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Topics: Perfectly matched layer, Bounded function, Weak formulation, Helmholtz equation and Sobolev space

AN EXACT BOUNDED PERFECTLY MATCHED LAYER FOR TIME-HARMONIC SCATTERING PROBLEMS

A. BERMÚDEZ*, L. HERVELLA-NIETO†, A. PRIETO‡, AND R. RODRÍGUEZ§

Abstract. The aim of this paper is to introduce an "exact" bounded perfectly matched layer (PML) for the scalar Helmholtz equation. This PML is based on using a non integrable absorbing function. "Exactness" must be understood in the sense that this technique allows exact recovering of the solution to time-harmonic scattering problems in unbounded domains. In spite of the singularity of the absorbing function, the coupled fluid/PML problem is well posed when the solution is sought in an adequate weighted Sobolev space. The resulting variational formulation can be numerically dealt with standard finite elements. The high accuracy of this approach is numerically demonstrated as compared with a classical PML technique.

Key words. Perfectly matched layer, time-harmonic scattering, Helmholtz equation

AMS subject classifications. 65N30 65N99 76Q05

1. Introduction. The typical first step for the numerical solution by finite elements or finite differences of any scattering problem in an unbounded domain is to truncate the computational domain, which entails an inherent difficulty: *to choose boundary conditions to replace the Sommerfeld radiation condition at infinity* (see for instance [21]).

There are several techniques to deal with this: boundary element methods, infinite element methods, Dirichlet-to-Neumann methods based on Fourier expansions, or the use of absorbing boundary conditions. The potential advantages of each of them have been widely studied in the literature [25, 35, 18].

If the domain of the original problem is truncated with a sphere, then the Dirichlet-to-Neumann (DtN) boundary condition is exactly known (see [21, 32]). However, this boundary condition involves an infinite series which must be truncated for numerical use, thus introducing errors. Moreover, the exact DtN condition is non local, leading to dense blocks in the linear system to be solved, when a finite element method is used.

As a partial solution to these drawbacks, local absorbing boundary conditions (ABCs) can be introduced to preserve the computational efficiency of the numerical method. Those of Bayliss and Turkel [5], Engquist and Majda [19], and Feng [20] are among the most widely used. However, in spite of the simple implementation of lowest order ABCs, good accuracy is only achieved for higher order ones [37], because these conditions are not fully non-reflecting on the truncated boundary of

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the computational domain. As a consequence, high accuracy using ABCs leads to a substantial computational cost and increases the difficulty of implementation.

An alternative approach to deal with the truncation of unbounded domains is the so called *perfectly matched layer* (PML) method introduced by Berenger [8, 9, 10]. It is based on simulating a layer of absorbing material surrounding the domain of interest, like a thin sponge which absorbs the scattered field radiated to the exterior of the domain. This method is known as "perfectly matched" because the interface between the fluid domain and the absorbing layer does not produce spurious reflections inside the domain of interest, as it is the case with ABCs.

This method has been applied to different problems. It was initially, settled for Maxwell's equations in electromagnetism [8, 7] and subsequently used for the scalar Helmholtz equation [36, 38, 22], advective acoustics [1, 6, 24], shallow water waves [34], elasticity [14, 4], poroelastic media [41], and other hyperbolic problems (see for instance [31] among many other authors). We focus our attention on wave propagation time-harmonic scattering problems in linear acoustics, i.e., on the scalar Helmholtz equation.

In the deduction of the PML [8], Berenger used an artificial splitting to force the tangential components of the velocities in the acoustic medium and in the PML layer to coincide on the interface for any frequency and any angle of incidence, thus guaranteeing absence of spurious reflections [25]. However, this non physical splitting has been shown unnecessary to state the PML equations. In fact, Chew and Weedon showed in [17] that the PML equations can be obtained using a complex-valued coordinate stretching, in the framework of time-harmonic wave propagation. Related to this, Lassas *et al.* [28, 29] showed that the PML, and in general a family of absorbing conditions, can be obtained by using complex Riemannian metric tensors.

Furthermore, in spite of the fact that the PML has been originally settled in Cartesian coordinates, Collino and Monk [12] proposed a similar complex-valued change of coordinates to build a PML on curvilinear coordinates. This is the point of view that we will follow in this paper.

In practice, since the PML has to be truncated at a finite distance of the domain of interest, its external boundary produces artificial reflections. Theoretically, these reflections are of minor importance because of the exponential decay of the acoustic waves inside the PML. In fact, for Helmholtz-type scattering problems, Lassas and Somersalo [27] proved, using boundary integral equation techniques, that the approximate solution obtained by the PML method converges exponentially to the exact solution in the computational domain as the thickness of the layer tends to infinity. This result was generalized by Hohage *et al.* [23] using techniques based on the pole condition. Similarly, Becache *et al.* [6] proved an analogous result for the convected Helmholtz equation.

When the problem is discretized to be numerically solved, the approximation error typically becomes larger. Increasing the thickness of the PML maybe a remedy, but not always available because of computational cost. An alternative is to take larger values for the absorbing function involved in the complex-valued coordinate stretching. However, Collino and Monk [13] showed that this methodology may produce an error growth in the discretized problem. Consequently, an optimization problem arises: given a data set and a mesh, to choose the optimal absorbing function to minimize the error.

In this framework, Asvadurov *et al.* [3] proposed a pure imaginary stretching to optimize the PML error. They recovered exponential error estimates using finite-difference grid optimization. However, to the best of the authors' knowledge, the optimization problem is still open in that there is no optimal criterion to choose the absorbing function independently of data and meshes.

The aim of this paper is to contribute to determining optimal absorbing functions for the PML. In fact, we propose to use a function with unbounded integral on the PML. We show that this choice leads to a theoretically exact bounded PML. More precisely, this kind of absorbing function on a circular annular layer allows recovering the exact solution of the time-harmonic scattering problem in the domain of interest, up to discretization errors, even though the thickness of the layer is finite. We will call "exact" PML methods to those based on such absorbing functions.

Standard PML techniques based on bounded absorbing functions, lead to partial differential equations in the PML with bounded coefficients. Thus, the theoretical procedure to prove the well-posedness of the coupled fluid/PML problems is based on the Freedholm alternative in standard Sobolev spaces. However, when a non integrable absorbing function is used, the coefficients in the PML equation become unbounded, and the natural functional framework involves a non-standard weighted Sobolev space. In this case standard arguments cannot be applied due to the lack of a compactness result. As an alternative, we reproduce the classical steps used for the Helmholtz equation, taking advantage of the series representation of the solution in the PML domain. Thus we prove a result of existence and uniqueness for the coupled fluid/PML problem.

The analysis of the theoretical error for other PML techniques is typically based on the construction of an analogous Dirichlet-to-Neumann operator using the solution in the PML. We also use this approach to prove that the solution in the fluid domain of the coupled fluid/PML problem is *exactly* equal to the solution of the scattering problem in an unbounded domain.

The outline of this paper is as follows: we recall in Section 2 how the classical time-harmonic scattering problem can be stated in a bounded domain by using a DtN operator. Section 3 is devoted to settling PML equations based on non integrable absorbing functions on an annular domain surrounding the physical one. Once the fundamental solution for the PML is calculated in Section 4, a Green's formula is proved in Section 5. We rewrite a classical "addition theorem" for the PML fundamental solution in Section 6. Using these tools, we prove a characterization theorem for the solution of the radial PML in Section 7 and derive a theorem of existence and uniqueness of solution for the PML problem. In Section 8, we use this result to recover the classical solution of the scattering problem by means of a coupled fluid/PML problem. We prove existence and uniqueness of solution for this coupled problem and write a variational formulation, as well. Finally, in Section 9 we report some numerical results obtained with a standard finite element method.

2. Scattering problem. Let Ω be a bounded two-dimensional open set with a Lipschitz boundary Γ. We aim to solve a scattering problem in the unbounded domain $\mathbb{R}^2 \setminus \overline{\Omega}$, which we assume connected (see Figure 2.1). Throughout the paper we use standard notation for Sobolev functional spaces. Consider the following Dirichlet boundary value problem for the Helmholtz partial differential equation, which models the wave propagation with frequency $\omega > 0$ and velocity of propagation $c > 0$:

Find $u \in \mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}^2 \setminus \overline{\Omega})$ *such that*

$$-\Delta u - k^2 u = 0 \qquad \text{in } \mathbb{R}^2 \setminus \overline{\Omega}, \tag{2.1}$$

$$u = f \qquad \text{on } \Gamma, \tag{2.2}$$

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u}{\partial r} - iku\right) = 0, \tag{2.3}$$

where $r := |\boldsymbol{x}|$ is the radial polar coordinate for $\boldsymbol{x} \in \mathbb{R}^2$, $k = \omega/c$ is the wave number, and $f \in \mathrm{H}^{\frac{1}{2}}(\Gamma)$ is the Dirichlet boundary data. Let us remark that we could analogously consider the corresponding Neumann boundary value problem. The existence of solution to both problems is well known in the literature (see for instance [40]).

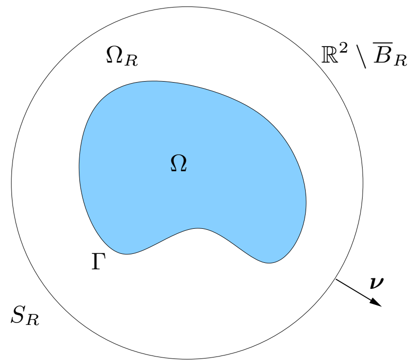

FIG. 2.1. *Scatterer and artificial circular boundary.*

Let $B_R := \left\{\boldsymbol{x} \in \mathbb{R}^2 : \ |\boldsymbol{x}| < R\right\}$ be an open ball of radius R such that $\overline{\Omega} \subset B_R$. Let $S_R := \left\{\boldsymbol{x} \in \mathbb{R}^2 : \ |\boldsymbol{x}| = R\right\}$ be its boundary and $\boldsymbol{\nu}$ its outward unit normal vector (see Figure 2.1). The DtN operator of the problem above is defined as follows:

$$\begin{array}{rccc} G: & \mathrm{H}^{\frac{1}{2}}(S_R) & \longrightarrow & \mathrm{H}^{-\frac{1}{2}}(S_R) \\ & g & \longmapsto & \left.\dfrac{\partial \tilde{u}}{\partial \boldsymbol{\nu}}\right|_{S_R} \end{array} \tag{2.4}$$

where $\tilde{u} \in \mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}^2 \setminus \overline{B}_R)$ is the unique solution of

$$-\Delta \tilde{u} - k^2 \tilde{u} = 0 \qquad \text{in } \mathbb{R}^2 \setminus \overline{B}_R,$$

$$\tilde{u} = g \qquad \text{on } S_R,$$

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial \tilde{u}}{\partial r} - ik\tilde{u}\right) = 0.$$

Let us recall that this operator is explicitly given by the following series (see [32, 35]):

$$Gg = \sum_{n=-\infty}^{\infty} g_n k \frac{[\mathrm{H}_n^{(1)}]'(kR)}{\mathrm{H}_n^{(1)}(kR)} \,\mathrm{e}^{in\theta},$$

where θ is the angular polar coordinate of $\boldsymbol{x}$, $g_n := 1/(2\pi R)\int_{S_R} g(\boldsymbol{x})\,\mathrm{e}^{-in\theta}\,dS$ is the n^{th} Fourier coefficient of g and $\mathrm{H}_n^{(j)}$ denotes the n^{th} Hankel function of j^{th} kind, $j = 1, 2$ (see for instance [39]).

Problem (2.1)–(2.3) can be equivalently settled in the bounded domain $\Omega_R := \{\boldsymbol{x} \in \mathbb{R}^2 \setminus \overline{\Omega} : |\boldsymbol{x}| < R\}$ by means of this DtN operator as follows:

Find $u \in \mathrm{H}^1(\Omega_R)$ *such that*

$$-\Delta u - k^2 u = 0 \qquad \text{in } \Omega_R, \tag{2.5}$$

$$u = f \qquad \text{on } \Gamma, \tag{2.6}$$

$$\frac{\partial u}{\partial \boldsymbol{\nu}} = G\left(u|_{S_R}\right) \qquad \text{on } S_R. \tag{2.7}$$

Clearly, if u is the solution of Problem (2.1)–(2.3), then $u|_{\Omega_R}$ is the unique solution of the problem above.

3. Statement of the PML equation. Radial PML methods are based on simulating dissipation in an annular domain, $\mathrm{D} := \{\boldsymbol{x} \in \mathbb{R}^2 : R < |\boldsymbol{x}| < R^\star\}$, surrounding the physical domain of interest. This can be done by means of a complex-valued radial stretching proposed by Collino and Monk [12], which leads to the following partial differential equation written in polar coordinates:

$$-\frac{1}{r}\left(\frac{\partial}{\partial r}\left(\frac{\hat{\gamma}(r)r}{\gamma(r)}\frac{\partial \hat{u}}{\partial r}\right) + \frac{\gamma(r)}{\hat{\gamma}(r)r}\frac{\partial^2 \hat{u}}{\partial \theta^2}\right) - \gamma(r)\hat{\gamma}(r)k^2\hat{u} = 0 \qquad \text{in } \mathrm{D}, \tag{3.1}$$

where

$$\gamma(r) := 1 + \frac{i}{\omega}\sigma(r) \qquad \text{and} \qquad \hat{\gamma}(r) := 1 + \frac{i}{\omega r}\int_R^r \sigma(s)\,ds, \tag{3.2}$$

with the so-called *absorbing function* $\sigma : [R, R^\star) \to [0, \infty)$ being monotonically increasing and smooth. In all what follows it is enough to consider $\sigma \in \mathcal{C}^{2,1}([R, R^\star))$. Notice that we do not assume that $\sigma(R) = 0$. This function has been typically chosen bounded (see [12]). As an alternative, we propose to choose a non integrable function, i.e., such that

$$\int_R^{R^\star} \sigma(s)\,ds = +\infty; \tag{3.3}$$

for example, $\sigma(r) := c/(R^\star - r)$.

