidential reasoning framework involved in the approach is suitable for representation and quantification of subjective judgments with uncertainty. The obtained two evidence combination algorithms are computationally useful for combining multiple uncertain subjective judgments. The presented examples have demonstrated the implementation process of the proposed approach, and perhaps its potential to treat uncertainty in hybrid MADM problems through multiple-factor analysis and evidential reasoning.

However, the approach reported in this paper is only at the early stage of its development. More work needs to be done for evolution of the approach into better approaches for dealing with more general problems. For instance, one question may be that only the parallel combination of factors is considered in the approach. In real world problems, however, multiple factors associated with the evaluations of an attribute may constitute a hierarchical structure [35]. In this case, sequential propagation of the evaluations for these factors may occur, which needs to be explored in further research.

In addition to this question, it may be argued that some technical details presented in this paper need more proper justification or more formal definition by using universally accepted rules or laws. For instance, the following questions may be proposed as well. Is the rationality assumption of Section III-A always rational? If not, is it possible to extend the approach on a more general basis so that more general problems could be treated? Are there any common rules to follow for assignment of the priority coefficient α_k defined in (25) as its value is important to conduct a rational transformation of the given confidence degrees and preference weights into the basic probability assignments? Some of these questions have actually been addressed to a large extent in the authors' current work [31], [32]. To answer all of these questions or similar ones, however, more effort should be placed on the work of other researchers such as Keefer *et al.* [13] and Miller *et al.* [18].

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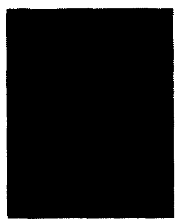

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