Under the previous assumptions on σ, $\lim_{r\to R^\star} |\gamma(r)| = \lim_{r\to R^\star} |\hat{\gamma}(r)| = +\infty$. Moreover, the coefficients of the differential equation (3.1) satisfy

$$\lim_{r\to R^\star}\left|\frac{\hat{\gamma}(r)r}{\gamma(r)}\right| = 0 \qquad \text{and} \qquad \lim_{r\to R^\star}\left|\frac{\gamma(r)}{\hat{\gamma}(r)r}\right| = +\infty. \tag{3.4}$$

Indeed, given $\epsilon > 0$, let $A_\epsilon := \int_R^{R^\star-\epsilon} \sigma(s)\,ds$. Because of (3.3), $\exists r \in (R^\star - \epsilon, R^\star)$ such that $\int_{R^\star-\epsilon}^r \sigma(s)\,ds \geq A_\epsilon$. Then

$$\int_R^r \sigma(s)\,ds = \int_R^{R^\star-\epsilon} \sigma(s)\,ds + \int_{R^\star-\epsilon}^r \sigma(s)\,ds \leq 2\int_{R^\star-\epsilon}^r \sigma(s)\,ds \leq 2\epsilon\sigma(r).$$

Hence,

$$\lim_{r\to R^\star} \frac{1}{\sigma(r)}\int_R^r \sigma(s)\,ds = 0,$$

which together with the definitions of γ and $\hat{\gamma}$, and the fact that $R \leq r \leq R^\star$ yield (3.4).

To end this section, we write the equation (3.1) in a more compact form. Let $\{\boldsymbol{e}_r, \boldsymbol{e}_\theta\}$ be the canonical polar-coordinates basis. Let us recall the expressions of the divergence and gradient operators in this basis:

$$\mathbf{grad}\, v = \frac{\partial v}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial v}{\partial \theta}\boldsymbol{e}_\theta,$$

and, for $\boldsymbol{w} = w_r \boldsymbol{e}_r + w_\theta \boldsymbol{e}_\theta$,

$$\operatorname{div} \boldsymbol{w} = \frac{1}{r}\left(\frac{\partial}{\partial r}(r w_r) + \frac{\partial w_\theta}{\partial \theta}\right).$$

It is straightforward to show that the PML equation (3.1) can be written as follows:

$$-\operatorname{div}(\boldsymbol{A}\,\mathbf{grad}\,\hat{u}) - \gamma(r)\hat{\gamma}(r)k^2\hat{u} = 0 \qquad \text{in D},$$

where

$$\boldsymbol{A} := \begin{pmatrix} a_{rr} & 0 \\ 0 & a_{\theta\theta} \end{pmatrix}, \qquad \text{with} \qquad a_{rr}(r) := \frac{\hat{\gamma}(r)}{\gamma(r)} \qquad \text{and} \qquad a_{\theta\theta}(r) := \frac{\gamma(r)}{\hat{\gamma}(r)}. \tag{3.5}$$

Let us emphasize that matrix $\boldsymbol{A}$ is written in the polar coordinates basis. Therefore, for the concrete evaluation of any expression involving $\boldsymbol{A}$ (like that above), all the vector fields and the differential operators must be written in the same basis.

4. PML fundamental solution. We consider the following complex change of variable, proposed by Collino and Monk in [12]:

$$\hat{r}(r) := \hat{\gamma}(r)r = r + \frac{i}{\omega}\int_R^r \sigma(s)ds, \qquad r \in [R, R^\star).$$

The assumed smoothness of σ is enough to ensure that $\hat{r} \in \mathcal{C}^2([R, R^\star))$. We also define a complex-valued function $\mathrm{d}(\cdot,\cdot)$ which plays a similar role to the standard Euclidean distance. This new "complex distance" between two points $\boldsymbol{x}$ and $\boldsymbol{y}$ in D, with respective polar coordinates $(r_{\boldsymbol{x}}, \theta_{\boldsymbol{x}})$ and $(r_{\boldsymbol{y}}, \theta_{\boldsymbol{y}})$, is given by

$$\mathrm{d}(\boldsymbol{x}, \boldsymbol{y}) := \sqrt{\hat{r}_{\boldsymbol{x}}^2 + \hat{r}_{\boldsymbol{y}}^2 - 2\hat{r}_{\boldsymbol{x}}\hat{r}_{\boldsymbol{y}}\cos(\theta_{\boldsymbol{y}} - \theta_{\boldsymbol{x}})}, \tag{4.1}$$

where $\hat{r}_{\boldsymbol{x}} = \hat{r}(r_{\boldsymbol{x}})$, $\hat{r}_{\boldsymbol{y}} = \hat{r}(r_{\boldsymbol{y}})$ and the square root is chosen so that $\mathrm{Re}(\sqrt{z}) > 0$ for $z \neq 0$. This makes sense since, for $\boldsymbol{x} \neq \boldsymbol{y}$, straightforward computations show that the radicand either has a positive imaginary part or is a positive real number. Consequently $\mathrm{d}(\boldsymbol{x}, \cdot)$ takes values in $\{z \in \mathbb{C} : \mathrm{Re}(z) > 0, \mathrm{Im}(z) \geq 0\}$.

Remark 4.1. *The usual distance between two points $\boldsymbol{x}$ and $\boldsymbol{y}$ can be written as follows:*

$$|\boldsymbol{x} - \boldsymbol{y}| = \left|r_{\boldsymbol{x}}\mathrm{e}^{i\theta_{\boldsymbol{x}}} - r_{\boldsymbol{y}}\mathrm{e}^{i\theta_{\boldsymbol{y}}}\right| = \sqrt{(r_{\boldsymbol{x}}\mathrm{e}^{i\theta_{\boldsymbol{x}}} - r_{\boldsymbol{y}}\mathrm{e}^{i\theta_{\boldsymbol{y}}})(r_{\boldsymbol{x}}\mathrm{e}^{-i\theta_{\boldsymbol{x}}} - r_{\boldsymbol{y}}\mathrm{e}^{-i\theta_{\boldsymbol{y}}})}.$$

If we substitute in the previous equation the radial coordinates $r_{\boldsymbol{x}}$ and $r_{\boldsymbol{y}}$ by the complex values $\hat{r}(r_{\boldsymbol{x}})$ and $\hat{r}(r_{\boldsymbol{y}})$, respectively, we obtain the complex-valued function

$$\sqrt{(\hat{r}_{\boldsymbol{x}}\mathrm{e}^{i\theta_{\boldsymbol{x}}} - \hat{r}_{\boldsymbol{y}}\mathrm{e}^{i\theta_{\boldsymbol{y}}})(\hat{r}_{\boldsymbol{x}}\mathrm{e}^{-i\theta_{\boldsymbol{x}}} - \hat{r}_{\boldsymbol{y}}\mathrm{e}^{-i\theta_{\boldsymbol{y}}})},$$

which coincides with $\mathrm{d}(\boldsymbol{x},\boldsymbol{y})$ *as defined by (4.1).*

Straightforward computations allow us to show the following result.

LEMMA 4.2. *For all* $\boldsymbol{x},\boldsymbol{y}\in\mathrm{D}$*, there holds:*

i) For fixed $\boldsymbol{x}\in\mathrm{D}$*, there exist three positive constants* C_1*,* C_2 *and* ρ*, which depend on* $\boldsymbol{x}$*, such that,*

$$C_1\,|\boldsymbol{x}-\boldsymbol{y}|\le|\mathrm{d}(\boldsymbol{x},\boldsymbol{y})|\le C_2\,|\boldsymbol{x}-\boldsymbol{y}|\,,$$

for all $\boldsymbol{y}\in\mathrm{D}$ *such that* $|\boldsymbol{y}-\boldsymbol{x}|<\rho$*.*

ii) $\mathrm{d}(\boldsymbol{x},\boldsymbol{y})=0$ *if and only if* $\boldsymbol{x}=\boldsymbol{y}$*.*

For fixed $\boldsymbol{x}\in\mathrm{D}$, $\mathrm{d}(\boldsymbol{x},\cdot)$ is infinitely differentiable with respect to $\theta_{\boldsymbol{y}}$, but the differentiability with respect to $r_{\boldsymbol{y}}$ depends on the regularity of σ. The assumed smoothness of σ is enough to ensure that $\mathrm{d}(\boldsymbol{x},\cdot)\in\mathcal{C}^2(\mathrm{D}\setminus\{\boldsymbol{x}\})$.

The following lemma collects several limits that will be used in the sequel. The corresponding proofs are straightforward. From now on, to simplify the notation, we denote $\gamma_{\boldsymbol{y}}=\gamma(r_{\boldsymbol{y}})$ and $\hat\gamma_{\boldsymbol{y}}=\hat\gamma(r_{\boldsymbol{y}})$. Accordingly, we denote $\boldsymbol{A}_{\boldsymbol{y}}$ instead of $\boldsymbol{A}$ when $\gamma_{\boldsymbol{y}}$ and $\hat\gamma_{\boldsymbol{y}}$ are used in the definition (3.5).

LEMMA 4.3. *For fixed* $\boldsymbol{x}\in\mathrm{D}$*, there holds uniformly in* $\theta_{\boldsymbol{y}}\in(-\pi,\pi]$*:*

$$|\mathrm{d}(\boldsymbol{x},\boldsymbol{y})|=O\left(\mathrm{Im}(\mathrm{d}(\boldsymbol{x},\boldsymbol{y}))\right),\qquad \text{as } r_{\boldsymbol{y}}\to R^\star,$$

$$\lim_{r_{\boldsymbol{y}}\to R^\star}\mathrm{Im}(\mathrm{d}(\boldsymbol{x},\boldsymbol{y}))=+\infty,$$

$$\lim_{r_{\boldsymbol{y}}\to R^\star}\frac{\mathrm{d}(\boldsymbol{x},\boldsymbol{y})}{\hat r_{\boldsymbol{y}}}=1,$$

$$\left|\frac{\partial\,\mathrm{d}(\boldsymbol{x},\boldsymbol{y})}{\partial\theta_{\boldsymbol{y}}}\right|=O(1),\qquad \text{as } r_{\boldsymbol{y}}\to R^\star.$$

Now we are in a position to compute a fundamental solution of the PML equation, i.e., a solution Φ (in the sense of distributions) of

$$-\mathrm{div}_{\boldsymbol{y}}(\boldsymbol{A}_{\boldsymbol{y}}\,\mathbf{grad}_{\boldsymbol{y}}\,\Phi(\boldsymbol{x},\boldsymbol{y}))-\gamma_{\boldsymbol{y}}\hat\gamma_{\boldsymbol{y}}k^2\Phi(\boldsymbol{x},\boldsymbol{y})=\delta_{\boldsymbol{x}}\qquad \text{in } \mathcal{D}'(\mathrm{D}),\tag{4.2}$$

$\delta_{\boldsymbol{x}}$ being the Dirac's delta supported at point $\boldsymbol{x}\in\mathrm{D}$.

THEOREM 4.4. *For fixed* $\boldsymbol{x}\in\mathrm{D}$*,*

$$\Phi^+(\boldsymbol{x},\boldsymbol{y}):=\frac{i}{4}\,\mathrm{H}_0^{(1)}(k\,\mathrm{d}(\boldsymbol{x},\boldsymbol{y})),\tag{4.3}$$

and

$$\Phi^-(\boldsymbol{x},\boldsymbol{y}):=-\frac{i}{4}\,\mathrm{H}_0^{(2)}(k\,\mathrm{d}(\boldsymbol{x},\boldsymbol{y})),\tag{4.4}$$

are solutions of (4.2).

Proof. Let $\boldsymbol{x}\in\mathrm{D}$ be fixed. First, we are going to prove that

$$-\frac{1}{r_{\boldsymbol{y}}}\left(\frac{\partial}{\partial r_{\boldsymbol{y}}}\left(\frac{\hat\gamma_{\boldsymbol{y}}r_{\boldsymbol{y}}}{\gamma_{\boldsymbol{y}}}\frac{\partial\Phi^\pm(\boldsymbol{x},\boldsymbol{y})}{\partial r_{\boldsymbol{y}}}\right)+\frac{\gamma_{\boldsymbol{y}}}{\hat\gamma_{\boldsymbol{y}}r_{\boldsymbol{y}}}\frac{\partial^2\Phi^\pm(\boldsymbol{x},\boldsymbol{y})}{\partial\theta_{\boldsymbol{y}}^2}\right)-\gamma_{\boldsymbol{y}}\hat\gamma_{\boldsymbol{y}}k^2\Phi^\pm(\boldsymbol{x},\boldsymbol{y})=0,\tag{4.5}$$

for all $\boldsymbol{y}\in\mathrm{D}\setminus\{\boldsymbol{x}\}$. Notice that $\mathrm{d}(\boldsymbol{x},\cdot)\in\mathcal{C}^2(\mathrm{D}\setminus\{\boldsymbol{x}\})$ and takes values in the half complex plane $\mathrm{Re}\,z>0$. Hence $\Phi^\pm\in\mathcal{C}^2(\mathrm{D}\setminus\{\boldsymbol{x}\})$, since the Hankel functions are analytical in the complex plane except along the negative real axis.

Since $\mathrm{H}_0^{(1)}(z)$ and $\mathrm{H}_0^{(2)}(z)$ are solutions of the following equation (see [2]):

$$v''(z) + \frac{1}{z}v'(z) + v(z) = 0, \qquad z \neq 0,$$

the change of variable $z = k\,\mathrm{d}(\boldsymbol{x}, \boldsymbol{y})$ allows us to conclude (4.5).

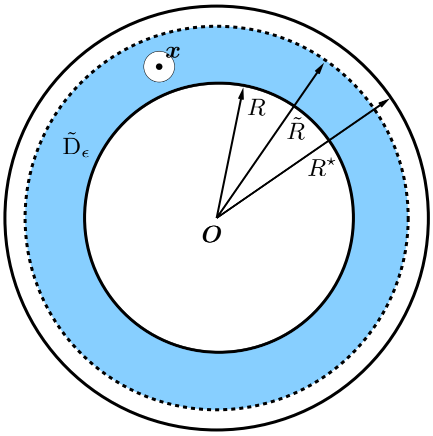

FIG. 4.1. *Domain* $\tilde{\mathrm{D}}_\epsilon$.

Next, let $\varphi \in \mathcal{D}(\mathrm{D})$ and $\tilde{R} \in (R, R^\star)$ be such that supp $\varphi \subset \tilde{\mathrm{D}} := \{\boldsymbol{x} \in \mathrm{D} : |\boldsymbol{x}| < \tilde{R}\}$. Let $\epsilon > 0$ be sufficiently small so that $S(\boldsymbol{x}, \epsilon) := \{\boldsymbol{x} \in \mathbb{R}^2 : |\boldsymbol{x} - \boldsymbol{y}| = \epsilon\} \subset \tilde{\mathrm{D}}$. Finally, let $\tilde{\mathrm{D}}_\epsilon = \{\boldsymbol{y} \in \mathrm{D} : |\boldsymbol{x} - \boldsymbol{y}| > \epsilon, |\boldsymbol{y}| < \tilde{R}\}$ (the blue region in Fig. 4.1) and $\boldsymbol{n}$ its outward unit normal vector. We have

$$\begin{aligned}
&-\langle \mathrm{div}_{\boldsymbol{y}}(\boldsymbol{A}_{\boldsymbol{y}}\,\mathbf{grad}_{\boldsymbol{y}}\,\Phi^\pm(\boldsymbol{x}, \boldsymbol{y})) + \gamma_{\boldsymbol{y}}\hat{\gamma}_{\boldsymbol{y}}k^2\Phi^\pm(\boldsymbol{x}, \boldsymbol{y}), \varphi\rangle \\
&\quad = -\lim_{\epsilon \to 0} \int_{\tilde{\mathrm{D}}_\epsilon} \Phi^\pm(\boldsymbol{x}, \boldsymbol{y})(\mathrm{div}_{\boldsymbol{y}}(\boldsymbol{A}_{\boldsymbol{y}}\,\mathbf{grad}_{\boldsymbol{y}}\,\varphi(\boldsymbol{y})) + \gamma_{\boldsymbol{y}}\hat{\gamma}_{\boldsymbol{y}}k^2\varphi(\boldsymbol{y}))\,d\boldsymbol{y} \\
&\quad = \lim_{\epsilon \to 0} \left(\int_{S(\boldsymbol{x},\epsilon)} \boldsymbol{A}_{\boldsymbol{y}}\,\mathbf{grad}_{\boldsymbol{y}}\,\Phi^\pm(\boldsymbol{x}, \boldsymbol{y}) \cdot \boldsymbol{n}\,\varphi(\boldsymbol{y})\,dS_{\boldsymbol{y}} \right. \\
&\qquad\qquad \left. - \int_{S(\boldsymbol{x},\epsilon)} \boldsymbol{A}_{\boldsymbol{y}}\,\mathbf{grad}_{\boldsymbol{y}}\,\varphi(\boldsymbol{y}) \cdot \boldsymbol{n}\,\Phi^\pm(\boldsymbol{x}, \boldsymbol{y})\,dS_{\boldsymbol{y}} \right) = \varphi(\boldsymbol{x}) = \langle \delta_{\boldsymbol{x}}, \varphi\rangle,
\end{aligned}$$

where we have used (4.5) and Lemma A.3 (which is proved in the appendix). Thus we conclude the theorem. □

We finish this section by studying some decay properties of the fundamental solution Φ^+.

LEMMA 4.5. *For fixed* $\boldsymbol{x} \in \mathrm{D}$, *there holds uniformly in* $\theta_{\boldsymbol{y}} \in (-\pi, \pi]$*:*

$$\lim_{r_{\boldsymbol{y}} \to R^\star} \sqrt{\hat{\gamma}_{\boldsymbol{y}}\gamma_{\boldsymbol{y}}}\,\Phi^+(\boldsymbol{x}, \boldsymbol{y}) = 0,$$

$$\lim_{r_{\boldsymbol{y}} \to R^\star} \sqrt{\frac{\hat{\gamma}_{\boldsymbol{y}}}{\gamma_{\boldsymbol{y}}}}\frac{\partial \Phi^+(\boldsymbol{x}, \boldsymbol{y})}{\partial r_{\boldsymbol{y}}} = 0,$$

$$\lim_{r_{\boldsymbol{y}} \to R^\star} \sqrt{\frac{\gamma_{\boldsymbol{y}}}{\hat{\gamma}_{\boldsymbol{y}}}}\frac{\partial \Phi^+(\boldsymbol{x}, \boldsymbol{y})}{\partial \theta_{\boldsymbol{y}}} = 0.$$

Proof. Let $\boldsymbol{x} \in \mathrm{D}$ fixed. Using the asymptotic classical estimates for Hankel functions (see [39]) and Lemma 4.3, we can check that $\Phi^+(\boldsymbol{x}, \cdot)$ and their derivatives go to zero exponentially and uniformly in all the directions, as $r_{\boldsymbol{y}} \to R^\star$. This allows us to conclude the three limits. ◻

5. PML Green's formula. The aim of this section is to obtain an integral representation of the solutions of the PML equation (3.1). We search for smooth solutions which, furthermore, belong to the functional space

$$\mathrm{V} := \left\{ v \in \mathcal{D}'(\mathrm{D}) : \|v\|_{\mathrm{V}}^2 := \int_R^{R^\star} \int_{-\pi}^{\pi} \left| \frac{\hat{\gamma}(r) r}{\gamma(r)} \right| \left| \frac{\partial v}{\partial r} \right|^2 d\theta\, dr \right.$$
$$\left. + \int_R^{R^\star} \int_{-\pi}^{\pi} \left| \frac{\gamma(r)}{\hat{\gamma}(r) r} \right| \left| \frac{\partial v}{\partial \theta} \right|^2 d\theta\, dr + \int_R^{R^\star} \int_{-\pi}^{\pi} |\hat{\gamma}(r)\gamma(r) r| \, |v|^2 \, d\theta\, dr < +\infty \right\}.$$

As a first step, we restrict our analysis to solutions of (3.1) in the space

$$\mathrm{W} := V \cap \mathcal{C}^1(\mathrm{D}^\star) \cap \mathcal{C}^2(D),$$

where

$$\mathrm{D}^\star := \mathrm{D} \cup S_R = \{\boldsymbol{x} \in \mathbb{R}^2 : R \le |\boldsymbol{x}| < R^\star\}.$$

Since the weights involved in the definition of V belong to $\mathrm{L}^1_{\mathrm{loc}}(R, R^\star)$ and are positive, V is a Banach space when endowed with the norm $\|\cdot\|_{\mathrm{V}}$ (see Kufner & Sändig [26]) and, moreover, $\mathrm{V} \subset \mathrm{H}^1_{\mathrm{loc}}(\mathrm{D}^\star)$ and so $v \in \mathrm{H}^1(K)$ even for compact sets K intersecting S_R.

First, we prove two preliminary results.

Lemma 5.1. *If $v \in \mathrm{V}$, then*

$$\lim_{\tilde{R} \to R^\star} \int_{S_{\tilde{R}}} |\hat{\gamma}| \, |v|^2 \, dS = 0.$$

Proof. For $v \in \mathrm{V}$, we define the complex-valued function F given by

$$F(r) := \int_{S_r} \hat{\gamma} \, |v|^2 \, dS = \int_{-\pi}^{\pi} r \hat{\gamma}(r) \, |v(r, \theta)|^2 \, d\theta.$$

From the definition of V, it is immediate to check that F and γF belong to $\mathrm{L}^1(R, R^\star)$. We also define

$$\begin{aligned} G(r) : &= \int_{-\pi}^{\pi} \frac{\partial}{\partial r} \left(r \hat{\gamma}(r) \, |v(r, \theta)|^2 \right) d\theta \\ &= \gamma(r) \int_{-\pi}^{\pi} |v(r, \theta)|^2 \, d\theta + 2 r \hat{\gamma}(r) \int_{-\pi}^{\pi} \mathrm{Im} \left(\frac{\partial v}{\partial r}(r, \theta) \bar{v}(r, \theta) \right) d\theta. \end{aligned}$$

For $v \in \mathrm{V}$, the first term in the above sum is integrable in $(R, R^\star)$. Regarding the second term, we have

$$\begin{aligned} &\int_R^{R^\star} \left| 2 r \hat{\gamma}(r) \int_{-\pi}^{\pi} \mathrm{Im} \left(\frac{\partial v}{\partial r}(r, \theta) \bar{v}(r, \theta) \right) d\theta \right| dr \\ &\quad \le 2 \left(\int_R^{R^\star} \int_{-\pi}^{\pi} \left| \frac{\hat{\gamma}(r) r}{\gamma(r)} \right| \left| \frac{\partial v}{\partial r} \right|^2 d\theta\, dr \right)^{\frac{1}{2}} \left(\int_R^{R^\star} \int_{-\pi}^{\pi} |\hat{\gamma}(r)\gamma(r) r| \, |v|^2 \, d\theta\, dr \right)^{\frac{1}{2}}, \end{aligned}$$

which is again finite for $v \in \mathrm{V}$. Thus $G \in \mathrm{L}^1(R, R^\star)$. Moreover, straightforward computations show that G is the distributional derivative of F. Hence $F \in \mathrm{W}^{1,1}(R, R^\star)$ and, consequently, $F \in \mathcal{C}([R, R^\star])$.

Now we can conclude the lemma by showing that $\lim_{r\to R^\star} F(r) = 0$. We proceed by contradiction. Suppose $\lim_{r\to R^\star} F(r) \neq 0$; in such a case, since $|\gamma| > 1$ and γ is not integrable near $r = R^\star$, $\int_R^{R^\star} |\gamma F|\, dr = \infty$, which would contradict the fact that $\gamma F \in \mathrm{L}^1(R, R^\star)$. ☐

LEMMA 5.2. *If* $\hat{u} \in \mathrm{W}$ *is a solution of (3.1) and* $v \in \mathrm{V}$*, then*

$$\lim_{\tilde{R}\to R^\star} \int_{S_{\tilde{R}}} \frac{\hat{\gamma}}{\gamma} \frac{\partial \hat{u}}{\partial r} v\, dS = 0.$$

Proof. Let $\tilde{R} \in (R, R^\star)$. Since $v \in \mathrm{V} \subset \mathrm{H}^1_{\mathrm{loc}}(\mathrm{D}^\star)$, if we multiply (3.1) by $v \in \mathrm{V}$ and integrate by parts in $\tilde{\mathrm{D}} := \{\boldsymbol{x} \in \mathbb{R}^2 : R < |\boldsymbol{x}| < \tilde{R}\}$, we obtain

(5.1)
$$\begin{aligned}\int_{S_{\tilde{R}}} \frac{\hat{\gamma}}{\gamma} \frac{\partial \hat{u}}{\partial r} v\, dS &= \int_{\tilde{\mathrm{D}}} \frac{\hat{\gamma}}{\gamma} \frac{\partial \hat{u}}{\partial r} \frac{\partial v}{\partial r} + \int_{\tilde{\mathrm{D}}} \frac{\gamma}{\hat{\gamma}} \frac{\partial \hat{u}}{\partial \theta} \frac{\partial v}{\partial \theta} - k^2 \int_{\tilde{\mathrm{D}}} \gamma\hat{\gamma}\hat{u}v + \int_{S_R} \frac{1}{\gamma} \frac{\partial \hat{u}}{\partial r} v\, dS \\ &= \int_R^{\tilde{R}} \left[\int_{S_r} \left(\frac{\hat{\gamma}}{\gamma} \frac{\partial \hat{u}}{\partial r} \frac{\partial v}{\partial r} + \frac{\gamma}{\hat{\gamma}} \frac{\partial \hat{u}}{\partial \theta} \frac{\partial v}{\partial \theta} - k^2 \gamma\hat{\gamma}\hat{u}v \right) dS \right] dr + \int_{S_R} \frac{1}{\gamma} \frac{\partial \hat{u}}{\partial r} v\, dS.\end{aligned}$$

Because of the definition of V, the expression between brackets above belongs to $\mathrm{L}^1(R, R^\star)$. Consequently, if we define

$$H(\tilde{R}) := \int_{S_{\tilde{R}}} \frac{\hat{\gamma}}{\gamma} \frac{\partial \hat{u}}{\partial r} v\, dS,$$

then, according to (5.1), $H \in \mathcal{C}([R, R^\star])$. On the other hand, from the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\int_R^{R^\star} |\gamma(r) H(r)|\, dr &= \int_R^{R^\star} \left| \int_{-\pi}^{\pi} \hat{\gamma}(r) r \frac{\partial \hat{u}}{\partial r} v\, d\theta \right| dr \\ &\leq \left(\int_R^{R^\star} \int_{-\pi}^{\pi} \left| \frac{\hat{\gamma}(r) r}{\gamma(r)} \right| \left| \frac{\partial \hat{u}}{\partial r} \right|^2 d\theta\, dr \right)^{\frac{1}{2}} \left(\int_R^{R^\star} \int_{-\pi}^{\pi} |\hat{\gamma}(r)\gamma(r) r|\, |v|^2\, d\theta\, dr \right)^{\frac{1}{2}}.\end{aligned}$$

which is finite for $\hat{u}$, $v \in \mathrm{V}$. Thus, $\gamma H \in \mathrm{L}^1(R, R^\star)$ and, hence, the same argument used to prove the previous lemma, allow us to conclude that $\lim_{r\to R^\star} H(r) = 0$. ☐

Now, the following step is to establish an integral representation of the solution of the PML equation.

THEOREM 5.3. *If* $\hat{u} \in \mathrm{W}$ *is a solution of (3.1), then the following Green's formula holds:*

$$\hat{u}(\boldsymbol{x}) = \frac{1}{\gamma(R)} \int_{S_R} \left(\frac{\partial \Phi^+(\boldsymbol{x}, \boldsymbol{y})}{\partial r_{\boldsymbol{y}}} \hat{u}(\boldsymbol{y}) - \frac{\partial \hat{u}}{\partial r_{\boldsymbol{y}}}(\boldsymbol{y}) \Phi^+(\boldsymbol{x}, \boldsymbol{y}) \right) dS_{\boldsymbol{y}}, \quad \boldsymbol{x} \in \mathrm{D}. \tag{5.2}$$

Proof. We fix an arbitrary $\boldsymbol{x} \in \mathrm{D}$ and use the notation from Fig. 4.1. As shown in the proof of Theorem 4.4, $\Phi^+(\boldsymbol{x}, \cdot)$ satisfies (4.5). Hence, since $\boldsymbol{A}_{\boldsymbol{y}}$ is diagonal, by using the Green's second theorem and (3.1), we have

$$\int_{\partial\tilde{\mathrm{D}}_\epsilon} \left(\boldsymbol{A_y}\,\mathbf{grad}_{\boldsymbol{y}}\,\hat{u}(\boldsymbol{y})\cdot\boldsymbol{n}\,\Phi^+(\boldsymbol{x},\boldsymbol{y}) - \boldsymbol{A_y}\,\mathbf{grad}_{\boldsymbol{y}}\,\Phi^+(\boldsymbol{x},\boldsymbol{y})\cdot\boldsymbol{n}\,\hat{u}(\boldsymbol{y})\right)\,dS_{\boldsymbol{y}}$$
$$= \int_{\tilde{\mathrm{D}}_\epsilon} \left(\mathrm{div}(\boldsymbol{A_y}\,\mathbf{grad}_{\boldsymbol{y}}\,\hat{u}(\boldsymbol{y}))\Phi^+(\boldsymbol{x},\boldsymbol{y}) - \mathrm{div}(\boldsymbol{A_y}\,\mathbf{grad}_{\boldsymbol{y}}\,\Phi^+(\boldsymbol{x},\boldsymbol{y}))\hat{u}(\boldsymbol{y})\right)\,d\boldsymbol{y} = 0, \tag{5.3}$$

where $\boldsymbol{n}$ is the outward unit normal vector to $\tilde{\mathrm{D}}_\epsilon$.

By using Lemma A.3 and (5.3), we obtain

$$\hat{u}(\boldsymbol{x}) = \lim_{\epsilon\to 0}\left(\int_{S(\boldsymbol{x},\epsilon)} \boldsymbol{A_y}\,\mathbf{grad}_{\boldsymbol{y}}\,\Phi^+(\boldsymbol{x},\boldsymbol{y})\cdot\boldsymbol{n}\,\hat{u}(\boldsymbol{y})\,dS_{\boldsymbol{y}} - \int_{S(\boldsymbol{x},\epsilon)} \boldsymbol{A_y}\,\mathbf{grad}_{\boldsymbol{y}}\,\hat{u}(\boldsymbol{y})\cdot\boldsymbol{n}\,\Phi^+(\boldsymbol{x},\boldsymbol{y})\,dS_{\boldsymbol{y}}\right)$$
$$= \frac{1}{\gamma(R)}\int_{S_R}\left(\frac{\partial\Phi^+(\boldsymbol{x},\boldsymbol{y})}{\partial r_{\boldsymbol{y}}}\,\hat{u}(\boldsymbol{y}) - \frac{\partial\hat{u}}{\partial r_{\boldsymbol{y}}}(\boldsymbol{y})\,\Phi^+(\boldsymbol{x},\boldsymbol{y})\right)\,dS_{\boldsymbol{y}}$$
$$- \int_{S_{\tilde{R}}} \frac{\hat{\gamma}_{\boldsymbol{y}}}{\gamma_{\boldsymbol{y}}}\frac{\partial\Phi^+(\boldsymbol{x},\boldsymbol{y})}{\partial r_{\boldsymbol{y}}}\,\hat{u}(\boldsymbol{y})\,dS_{\boldsymbol{y}} + \int_{S_{\tilde{R}}} \frac{\hat{\gamma}_{\boldsymbol{y}}}{\gamma_{\boldsymbol{y}}}\frac{\partial\hat{u}}{\partial r_{\boldsymbol{y}}}(\boldsymbol{y})\,\Phi^+(\boldsymbol{x},\boldsymbol{y})\,dS_{\boldsymbol{y}}.$$

To conclude the proof, it is enough to show that the last two integrals goes to zero as $\tilde{R}\to R^\star$. For the first one we use Lemmas 4.5 and 5.1. For the second one, first we replace $\Phi^+(\boldsymbol{x},\boldsymbol{y})$ by $\zeta(\boldsymbol{y})\Phi^+(\boldsymbol{x},\boldsymbol{y})$ where ζ is a smooth cutoff in a neighborhood of $\boldsymbol{x}$ which takes the value 1 in a neighborhood of $S_{R^\star}$ including $S_{\tilde{R}}$. Thus, the value of the integral does not change and $\zeta(\boldsymbol{y})\Phi^+(\boldsymbol{x},\boldsymbol{y})\in\mathrm{V}$, because of Lemma 4.5. Hence, the integral goes to zero as $\tilde{R}\to R^\star$ as a consequence of Lemma 5.2. ∎

6. Addition theorem. To characterize the solution of the PML equation, it is useful to write the fundamental solution as a series involving Bessel functions of first kind and order n, which we denote as usual by J_n.

THEOREM 6.1. *Let $\boldsymbol{x}\in\mathrm{D}$ be fixed. For all $\boldsymbol{y}\in\mathrm{D}$ such that $|\boldsymbol{y}|<|\boldsymbol{x}|$, there holds:*

$$\Phi^+(\boldsymbol{x},\boldsymbol{y}) = \frac{i}{4}\sum_{n=-\infty}^{\infty}\mathrm{H}_n^{(1)}\left(k\hat{r}_{\boldsymbol{x}}\right)\mathrm{J}_n\left(k\hat{r}_{\boldsymbol{y}}\right)\mathrm{e}^{in(\theta_{\boldsymbol{x}}-\theta_{\boldsymbol{y}})}. \tag{6.1}$$

This series and its term by term first derivatives with respect to $r_{\boldsymbol{y}}$ are absolutely and uniformly convergent on compact subsets of the set $\{\boldsymbol{y}\in\mathbb{R}^2 : R\le|\boldsymbol{y}|<|\boldsymbol{x}|\}$.

Proof. Let $\tilde{R}\in[R,|\boldsymbol{x}|)$. First, we define the following functions:

$$\phi(\boldsymbol{y}) := \begin{cases} \frac{i}{4}\,\mathrm{H}_0^{(1)}(k\,|\boldsymbol{x}-\boldsymbol{y}|), & \text{if } 0\le|\boldsymbol{y}|<R,\\ \Phi^+(\boldsymbol{x},\boldsymbol{y}), & \text{if } R\le|\boldsymbol{y}|<R^\star,\end{cases}$$

and, for each $n\in\mathbb{N}$,

$$u_n(\boldsymbol{y}) := \begin{cases} \mathrm{J}_n\left(kr_{\boldsymbol{y}}\right)\mathrm{e}^{in\theta_{\boldsymbol{y}}}, & \text{if } 0\le|\boldsymbol{y}|<R,\\ \mathrm{J}_n\left(k\hat{r}_{\boldsymbol{y}}\right)\mathrm{e}^{in\theta_{\boldsymbol{y}}}, & \text{if } R\le|\boldsymbol{y}|\le\tilde{R}.\end{cases}$$

All these functions are continuous for $|\boldsymbol{y}|<\tilde{R}$, analytic for $|\boldsymbol{y}|<R$ and $\mathcal{C}^2$ for $R\le|\boldsymbol{y}|\le\tilde{R}$. Moreover, they satisfy

$$\lim_{r\to R^+}\frac{1}{\gamma}\frac{\partial\phi}{\partial r} = \lim_{r\to R^-}\frac{\partial\phi}{\partial r} \qquad\text{and}\qquad \lim_{r\to R^+}\frac{1}{\gamma}\frac{\partial u_n}{\partial r} = \lim_{r\to R^-}\frac{\partial u_n}{\partial r},\ n\in\mathbb{N}.$$

Furthermore, straightforward computations allow us to show that all of them are solutions of the PML equation (3.1) in $R \leq |\boldsymbol{y}| < \tilde{R}$ and solutions of the Helmholtz equation in $0 < |\boldsymbol{y}| < R$.

Next, we proceed as in the proof of the addition theorem for the Helmholtz equation (see [15]), taking care that the Helmholtz equation is substituted by the PML equation for $R \leq |\boldsymbol{y}| < \tilde{R}$. Thus we obtain

$$\int_{S_{\tilde{R}}} \frac{\hat{\gamma}}{\gamma} \left(u_n \frac{\partial \phi}{\partial r} - \frac{\partial u_n}{\partial r} \phi \right) dS = 0. \tag{6.2}$$

On the other hand, straightforward computations allow us to show that, for all $n \in \mathbb{N}$, if we define

$$v_n(\boldsymbol{y}) := \mathrm{H}_n^{(1)}\left(k\hat{r}_{\boldsymbol{y}}\right) \mathrm{e}^{in\theta_{\boldsymbol{y}}}, \qquad \boldsymbol{y} \in \mathrm{D}, \tag{6.3}$$

then v_n are solutions of the PML equation (3.1) and belong to W. By applying the analogous of Theorem 5.3 for v_n instead of $\hat{u}$, on the annular domain $\tilde{R} < |\boldsymbol{y}| < R^\star$ instead of D, and taking into account that $\phi = \Phi^+(\boldsymbol{x}, \cdot)$ in this domain, we obtain

$$\int_{S_{\tilde{R}}} \frac{\hat{\gamma}}{\gamma} \left(v_n \frac{\partial \phi}{\partial r} - \frac{\partial v_n}{\partial r} \phi \right) dS = v_n(\boldsymbol{x}). \tag{6.4}$$

Now, multiplying equation (6.2) by $\mathrm{H}_n^{(1)}(k\hat{r}(\tilde{R}))$ and (6.4) by $\mathrm{J}_n(k\hat{r}(\tilde{R}))$, we have:

$$\int_{S_{\tilde{R}}} \frac{\hat{\gamma}}{\gamma} \left(\mathrm{H}_n^{(1)}(k\hat{r}(\tilde{R})) u_n \frac{\partial \phi}{\partial r} - \mathrm{H}_n^{(1)}(k\hat{r}(\tilde{R})) \frac{\partial u_n}{\partial r} \phi \right) dS = 0,$$
$$\int_{S_{\tilde{R}}} \frac{\hat{\gamma}}{\gamma} \left(\mathrm{J}_n(k\hat{r}(\tilde{R})) v_n \frac{\partial \phi}{\partial r} - \mathrm{J}_n(k\hat{r}(\tilde{R})) \frac{\partial v_n}{\partial r} \phi \right) dS = \mathrm{J}_n(k\hat{r}(\tilde{R})) v_n(\boldsymbol{x}).$$

If we subtract the first from the second equation, taking into account that $\mathrm{H}_n^{(1)}(k\hat{r}(\tilde{R}))u_n = \mathrm{J}_n(k\hat{r}(\tilde{R}))v_n$ on $S_{\tilde{R}}$, we have

$$\begin{aligned} \mathrm{J}_n(k\hat{r}(\tilde{R}))v_n(\boldsymbol{x}) &= \Big(\mathrm{H}_n^{(1)}(k\hat{r}(\tilde{R}))k\gamma(\tilde{R})\,\mathrm{J}'{}_n(k\hat{r}(\tilde{R})) \\ &\qquad - \mathrm{J}_n(k\hat{r}(\tilde{R}))k\gamma(\tilde{R})[\mathrm{H}_n^{(1)}]'(k\hat{r}(\tilde{R})) \Big) \frac{\hat{\gamma}(\tilde{R})}{\gamma(\tilde{R})} \int_{-\pi}^{\pi} \phi\, \mathrm{e}^{in\theta}\, d\theta \\ &= -\frac{2i}{\pi} \int_{-\pi}^{\pi} \phi\, \mathrm{e}^{in\theta}\, d\theta, \end{aligned} \tag{6.5}$$

where we have used the explicit value of the Wronskian $\mathrm{H}_n^{(1)}(z)\,\mathrm{J}'{}_n(z) - \mathrm{J}_n(z)[\mathrm{H}_n^{(1)}]'(z) = -2i/(\pi z)$ (see [2]).

Since $\phi \in \mathcal{C}(\mathrm{D}^\star)$, $\phi|_{S_{\tilde{R}}}$ admits a Fourier series, i.e.,

$$\phi(\boldsymbol{y}) = \sum_{n=-\infty}^{\infty} \phi_n \mathrm{e}^{-in\theta_{\boldsymbol{y}}}, \qquad \boldsymbol{y} \in S_{\tilde{R}}, \tag{6.6}$$

where, from (6.5) and (6.3),

$$\phi_n := \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi\, \mathrm{e}^{in\theta}\, d\theta = \frac{i}{4} \mathrm{H}_n^{(1)}\left(k\hat{r}_{\boldsymbol{x}}\right) \mathrm{J}_n(k\hat{r}(\tilde{R}))\mathrm{e}^{in\theta_{\boldsymbol{x}}}. \tag{6.7}$$

Finally, we conclude (6.1) from (6.6) and (6.7), since $\hat{r}(\tilde{R}) = \hat{r}_{\boldsymbol{y}}$, for $\boldsymbol{y} \in S_{\tilde{R}}$.

The uniform convergence of the series (6.1) and its term by term first derivatives on compact subsets of $\{\boldsymbol{y} \in \mathbb{R}^2 : R \leq |\boldsymbol{y}| < |\boldsymbol{x}|\}$ is straightforward from the uniform convergence of the analogous series in the addition theorem for the fundamental solution of the Helmholtz equation (see [15]), and the fact that $|\hat{r}(r)|$ is a monotonically increasing function. □

7. Existence and uniqueness of solutions for the PML equation. Now we are able to characterize the smooth solutions of the PML equation (3.1):

THEOREM 7.1. *If* $\hat{u} \in \mathrm{W}$ *is a solution of (3.1), then there exists a sequence* $\{a_n\}$ *such that, for all* $\boldsymbol{x} \in \mathrm{D}$,

$$\hat{u}(\boldsymbol{x}) = \sum_{n=-\infty}^{\infty} a_n \, \mathrm{H}_n^{(1)}(k\hat{r}_{\boldsymbol{x}}) \, \mathrm{e}^{in\theta_{\boldsymbol{x}}}.$$

Proof. Let $u \in \mathrm{W}$ be a solution of (3.1). For fixed $\boldsymbol{x} \in \mathrm{D}$, if we apply the Green's formula (5.2) and Theorem 6.1, then we have

$$\begin{aligned}
\hat{u}(\boldsymbol{x}) &= \frac{1}{\gamma(R)} \int_{S_R} \left(\frac{\partial \Phi^+(\boldsymbol{x},\boldsymbol{y})}{\partial r_{\boldsymbol{y}}} \, \hat{u}(\boldsymbol{y}) - \frac{\partial \hat{u}}{\partial r_{\boldsymbol{y}}}(\boldsymbol{y}) \, \Phi^+(\boldsymbol{x},\boldsymbol{y}) \right) dS_{\boldsymbol{y}} \\
&= \frac{1}{\gamma(R)} \int_{S_R} \left(\frac{i}{4} \sum_{n=-\infty}^{\infty} k\gamma(R) \, \mathrm{H}_n^{(1)}(k\hat{r}_{\boldsymbol{x}}) \, \mathrm{J}_n{}'(kR) \mathrm{e}^{in(\theta_{\boldsymbol{x}}-\theta_{\boldsymbol{y}})} \, \hat{u}(\boldsymbol{y}) \right. \\
&\qquad \left. - \frac{\partial \hat{u}}{\partial r_{\boldsymbol{y}}}(\boldsymbol{y}) \, \frac{i}{4} \sum_{n=-\infty}^{\infty} \mathrm{H}_n^{(1)}(k\hat{r}_{\boldsymbol{x}}) \, \mathrm{J}_n(kR) \mathrm{e}^{in(\theta_{\boldsymbol{x}}-\theta_{\boldsymbol{y}})} \right) dS_{\boldsymbol{y}} \\
&= \sum_{n=-\infty}^{\infty} a_n \, \mathrm{H}_n^{(1)}(k\hat{r}_{\boldsymbol{x}}) \, \mathrm{e}^{in\theta_{\boldsymbol{x}}},
\end{aligned}$$

where

$$a_n = \frac{i}{4} \frac{1}{\gamma(R)} \int_{S_R} \left(k\gamma(R) \, \mathrm{J}_n{}'(kR) \mathrm{e}^{-in\theta_{\boldsymbol{y}}} \, \hat{u}(\boldsymbol{y}) - \frac{\partial \hat{u}}{\partial r_{\boldsymbol{y}}}(\boldsymbol{y}) \, \mathrm{J}_n(kR) \mathrm{e}^{-in\theta_{\boldsymbol{y}}} \right) dS_{\boldsymbol{y}}. \qquad \square$$

Now, we prove the existence and uniqueness of smooth solutions of the following problem for the PML equation with Dirichlet data g:

Find $\hat{u} \in \mathrm{W}$ *such that*

$$-\operatorname{div}(\boldsymbol{A} \, \mathbf{grad} \, \hat{u}) - \gamma \hat{\gamma} k^2 \hat{u} = 0 \qquad \text{in } \mathrm{D}, \tag{7.1}$$

$$\hat{u} = g \qquad \text{on } S_R. \tag{7.2}$$

THEOREM 7.2. *If* $g \in \mathrm{H}^s(S_R)$ *with* $s > 3/2$, *then there exists a unique solution* $\hat{u} \in \mathrm{W}$ *of (7.1)-(7.2). Moreover, this solution is given by*

$$\hat{u}(\boldsymbol{x}) = \sum_{n=-\infty}^{\infty} \frac{g_n}{\mathrm{H}_n^{(1)}(kR)} \, \mathrm{H}_n^{(1)}(k\hat{r}_{\boldsymbol{x}}) \, \mathrm{e}^{in\theta_{\boldsymbol{x}}}, \tag{7.3}$$

where g_n *are the Fourier coefficients of* g: $g_n = 1/(2\pi R) \int_{S_R} g(\boldsymbol{x}) \mathrm{e}^{-in\theta_{\boldsymbol{x}}} \, dS$. *Moreover the series and its term by term first derivatives converge uniformly on compact subsets of* $\mathrm{D}^\star$.

Proof. First, we are going to prove that $\hat{u}$ as defined by (7.3) belongs to W. We split the proof into three steps. The first one consists in proving that $\hat{u} \in \mathcal{C}^2(\mathrm{D})$. This step is essentially identical to what is known for the Helmholtz problem. Thus, taking into account classical estimates of the Hankel functions of first kind for large order (see [39]), for $\boldsymbol{x}$ in any compact subset of D and N large enough,

$$\left(\sum_{|n|\geq N}\left|\frac{g_n}{\mathrm{H}_n^{(1)}(kR)}\,\mathrm{H}_n^{(1)}\left(k\hat{r}_{\boldsymbol{x}}\right)\mathrm{e}^{in\theta_{\boldsymbol{x}}}\right|\right)^2 \leq \sum_{|n|\geq N}\left|\frac{\mathrm{H}_n^{(1)}\left(k\hat{r}_{\boldsymbol{x}}\right)}{\mathrm{H}_n^{(1)}(kR)}\right|^2 \sum_{|n|\geq N}|g_n|^2$$
$$\leq C\,\|g\|^2_{\mathrm{L}^2(S_R)}\sum_{|n|\geq N}\left(\frac{R}{|\hat{r}_{\boldsymbol{x}}|}\right)^{2|n|},$$

where we recall that $|\hat{r}_{\boldsymbol{x}}| > R$ (here and thereafter C denotes a generic constant, not necessarily the same at each occurrence). From this, we conclude the uniform and absolute convergence of the series (7.3) on compact subsets of D. Analogous procedures allow us to prove the uniform convergence of the corresponding series for the first and the second derivatives. Since each term in each series is continuous, we conclude that $\hat{u} \in \mathcal{C}^2(\mathrm{D})$.

The second step consists in proving that $\hat{u} \in \mathcal{C}(\mathrm{D}^\star)$. Since $g \in \mathrm{H}^s(S_R)$ with $s > 1/2$, for $\boldsymbol{x}$ in any compact subset of $\mathrm{D}^\star$, we have for N large enough

$$\left(\sum_{|n|\geq N}\left|\frac{g_n}{\mathrm{H}_n^{(1)}(kR)}\,\mathrm{H}_n^{(1)}\left(k\hat{r}_{\boldsymbol{x}}\right)\mathrm{e}^{in\theta_{\boldsymbol{x}}}\right|\right)^2 \leq \sum_{|n|\geq N}\left|\frac{1}{n^s}\frac{\mathrm{H}_n^{(1)}\left(k\hat{r}_{\boldsymbol{x}}\right)}{\mathrm{H}_n^{(1)}(kR)}\right|^2 \sum_{|n|\geq N} n^{2s}\,|g_n|^2$$
$$\leq C\sum_{|n|\geq N}\left|\frac{1}{n^s}\right|^2\sum_{|n|\geq N} n^{2s}\,|g_n|^2 \leq C\,\|g\|^2_{\mathrm{H}^s(S_R)}\sum_{|n|\geq N}\frac{1}{n^{2s}},$$

the latter because of the decay behavior of the Fourier coefficients of functions in $\mathrm{H}^s(S_R)$ (see [30]). This allows us to conclude that $\hat{u} \in \mathcal{C}(\mathrm{D}^\star)$.

The same arguments as above applied to the term by term derivatives of the series allow us to show that, for $g \in \mathrm{H}^s(S_R)$ with $s > 3/2$, $\partial\hat{u}/\partial r$ and $\partial\hat{u}/\partial\theta$ belong to $\mathcal{C}(\mathrm{D}^\star)$, too.

From the previous steps, clearly $\hat{u} \in \mathrm{H}^1_{\mathrm{loc}}(\mathrm{D}^\star)$. Hence, since the weights in the norm of V are positive bounded functions in compact subsets of $\mathrm{D}^\star$, in order to prove that $\hat{u} \in \mathrm{V}$ we only need to prove that there exists $\delta > 0$ such that

$$\int_{R^\star-\delta}^{R^\star}\int_{-\pi}^{\pi}\left(\left|\frac{\hat{\gamma}(r)r}{\gamma(r)}\right|\left|\frac{\partial\hat{u}}{\partial r}\right|^2 + \left|\frac{\gamma(r)}{\hat{\gamma}(r)r}\right|\left|\frac{\partial\hat{u}}{\partial\theta}\right|^2 + |\hat{\gamma}(r)\gamma(r)r|\,|\hat{u}|^2\right)d\theta\,dr < +\infty. \tag{7.4}$$

Since $\mathrm{Im}(\hat{r}_{\boldsymbol{x}}) \to +\infty$, as $r_{\boldsymbol{x}} \to R^\star$, using standard estimates for $\mathrm{H}_0^{(1)}$ and $[\mathrm{H}_0^{(1)}]'$, and uniform in n estimates for $\mathrm{H}_n^{(1)}$ and $[\mathrm{H}_n^{(1)}]'$ (see [16]), it is straightforward to prove that the following limits hold uniformly in $\theta_{\boldsymbol{x}} \in (-\pi, \pi]$ and $n \in \mathbb{N}$:

$$\lim_{r_{\boldsymbol{x}} \to R^\star} \sqrt{\hat{\gamma}_{\boldsymbol{x}} \gamma_{\boldsymbol{x}}} \frac{\mathrm{H}_n^{(1)}(k\hat{r}_{\boldsymbol{x}})}{\mathrm{H}_n^{(1)}(kR)} = 0,$$

$$\lim_{r_{\boldsymbol{x}} \to R^\star} \frac{1}{n} \sqrt{\frac{\hat{\gamma}_{\boldsymbol{x}}}{\gamma_{\boldsymbol{x}}}} \frac{\partial}{\partial r_{\boldsymbol{x}}} \left(\frac{\mathrm{H}_n^{(1)}(k\hat{r}_{\boldsymbol{x}})}{\mathrm{H}_n^{(1)}(kR)} \right) = 0,$$

$$\lim_{r_{\boldsymbol{x}} \to R^\star} \frac{1}{n} \sqrt{\frac{\gamma_{\boldsymbol{x}}}{\hat{\gamma}_{\boldsymbol{x}}}} \frac{\mathrm{H}_n^{(1)}(k\hat{r}_{\boldsymbol{x}})}{\mathrm{H}_n^{(1)}(kR)} = 0.$$

Hence, for δ small enough, we have

$$\int_{R^\star-\delta}^{R^\star} \int_{-\pi}^{\pi} |\hat{\gamma}(r)\gamma(r)r| \, |\hat{u}|^2 \, d\theta \, dr \le C\delta \int_{-\pi}^{\pi} \left| \sum_{n=-\infty}^{\infty} g_n \mathrm{e}^{in\theta} \right|^2 d\theta = C\delta \, \|g\|^2_{\mathrm{L}^2(S_R)},$$

$$\int_{R^\star-\delta}^{R^\star} \int_{-\pi}^{\pi} \left| \frac{\hat{\gamma}(r)r}{\gamma(r)} \right| \left| \frac{\partial \hat{u}}{\partial r} \right|^2 d\theta \, dr \le C\delta \int_{-\pi}^{\pi} \left| \sum_{n=-\infty}^{\infty} n g_n \mathrm{e}^{in\theta} \right|^2 d\theta \le C\delta \, \|g\|^2_{\mathrm{H}^1(S_R)},$$

$$\int_{R^\star-\delta}^{R^\star} \int_{-\pi}^{\pi} \left| \frac{\gamma(r)}{\hat{\gamma}(r)r} \right| \left| \frac{\partial \hat{u}}{\partial \theta} \right|^2 d\theta \, dr \le C\delta \int_{-\pi}^{\pi} \left| \sum_{n=-\infty}^{\infty} n g_n \mathrm{e}^{in\theta} \right|^2 d\theta \le C\delta \, \|g\|^2_{\mathrm{H}^1(S_R)},$$

which allow us to conclude that the integral (7.4) is finite. Therefore, from the three previous steps we deduce that $\hat{u} \in \mathrm{W}$.

Next, since $\hat{u}$ is continuous on S_R, evaluating (7.3) for $\boldsymbol{x} \in S_R$ we have $\hat{u}(\boldsymbol{x}) = \sum_{n=-\infty}^{\infty} g_n \mathrm{e}^{in\theta}$, and (7.2) follows from the convergence of the Fourier series of g.

On the other hand, straightforward computations allow us to show that each term in the series defining $\hat{u}$ is a solution of (7.1). Thus, $\hat{u}$ is a solution too, because we have already shown the uniform convergence on compact subsets of D of the series and its term by term first and second derivatives.

Finally, $\hat{u}$ is the unique solution of (7.1)-(7.2) in W because of Theorem 7.1 and the uniqueness of the Fourier expansion of g. □

8. Coupled fluid/PML problem. Our next goal is to study the coupled fluid/PML problem and to prove that the solution of the classical scattering problem is recovered when the PML is used.

Theorem 7.2 allows us to define a "Dirichlet-to-Neumann" PML operator,

$$\hat{G} : \mathrm{H}^{\frac{1}{2}}(S_R) \to \mathrm{H}^{-\frac{1}{2}}(S_R).$$

First it is defined for sufficiently smooth data as follows:

$$\hat{G}(g) = \frac{1}{\gamma(R)} \left. \frac{\partial \hat{u}}{\partial r} \right|_{S_R}, \qquad g \in \mathrm{H}^s(S_R), \text{ with } s > 3/2, \tag{8.1}$$

where $\hat{u}$ is the unique solution in W of (7.1)-(7.2). This definition can be extended to $g \in \mathrm{H}^{\frac{1}{2}}(S_R)$ by means of a density argument, because of the following result:

THEOREM 8.1. *There exists a unique bounded linear operator $\hat{G} : \mathrm{H}^{\frac{1}{2}}(S_R) \to \mathrm{H}^{-\frac{1}{2}}(S_R)$ satisfying (8.1), which coincides with G as defined by (2.4).*

Proof. If $g \in \mathrm{H}^s(S_R)$, $s > 3/2$, and G is defined by (2.4), then $\hat{G}g = Gg$. Indeed,

$$\hat{G}g = \frac{1}{\gamma(R)} \frac{\partial \hat{u}}{\partial r}\bigg|_{S_R} = \frac{1}{\gamma(R)} \sum_{n=-\infty}^{\infty} \frac{g_n k}{\mathrm{H}_n^{(1)}(kR)} \frac{d\hat{r}}{dr}(R)[\mathrm{H}_n^{(1)}]'(k\hat{r}_{\boldsymbol{x}})\mathrm{e}^{in\theta}$$

$$= \sum_{n=-\infty}^{\infty} \frac{g_n k}{\mathrm{H}_n^{(1)}(kR)} [\mathrm{H}_n^{(1)}]'(kR)\mathrm{e}^{in\theta} = Gg.$$

Consequently, the definition of $\hat{G}$ extends uniquely to the whole space $\mathrm{H}^{\frac{1}{2}}(S_R)$ and $\hat{G} = G$. □

Therefore, $\hat{G}$ can be equivalently used instead of G in the definition of problem (2.5)-(2.7). Moreover we have the following result.

THEOREM 8.2. *For $f \in \mathrm{H}^{\frac{1}{2}}(\Gamma)$, there exists a unique solution $(u, \hat{u}) \in \mathrm{H}^1(\Omega_R) \times \mathrm{V}$ of the following problem:*

$$-\Delta u - k^2 u = 0 \qquad \text{in } \Omega_R, \tag{8.2}$$

$$-\operatorname{div}(\boldsymbol{A}\,\mathbf{grad}\,\hat{u}) - \gamma\hat{\gamma}k^2\hat{u} = 0 \qquad \text{in D}, \tag{8.3}$$

$$u = f \qquad \text{on } \Gamma, \tag{8.4}$$

$$\frac{\partial u}{\partial \boldsymbol{\nu}} = \boldsymbol{A}\,\mathbf{grad}\,\hat{u} \cdot \boldsymbol{\nu} \qquad \text{in } \mathrm{H}^{-\frac{1}{2}}(S_R), \tag{8.5}$$

$$u = \hat{u} \qquad \text{on } S_R. \tag{8.6}$$

Moreover, u coincides with the solution of (2.5)-(2.7) and, hence, it coincides with the solution of the scattering problem (2.1)-(2.3) in Ω_R.

Proof. Let $u \in \mathrm{H}^1(\Omega_R)$ be the solution of (2.5)-(2.7). Then u is the restriction to Ω_R of the solution of (2.1)-(2.3). Hence $u|_{S_R}$ is arbitrarily smooth. Thus, $\hat{G}(u|_{S_R}) = (1/\gamma(R))\partial\hat{u}/\partial r$, with $\hat{u} \in \mathrm{W}$ being the solution of (7.1)-(7.2). Therefore $(u, \hat{u}) \in \mathrm{H}^1(\Omega_R) \times \mathrm{V}$ is a solution of the coupled fluid/PML problem.

To prove the uniqueness, it is enough to show that the solution $(u_o, \hat{u}_o)$ of problem (8.2)-(8.6) with $f = 0$ vanishes. By applying local regularity up to the boundary results for transmission problems (in particular Theorem 4.20 from [33]), we conclude that $\hat{u}_o \in \mathcal{C}^2(\mathrm{D}^\star)$. Notice that this regularity comes from the assumed smoothness on the absorbing function: $\sigma \in \mathcal{C}^{2,1}(\mathrm{D})$. Hence $\hat{u}_o \in \mathrm{W}$ and

$$\boldsymbol{A}\,\mathbf{grad}\,\hat{u}_o \cdot \boldsymbol{\nu} = \frac{1}{\gamma(R)} \frac{\partial \hat{u}_o}{\partial r} \qquad \text{on } S_R.$$

Consequently, (8.3) and (8.6) imply that $\hat{G}(u_o|_{S_R}) = \boldsymbol{A}\,\mathbf{grad}\,\hat{u}_o \cdot \boldsymbol{\nu}$ and, because of (8.5) and Theorem 8.1, we have

$$\frac{\partial u_o}{\partial \boldsymbol{\nu}} = \hat{G}(u_o|_{S_R}) = G(u_o|_{S_R}) \qquad \text{on } S_R.$$

Therefore, u_o is the unique solution of (2.5)-(2.7) with $f = 0$ and hence $u_o = 0$. Finally, as a consequence of Theorem 7.2, $\hat{u}_o = 0$, too. □

Finally we write a variational formulation of the coupled fluid/PML problem (8.2)-(8.6), which will be used in the following section to introduce a convenient finite element discretization. For this purpose, we introduce the functional space

$$\mathrm{H} := \left\{(v, \hat{v}) \in \mathrm{H}^1(\Omega_R) \times \mathrm{V} : v = \hat{v} \text{ on } S_R\right\}.$$

Let $(u,\hat{u}) \in \mathrm{H}$ be the solution of problem (8.2)-(8.6) and $(v,\hat{v}) \in \mathrm{H}$. Integrating by parts (8.2) in Ω_R and (8.3) in $\tilde{\mathrm{D}} = \{\boldsymbol{x} \in \mathbb{R}^2 : R < |\boldsymbol{x}| < \tilde{R}\}$ with $\tilde{R} \in (R, R^\star)$, since u and $\hat{u}$ are smooth, we obtain

$$\int_{\Omega_R} \mathbf{grad}\, u \cdot \mathbf{grad}\, v - k^2 \int_{\Omega_R} uv - \int_{S_R} \frac{\partial u}{\partial r} v\, dS$$
$$+ \int_{\tilde{\mathrm{D}}} \boldsymbol{A}\, \mathbf{grad}\, \hat{u} \cdot \mathbf{grad}\, \hat{v} - k^2 \int_{\tilde{\mathrm{D}}} \gamma\hat{\gamma}\hat{u}\hat{v} + \int_{S_R} \frac{1}{\gamma} \frac{\partial \hat{u}}{\partial r} \hat{v}\, dS - \int_{S_{\tilde{R}}} \frac{\hat{\gamma}}{\gamma} \frac{\partial \hat{u}}{\partial r} \hat{v}\, dS = 0.$$

Since $(v,\hat{v}) \in \mathrm{H}$, the boundary terms on S_R cancel out because of (8.5). Moreover, as $\tilde{R}$ goes to $R^\star$, using Lemma 5.2 we have

$$\int_{\Omega_R} \mathbf{grad}\, u \cdot \mathbf{grad}\, v + \int_{\mathrm{D}} \boldsymbol{A}\, \mathbf{grad}\, \hat{u} \cdot \mathbf{grad}\, \hat{v} - k^2 \left(\int_{\Omega_R} uv + \int_{\mathrm{D}} \gamma\hat{\gamma}\hat{u}\hat{v} \right) = 0.$$

Thus we are lead to the following variational formulation of problem (8.2)-(8.6):

For $f \in \mathrm{H}^{\frac{1}{2}}(\Gamma)$, find $(u,\hat{u}) \in \mathrm{H}$ such that $u = f$ on Γ and

$$\int_{\Omega_R} \mathbf{grad}\, u \cdot \mathbf{grad}\, v + \int_{\mathrm{D}} \boldsymbol{A}\, \mathbf{grad}\, \hat{u} \cdot \mathbf{grad}\, \hat{v} - k^2 \left(\int_{\Omega_R} uv + \int_{\mathrm{D}} \gamma\hat{\gamma}\hat{u}\hat{v} \right) = 0 \qquad \forall\, (v,\hat{v}) \in \mathrm{H}_0. \tag{8.7}$$

The space H_0 above is given by $\mathrm{H}_0 := \{(v,\hat{v}) \in \mathrm{H} : v = 0 \text{ on } \Gamma\}$.

COROLLARY 8.3. *The solution of the coupled fluid/PML problem (8.2)-(8.6) is the unique solution of the variational problem (8.7).*

Proof. We have already shown that the solution of problem (8.2)-(8.6) satisfies (8.7). The converse follows from standard arguments. □

9. Discretization and numerical results. In this section, we introduce a finite element discretization of problem (8.7). For this purpose, we use meshes $\mathcal{T}_h$ of curved elements which correspond to standard quadrilaterals in polar coordinates. As usual, h denotes the mesh-size. Each element must be completely contained either in $\bar{\Omega}_R$ or in $\bar{\mathrm{D}}$. Moreover, we take advantage of the fact that D is an annular domain by using curved rectangles in D (see Fig. 9.1). We use bilinear elements in polar coordinates; namely, for $K \in \mathcal{T}_h$, let

$$\mathcal{Q}_1(K) := \{v_h \in \mathcal{C}(K) : v_h(\boldsymbol{y}) = a r_{\boldsymbol{y}} \theta_{\boldsymbol{y}} + b r_{\boldsymbol{y}} + c\theta_{\boldsymbol{y}} + d,\ \boldsymbol{y} \in K,\ a,b,c,d \in \mathbb{C}\}.$$

Thus, the finite-element space is

$$\mathrm{H}_h := \{(v_h,\hat{v}_h) \in \mathcal{C}(\Omega_R) \times \mathcal{C}(\mathrm{D}) :\ \hat{v}_h = 0 \text{ on } S_{R^\star},$$
$$v_h = \hat{v}_h \text{ on } S_R,\ v_h|_K, \hat{v}_h|_K \in \mathcal{Q}_1(K)\ \forall K \in \mathcal{T}_h\}.$$

From Lemma 5.1, the boundary condition $\hat{v}_h = 0$ on $S_{R^\star}$ in the definition of H_h turns out necessary for $\mathrm{H}_h \subset \mathrm{H}$. For a non integrable absorbing function σ in (3.2) as that in (9.1) below, this boundary condition is also sufficient (see [11] for other feasible choices of σ for which $\mathrm{H}_h \subset \mathrm{H}$).

Let f_h be a convenient approximation of f in the space of the traces on Γ of functions in H_h. The discrete variational problem associated with the coupled fluid/PML problem is the following:

Find $(u_h, \hat{u}_h) \in \mathrm{H}_h$ *such that* $u_h = f_h$ *on* Γ *and*

$$\int_{\Omega_R} \mathbf{grad}\, u_h \cdot \mathbf{grad}\, v_h + \int_{\mathrm{D}} \boldsymbol{A}\, \mathbf{grad}\, \hat{u}_h \cdot \mathbf{grad}\, \hat{v}_h - k^2 \left(\int_{\Omega_R} u_h v_h + \int_{\mathrm{D}} \gamma \hat{\gamma} \hat{u}_h \hat{v}_h \right) = 0 \qquad \forall\, (v_h, \hat{v}_h) \in \mathrm{H}_h \cap \mathrm{H}_0.$$

In what follows we report some numerical results obtained with a computer code implementing the perfectly matched layer method with a non integrable absorbing function σ. In all the numerical tests we have used

$$\sigma(s) = \frac{c}{R^\star - s}; \tag{9.1}$$

let us recall that c is the velocity of propagation in Ω_R.

To illustrate the performance of the PML method with an non integrable σ, we consider a simple problem for which we have a closed form solution. It is well known that the function

$$u(\boldsymbol{y}) = \frac{i}{4} \mathrm{H}_0^{(1)} \left(k \left| \boldsymbol{y} \right| \right)$$

satisfies the scattering problem (2.1)-(2.3). Therefore, if we take $f := u|_\Gamma$, then u is the component in the fluid domain Ω_R of the unique solution of (8.2)-(8.6).

In this numerical experiment we have taken $k = \omega/c$ with $c = 343\,\mathrm{m/s}$ and frequency $\omega = 750\,\mathrm{rad/s}$. We have used the computational domain shown in Fig. 9.1 where $R_\star = 1\,\mathrm{m}$, $R = 2.25\,\mathrm{m}$ and $R^\star = 3.5\,\mathrm{m}$.

To evaluate the integrals involved in the finite element method, we have used a Simpson adaptive rule, to reduce the numerical errors arising from the quadrature rules as much as possible. However, it is shown in [11] that standard quadrature rules lead to numerical results essentially of the same accuracy.

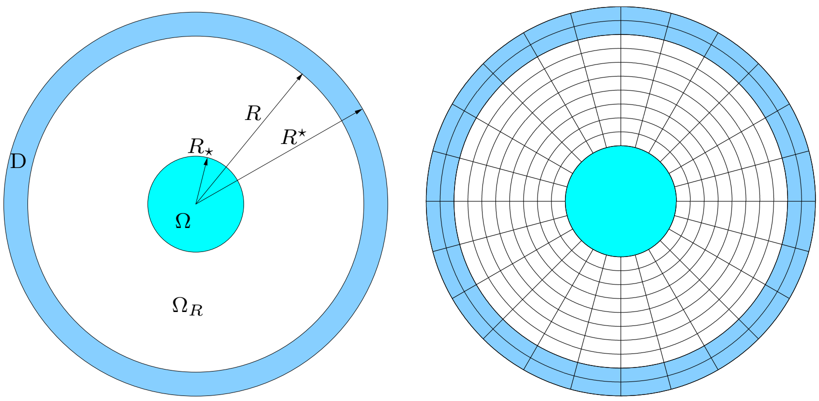

FIG. 9.1. *Domains and mesh ($N = 2$) in the scattering problem.*

We have used uniform refinements of the mesh shown in Fig. 9.1; the number N of elements through the thickness of the PML is used to label each mesh. Specifically, meshes corresponding to $N = 2$, 4 and 8 have 264, 1008 and 3936 degrees of freedom, respectively.

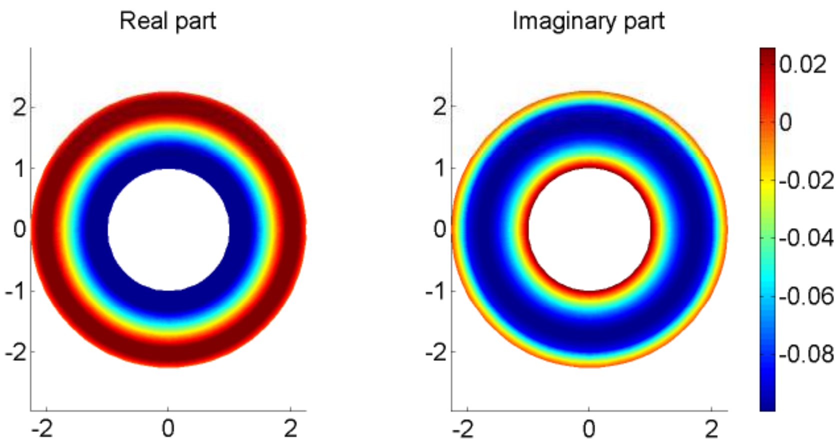

FIG. 9.2. *Solution of the fluid/PML coupled problem. Mesh* $N = 8$, $\omega = 750\,rad/s$.

We show in Fig. 9.2 the real and imaginary parts of the solution computed for the fluid/PML coupled problem with the mesh corresponding to $N = 8$ and $\omega = 750\,\mathrm{rad/s}$. The solution is plotted in the fluid domain and in the PML.

To measure the accuracy we have estimated the relative error in the L^2-norm in Ω_R as follows:

$$\text{Error} = \frac{\|u_h - \Pi_h u\|_{\mathrm{L}^2(\Omega_R)}}{\|\Pi_h u\|_{\mathrm{L}^2(\Omega_R)}}, \tag{9.2}$$

where u_h is the numerical solution in Ω_R and $\Pi_h u$ is the Lagrange interpolant of the exact solution u.

To assess the order of convergence of the proposed numerical method, we show in Fig. 9.3 the error curve (log-log plot of error versus mesh-size) computed in the fluid domain Ω_R. It can be seen from this figure that an order of convergence $\mathcal{O}(h^2)$ is achieved. Let us recall that this is the optimal order for the used finite elements in L^2-norm.

Further numerical examples on more complex geometries and with different data have been reported in [11], where we have implemented a Cartesian PML using non integrable absorbing functions.

To end this section, we compare the numerical performance of this PML technique with that of the classical one based on a quadratic function (see for instance [8] or [12]):

$$\sigma_{\mathrm{Q}}(s) = c\sigma^\star(s - R)^2. \tag{9.3}$$

As shown in [13], for a given problem and a given mesh there is an optimal value of $\sigma^\star$ leading to minimal errors. Such optimal value depends strongly on the problem data as well as on the particular mesh. Thus, in practice, it is necessary to tune it. No theoretical procedure for such a tuning is known to date.

In Table 9.1, we compare the errors of the PML methods with the unbounded absorbing function (9.1) and with the quadratic absorbing function (9.3), for $\omega = 750\,\mathrm{rad/s}$. For the latter, we have used the optimally tuned value of $\sigma^\star$, which is also

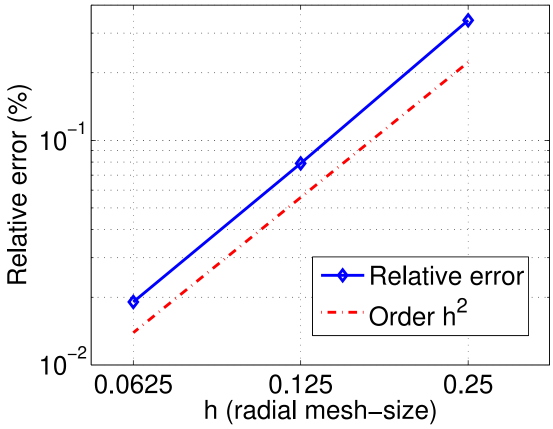

Fig. 9.3. *Error curve for the fluid/PML coupled problem* ($\omega = 750\,\mathrm{rad/s}$).

reported in the table (and which can be seen that changes significantly from one mesh to the others).

Table 9.1
Comparison of PML methods with non integrable and quadratic absorbing functions.

	Non integrable σ (9.1)	Quadratic σ_{Q} (9.3)	
Mesh	Error(%)	Error(%)	$\sigma^\star$
$N = 2$	0.342	11.346	415.54
$N = 4$	0.079	3.247	565.07
$N = 8$	0.019	0.970	702.32

Table 9.1 shows that the errors of the PML method with the non integrable absorbing function are noticeable smaller than those of the classical PML technique. On the other hand, another benefit of our proposed PML method is that there is no need of fitting any non-physical parameter.

Appendix A. Technical results. In this appendix we collect some technical results that have been used along the proof of Theorems 4.4 and 5.3.

First, we recall some basic results about the relation between polar coordinates centered at different points. For a fixed point $\boldsymbol{x} \in \mathrm{D}$, we introduce $(\rho_{\boldsymbol{y}}, \phi_{\boldsymbol{y}})$ as the coordinates of point $\boldsymbol{y}$ in polar coordinates centered at $\boldsymbol{x}$:

$$\boldsymbol{y} = \boldsymbol{x} + \rho_{\boldsymbol{y}}(\cos\phi_{\boldsymbol{y}}, \sin\phi_{\boldsymbol{y}}).$$

We denote by $\{\boldsymbol{e}_\rho, \boldsymbol{e}_\phi\}$ the canonical basis of the second system of coordinates. For each point $\boldsymbol{y} \in \mathrm{D}$, we have

$$\boldsymbol{e}_r = \cos(\phi_{\boldsymbol{y}} - \theta_{\boldsymbol{y}})\boldsymbol{e}_\rho - \sin(\phi_{\boldsymbol{y}} - \theta_{\boldsymbol{y}})\boldsymbol{e}_\phi, \tag{A.1}$$

$$\boldsymbol{e}_\theta = \sin(\phi_{\boldsymbol{y}} - \theta_{\boldsymbol{y}})\boldsymbol{e}_\rho + \cos(\phi_{\boldsymbol{y}} - \theta_{\boldsymbol{y}})\boldsymbol{e}_\phi. \tag{A.2}$$

On the other hand, explicit computations lead to

$$\frac{\partial r_{\boldsymbol{y}}}{\partial \rho_{\boldsymbol{y}}} = \cos(\phi_{\boldsymbol{y}} - \theta_{\boldsymbol{y}}), \qquad \frac{\partial \theta_{\boldsymbol{y}}}{\partial \rho_{\boldsymbol{y}}} = \frac{1}{r_{\boldsymbol{y}}}\sin(\phi_{\boldsymbol{y}} - \theta_{\boldsymbol{y}}). \tag{A.3}$$

The following lemma collects several limits that will be used in the proof of Lemma A.3. The corresponding proofs are straightforward.

LEMMA A.1. *For fixed* $\boldsymbol{x} \in \mathrm{D}$ *and* $\phi_{\boldsymbol{y}} \in (-\pi, \pi]$,

$$\lim_{\rho_{\boldsymbol{y}} \to 0} \frac{\partial \hat{r}_{\boldsymbol{y}}}{\partial \rho_{\boldsymbol{y}}} = \gamma_{\boldsymbol{x}} \cos(\phi_{\boldsymbol{y}} - \theta_{\boldsymbol{x}}),$$
$$\lim_{\rho_{\boldsymbol{y}} \to 0} \frac{\partial\, \mathrm{d}(\boldsymbol{x}, \boldsymbol{y})}{\partial \rho_{\boldsymbol{y}}} = \sqrt{\gamma_{\boldsymbol{x}}^2 \cos^2(\phi_{\boldsymbol{y}} - \theta_{\boldsymbol{x}}) + \hat{\gamma}_{\boldsymbol{x}}^2 \sin^2(\phi_{\boldsymbol{y}} - \theta_{\boldsymbol{x}})},$$
$$\lim_{\rho_{\boldsymbol{y}} \to 0} \frac{\hat{r}_{\boldsymbol{y}} - \hat{r}_{\boldsymbol{x}} \cos(\theta_{\boldsymbol{y}} - \theta_{\boldsymbol{x}})}{r_{\boldsymbol{y}} - r_{\boldsymbol{x}} \cos(\theta_{\boldsymbol{y}} - \theta_{\boldsymbol{x}})} = \gamma_{\boldsymbol{x}}.$$

The following integral will be also used below.

LEMMA A.2. *For* $a \in \mathbb{C}$ *with* $\mathrm{Re}(a) \neq 0$*, there holds*

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{a \cos^2 \theta + a^{-1} \sin^2 \theta}\, d\theta = \mathrm{sign}(\mathrm{Re}(a)). \tag{A.4}$$

Proof. First, using the change of variable $s = \tan\theta$, it is easy to see that

$$\int_{-\pi}^{\pi} \frac{1}{a \cos^2 \theta + a^{-1} \sin^2 \theta}\, d\theta = \frac{2}{a} \int_{-\infty}^{\infty} \frac{1}{a^{-2} + s^2} ds.$$

To evaluate the improper integral, we apply the residue theorem. The residues of the integrand are $\pm \frac{a}{2i}$. We have to distinguish two cases depending on the sign of $\mathrm{Im}(ia^{-1})$. For instance, if $\mathrm{Im}(ia^{-1}) > 0$, then the standard procedure leads to

$$\frac{2}{a} \int_{-\infty}^{\infty} \frac{1}{a^{-2} + s^2} ds = \frac{2}{a} \left(\frac{a}{2i} 2\pi i \right) = \mathrm{sign}(\mathrm{Im}(ia^{-1})) 2\pi,$$

and analogously for $\mathrm{Im}(ia^{-1}) < 0$. We conclude the result since $\mathrm{sign}(\mathrm{Im}(ia^{-1})) = \mathrm{sign}(\mathrm{Re}(a))$. □

Now, we are in a position to prove the next lemma which has been used in Theorems 4.4 and 5.3.

LEMMA A.3. *For* $\boldsymbol{x} \in \mathrm{D}$ *fixed, if* $\Phi^{\pm}$ *are the fundamental solutions given by (4.3) and (4.4), and* $\varphi \in \mathcal{C}^1(\mathrm{D})$*, then*

$$\lim_{\epsilon \to 0} \left(\int_{S(\boldsymbol{x},\epsilon)} \boldsymbol{A}_{\boldsymbol{y}}\, \mathbf{grad}_{\boldsymbol{y}}\, \Phi^{\pm}(\boldsymbol{x}, \boldsymbol{y}) \cdot \boldsymbol{n}\, \varphi(\boldsymbol{y})\, dS_{\boldsymbol{y}} - \int_{S(\boldsymbol{x},\epsilon)} \boldsymbol{A}_{\boldsymbol{y}}\, \mathbf{grad}_{\boldsymbol{y}}\, \varphi(\boldsymbol{y}) \cdot \boldsymbol{n}\, \Phi^{\pm}(\boldsymbol{x}, \boldsymbol{y})\, dS_{\boldsymbol{y}} \right) = \varphi(\boldsymbol{x}), \tag{A.5}$$

where $S(\boldsymbol{x}, \epsilon) = \{\boldsymbol{y} \in \mathbb{R}^2 : |\boldsymbol{x} - \boldsymbol{y}| = \epsilon\}$ *and* $\boldsymbol{n}$ *is its inward unit normal vector.*

Proof. We prove the lemma for Φ^+. An analogous proof is valid for Φ^-.

First, we check that the limit of the second integral in (A.5) is zero. Since $\varphi \in \mathcal{C}^1(\mathrm{D})$ and the coefficients of $\boldsymbol{A}_{\boldsymbol{y}}$ are bounded for $\boldsymbol{y}$ such that $|\boldsymbol{x} - \boldsymbol{y}| \leq \epsilon$, we only have to prove that

$$\lim_{\epsilon \to 0} \int_{S(\boldsymbol{x},\epsilon)} \Phi^{+}(\boldsymbol{x}, \boldsymbol{y})\, dS_{\boldsymbol{y}} = 0.$$

The above limit is easy to check by using Lemma 4.2 and the following estimate:

$$\left|\Phi^+(\boldsymbol{x},\boldsymbol{y}) - \frac{i}{4}\left(\frac{2i}{\pi}\log\frac{k\,\mathrm{d}(\boldsymbol{x},\boldsymbol{y})}{2} + \frac{2C_e i}{\pi} + 1\right)\right| \le C\left|\mathrm{d}(\boldsymbol{x},\boldsymbol{y})\right|^2\left|\log\mathrm{d}(\boldsymbol{x},\boldsymbol{y})\right|,$$

for $|\boldsymbol{x}-\boldsymbol{y}|$ small enough, which in its turn follows from the asymptotic behavior of Hankel functions (see [39]). In the above expression, C_e is the Euler's constant.

Regarding the first integral in (A.5), since $\boldsymbol{n} = -\boldsymbol{e}_\rho$ on $S(\boldsymbol{x},\epsilon)$, from (4.3), (A.1) and (A.2), we have

(A.6)

$$\begin{aligned}\boldsymbol{A}_{\boldsymbol{y}}\,\mathbf{grad}_{\boldsymbol{y}}\,\Phi^+(\boldsymbol{x},\boldsymbol{y})\cdot\boldsymbol{n} &= \left(\frac{\hat{\gamma}_{\boldsymbol{y}}}{\gamma_{\boldsymbol{y}}}\frac{\partial\Phi^+(\boldsymbol{x},\boldsymbol{y})}{\partial r_{\boldsymbol{y}}}\boldsymbol{e}_r + \frac{\gamma_{\boldsymbol{y}}}{\hat{r}_{\boldsymbol{y}}}\frac{\partial\Phi^+(\boldsymbol{x},\boldsymbol{y})}{\partial\theta_{\boldsymbol{y}}}\boldsymbol{e}_\theta\right)\cdot(-\boldsymbol{e}_\rho)\\ &= -k\frac{i}{4}[\mathrm{H}_0^{(1)}]'(k\,\mathrm{d}(\boldsymbol{x},\boldsymbol{y}))\left[\frac{\hat{\gamma}_{\boldsymbol{y}}}{\gamma_{\boldsymbol{y}}}\frac{\partial\,\mathrm{d}(\boldsymbol{x},\boldsymbol{y})}{\partial r_{\boldsymbol{y}}}\cos(\phi_{\boldsymbol{y}}-\theta_{\boldsymbol{y}}) + \frac{\gamma_{\boldsymbol{y}}}{\hat{r}_{\boldsymbol{y}}}\frac{\partial\,\mathrm{d}(\boldsymbol{x},\boldsymbol{y})}{\partial\theta_{\boldsymbol{y}}}\sin(\phi_{\boldsymbol{y}}-\theta_{\boldsymbol{y}})\right]\\ &=: -k\frac{i}{4}[\mathrm{H}_0^{(1)}]'(k\,\mathrm{d}(\boldsymbol{x},\boldsymbol{y}))M(\boldsymbol{x},\boldsymbol{y}),\end{aligned}$$

where we denote by $M(\boldsymbol{x},\boldsymbol{y})$ the expression between brackets above. By using the following elementary identities (see Fig. A.1):

$$\begin{aligned}|\boldsymbol{x}-\boldsymbol{y}|\cos(\phi_{\boldsymbol{y}}-\theta_{\boldsymbol{y}}) &= r_{\boldsymbol{y}} - r_{\boldsymbol{x}}\cos(\theta_{\boldsymbol{y}}-\theta_{\boldsymbol{x}}),\\ |\boldsymbol{x}-\boldsymbol{y}|\sin(\phi_{\boldsymbol{y}}-\theta_{\boldsymbol{y}}) &= r_{\boldsymbol{x}}\sin(\theta_{\boldsymbol{y}}-\theta_{\boldsymbol{x}}),\end{aligned}$$

we obtain

$$\text{(A.7)}\quad \begin{aligned}M(\boldsymbol{x},\boldsymbol{y}) = \Bigg(\hat{\gamma}_{\boldsymbol{y}}\frac{\hat{r}_{\boldsymbol{y}} - \hat{r}_{\boldsymbol{x}}\cos(\theta_{\boldsymbol{y}}-\theta_{\boldsymbol{x}})}{r_{\boldsymbol{y}} - r_{\boldsymbol{x}}\cos(\theta_{\boldsymbol{y}}-\theta_{\boldsymbol{x}})}&\cos^2(\phi_{\boldsymbol{y}}-\theta_{\boldsymbol{y}})\\ &+\frac{\gamma_{\boldsymbol{y}}\hat{r}_{\boldsymbol{x}}}{r_{\boldsymbol{x}}}\sin^2(\phi_{\boldsymbol{y}}-\theta_{\boldsymbol{y}})\Bigg)\frac{|\boldsymbol{x}-\boldsymbol{y}|}{\mathrm{d}(\boldsymbol{x},\boldsymbol{y})},\end{aligned}$$

which, in particular, together with Lemma 4.2 and the third limit in Lemma A.1, show that $M(\boldsymbol{x},\boldsymbol{y})$ is bounded for $|\boldsymbol{x}-\boldsymbol{y}|$ small enough.

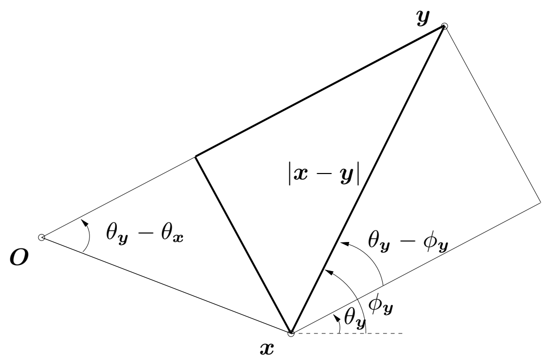

FIG. A.1. *Polar coordinates systems centered at the origin $\boldsymbol{O}$ and at point $\boldsymbol{x}$*

On the other hand, by using a classical estimate of $[\mathrm{H}_0^{(1)}]'(z)$ (see [39]) and Lemma 4.2, we have

$$\text{(A.8)}\qquad \left|[\mathrm{H}_0^{(1)}]'(k\,\mathrm{d}(\boldsymbol{x},\boldsymbol{y})) - \frac{2i}{\pi}\frac{1}{k\,\mathrm{d}(\boldsymbol{x},\boldsymbol{y})}\right| \le C\,|\boldsymbol{x}-\boldsymbol{y}|\left|\log(k\,\mathrm{d}(\boldsymbol{x},\boldsymbol{y}))\right|,$$

for $|\boldsymbol{x}-\boldsymbol{y}|$ small enough. Because of this, we proceed from (A.6) as follows:

(A.9)
$$\int_{S(\boldsymbol{x},\epsilon)} \boldsymbol{A}_{\boldsymbol{y}}\, \mathbf{grad}_{\boldsymbol{y}} \Phi^{+}(\boldsymbol{x},\boldsymbol{y})\cdot\boldsymbol{n}\,\varphi(\boldsymbol{y}) dS_{\boldsymbol{y}} = -\int_{-\pi}^{\pi} \epsilon k \frac{i}{4} M(\boldsymbol{x},\boldsymbol{y}) \frac{2i}{\pi}\frac{1}{k\,\mathrm{d}(\boldsymbol{x},\boldsymbol{y})}\varphi(\boldsymbol{y})\, d\phi_{\boldsymbol{y}}$$
$$-\int_{-\pi}^{\pi} \epsilon k \frac{i}{4} M(\boldsymbol{x},\boldsymbol{y}) \left([\mathrm{H}_0^{(1)}]'(k\,\mathrm{d}(\boldsymbol{x},\boldsymbol{y})) - \frac{2i}{\pi}\frac{1}{k\,\mathrm{d}(\boldsymbol{x},\boldsymbol{y})}\right)\varphi(\boldsymbol{y})\, d\phi_{\boldsymbol{y}}.$$

Manipulating the second integral from (A.8), we obtain

$$\text{(A.10)}\quad \left|\int_{-\pi}^{\pi} \epsilon k \frac{i}{4} M(\boldsymbol{x},\boldsymbol{y}) \left([\mathrm{H}_0^{(1)}]'(k\,\mathrm{d}(\boldsymbol{x},\boldsymbol{y})) - \frac{2i}{\pi}\frac{1}{k\,\mathrm{d}(\boldsymbol{x},\boldsymbol{y})}\right)\varphi(\boldsymbol{y})\, d\phi_{\boldsymbol{y}}\right|$$
$$\leq C\int_{-\pi}^{\pi} \frac{k}{4}\,|M(\boldsymbol{x},\boldsymbol{y})|\,\epsilon^2\,|\log(k\,\mathrm{d}(\boldsymbol{x},\boldsymbol{y}))|\,|\varphi(\boldsymbol{y})|\, d\phi_{\boldsymbol{y}} \longrightarrow 0 \qquad \text{as } \epsilon\to 0,$$

since, $M(\boldsymbol{x},\boldsymbol{y})$ is bounded for $|\boldsymbol{x}-\boldsymbol{y}|\leq\epsilon$ and $|\mathrm{d}(\boldsymbol{x},\boldsymbol{y})| = O(\epsilon)$ uniformly in all directions (see Lemma 4.2).

Thus, we only have to calculate the limit of the remaining integral in (A.9). For this purpose, we calculate the following limit as $\rho_{\boldsymbol{y}} = |\boldsymbol{x}-\boldsymbol{y}| \to 0$ for fixed $\phi_{\boldsymbol{y}}\in(-\pi,\pi]$:

$$\lim_{\rho_{\boldsymbol{y}}\to 0}\frac{\rho_{\boldsymbol{y}} M(\boldsymbol{x},\boldsymbol{y})}{\mathrm{d}(\boldsymbol{x},\boldsymbol{y})} = \left(\hat\gamma_{\boldsymbol{x}}\cos(\phi_{\boldsymbol{y}}-\theta_{\boldsymbol{x}}) \lim_{\rho_{\boldsymbol{y}}\to 0} \frac{\dfrac{\partial \hat r_{\boldsymbol{y}}}{\partial\rho_{\boldsymbol{y}}} + \hat r_{\boldsymbol{x}}\sin(\theta_{\boldsymbol{y}}-\theta_{\boldsymbol{x}})\dfrac{\partial\theta_{\boldsymbol{y}}}{\partial\rho_{\boldsymbol{y}}}}{\dfrac{\partial\,\mathrm{d}(\boldsymbol{x},\boldsymbol{y})}{\partial\rho_{\boldsymbol{y}}}}\right.$$
$$\left. +\gamma_{\boldsymbol{x}}\hat r_{\boldsymbol{x}}\sin(\phi_{\boldsymbol{y}}-\theta_{\boldsymbol{x}}) \lim_{\rho_{\boldsymbol{y}}\to 0} \frac{\cos(\theta_{\boldsymbol{y}}-\theta_{\boldsymbol{x}})\dfrac{\partial\theta_{\boldsymbol{y}}}{\partial\rho_{\boldsymbol{y}}}}{\dfrac{\partial\,\mathrm{d}(\boldsymbol{x},\boldsymbol{y})}{\partial\rho_{\boldsymbol{y}}}}\right) \lim_{\rho_{\boldsymbol{y}}\to 0}\left(\frac{\partial\,\mathrm{d}(\boldsymbol{x},\boldsymbol{y})}{\partial\rho_{\boldsymbol{y}}}\right)^{-1}$$
$$=\gamma_{\boldsymbol{x}}\hat\gamma_{\boldsymbol{x}} \lim_{\rho_{\boldsymbol{y}}\to 0}\left(\frac{\partial\,\mathrm{d}(\boldsymbol{x},\boldsymbol{y})}{\partial\rho_{\boldsymbol{y}}}\right)^{-2} = \frac{\gamma_{\boldsymbol{x}}\hat\gamma_{\boldsymbol{x}}}{\gamma_{\boldsymbol{x}}^2\cos^2(\phi_{\boldsymbol{y}}-\theta_{\boldsymbol{x}}) + \hat\gamma_{\boldsymbol{x}}^2\sin^2(\phi_{\boldsymbol{y}}-\theta_{\boldsymbol{x}})},$$

where we have used L'Hôpital's rule, Lemma A.1 and (A.3).

Therefore, since $\rho_{\boldsymbol{y}} = \epsilon$ on $S(\boldsymbol{x},\epsilon)$, by using the above limit, the boundedness of $M(\boldsymbol{x},\boldsymbol{y})$ and Lemma 4.2, we have from (A.9) and (A.10)

$$\lim_{\epsilon\to 0}\int_{S(\boldsymbol{x},\epsilon)} \boldsymbol{A}_{\boldsymbol{y}}\, \mathbf{grad}_{\boldsymbol{y}}\, \Phi^{+}(\boldsymbol{x},\boldsymbol{y})\cdot\boldsymbol{n}\,\varphi(\boldsymbol{y}) dS_{\boldsymbol{y}}$$
$$=\varphi(\boldsymbol{x})\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\gamma_{\boldsymbol{x}}\hat\gamma_{\boldsymbol{x}}}{\gamma_{\boldsymbol{x}}^2\cos^2(\phi_{\boldsymbol{y}}-\theta_{\boldsymbol{x}}) + \hat\gamma_{\boldsymbol{x}}^2\sin^2(\phi_{\boldsymbol{y}}-\theta_{\boldsymbol{x}})}\, d\phi_{\boldsymbol{y}}$$
$$=\operatorname{sign}\left(\operatorname{Re}\left(\frac{\gamma_{\boldsymbol{x}}}{\hat\gamma_{\boldsymbol{x}}}\right)\right)\varphi(\boldsymbol{x})$$
$$=\varphi(\boldsymbol{x}),$$

because of Lemma A.2 with $a = \gamma_{\boldsymbol{x}}/\hat\gamma_{\boldsymbol{x}}$, which can be shown that has a positive real part. ∎

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