AN EXACT GENERAL SOLUTION FOR THE TEMPERATURE DISTRIBUTION AND THE COMPOSITE RADIATION CONVECTION HEAT EXCHANGE ALONG A CONSTANT CROSS-SECTIONAL AREA FIN*

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Abstract. An exact general solution is obtained for the nonlinear differential equation governing the one-dimensional steady-state heat exchange by composite radiation and convection along constant area fins with uniform temperature at the base and an arbitrary temperature gradient at the other end. The fin can dissipate or receive energy. The fin is assumed to have constant thermal properties, and the radiant interaction between the fin and the base surfaces is neglected.

Introduction. Although the subject of heat transfer from fins and extended surfaces has been studied analytically and experimentally for almost two centuries [1], the subject of radiating fins has only recently come under extensive study because of the interest in space and space travel. The assumption is usually made that the end of the fin farthest from the base is insulated.

Numerical solutions of the radiating fin problem using difference equations and computers have been published by Chambers and Somers [2], Lieblein [3], Bartes and Sellers [4], and Callinan and Berggren [5]. Wilkins [6] and Liu [7] treated the problem of the minimum mass fin. Mackay [8] has outlined a method of successive approximations for use with a digital computer to solve the problem including variable area. Shouman [9] has shown that the problem can be solved. Hung and Appl [10] considered the heat generation and the effect of the variation of thermal properties with temperature.

Since convection heat transfer sometimes accounts for a significant portion of the exchange of heat, the purpose of this paper is to present the exact and general solutions for heat exchange by means of radiation-convection along a constant area fin with constant thermal properties.

The problem and its solution. The general case of a fin with a constant cross-sectional area of arbitrary shape is considered herein. For reasons which will be made clear later, the positive x axis is chosen in the direction of increasing temperature for the case in which the fin transfers heat to the surroundings and in the direction of decreasing temperature for the case in which the fin receives heat from the surroundings. Assuming constant thermal properties, the steady-state one-dimensional heat flow equation is written as

$$\frac{d^2T}{dx^2} - \frac{hP}{kA}(T - T_s) - \frac{\sigma EP}{kA}(T^4 - T_s^4) = 0. \tag{1}$$

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If Eq. (1) is integrated, the following expression is obtained.

$$\left(\frac{dT}{dx}\right)^2 = \frac{2\sigma EP}{5kA}(T^5 - 5T_s^4 T) + \frac{hP}{kA}(T^2 - 2T_s T) + C'. \tag{2}$$

Introducing $\theta = T/T_0$ where T_0 is the temperature at $x = 0$, $r = 5h/(2\sigma E T_0^3)$ and applying the boundary condition

$$(d\theta/dx)^2_{x=0} = g(2\sigma EPT_0^3/5kA)$$

the solution to Eq. (2) can be manipulated into the form

$$\left(\frac{2\sigma EPT_0^3}{5kA}\right)^{1/2} x = \pm \int_1^\theta [(\theta^5 - 1) - 5\theta_s^4(\theta - 1) + r(\theta - 1)(\theta - 2\theta_s + 1) + g]^{-1/2}\, d\theta. \tag{3}$$

Substituting for $C = g + 5\theta_s^4 - 1 + r(2\theta_s - 1)$, Eq. (3) can be written as follows:

$$\left(\frac{2\sigma EPT_0^3}{5kA}\right)^{1/2} x = \pm \int_1^\theta [\theta^5 - 5\theta_s^4\theta + r(\theta^2 - 2\theta_s\theta) + C]^{-1/2}\, d\theta. \tag{4}$$

The positive and negative signs in Eqs. (3) and (4) correspond respectively to the cases of heat transfer to and from the surroundings. The integral in the right-hand member can be evaluated numerically or graphically for the case in which $g > 0$. Note from Eq. (3) that for $g = 0$, a singularity exists at $\theta = 1$, so the evaluation of the integral requires special consideration. If Φ and Λ are defined as:

$$\Phi(\theta, \theta_s, r, C) = \int_\theta^\infty [\theta^5 - 5\theta_s^4\theta + r(\theta^2 - 2\theta_s\theta) + C]^{-1/2}\, d\theta \tag{5}$$

and

$$\Lambda(\theta, \theta_s, r, C) = \int_0^\theta [\theta^5 - 5\theta_s^4\theta + r(\theta^2 - 2\theta_s\theta) + C]^{-1/2}\, d\theta \tag{6}$$

and Φ and Λ are proved to be finite for the range of interest, the solution can be written for the fin transferring heat as

$$(2\sigma EPT_0^3/5kA)^{1/2}x = \Phi(1, \theta_s, r, C) - \Phi(\theta, \theta_s, r, C) \tag{7}$$

and for the fin receiving heat as

$$(2\sigma EPT_0^3/5kA)^{1/2}x = \Lambda(1, \theta_s, r, C) - \Lambda(\theta, \theta_s, r, C). \tag{8}$$

Evaluation of the functions Φ and Λ. An examination of Φ and Λ shows that both functions have an upper bound which exists for $r = 0$, $g = 0$ and $\theta_s = 1$ and also that the functions are finite in the entire field except at the point $\theta = 1$ when $\theta_s = 1$ and $g = 0$.

The Function Φ. After introducing the variable $\lambda = 1/\theta$, Eq. (5) can be written as follows:

$$\Phi(\lambda, \theta_s, r, C) = \frac{2}{3}\int_0^\lambda [1 - 5\theta_s^4\lambda^4 + r\lambda^3(1 - 2\theta_s\lambda) + C\lambda^5]^{-1/2}\, d(\lambda^{3/2}). \tag{9}$$

This form can be used when $g > 0$ and the integral can be determined to any desired accuracy by using $\lambda^{3/2}$ as the independent variable. When $g = 0$, a singularity exists at $\lambda = 1$. However, integration by parts leads to the following form:

$$\Phi(\lambda, \theta_s, r, C) = \frac{8}{3}\int_0^\lambda \frac{[1 - 5\theta_s^4\lambda^4 + r\lambda^3(1 - 2\theta_s\lambda) + C\lambda^5]^{1/2}(10 + r\lambda^3)\, d(\lambda^{3/2})}{[5(1 - \theta_s^4\lambda^4) + 2r\lambda^3(1 - \theta_s\lambda)]^2} - \frac{2\lambda^{3/2}[1 - 5\theta_s^4\lambda^4 + r\lambda^3(1 - 2\theta_s\lambda) + C\lambda^5]^{1/2}}{5(1 - \theta_s^4\lambda^4) + 2r\lambda^3(1 - \theta_s\lambda)}. \tag{10}$$

In this form Φ is finite over the entire range of interest except at $\lambda = 1$, $\theta_s = 1$, and $g = 0$. Hence the same numerical procedure can be applied as in the case of Eq. (9).

The Function Λ. When $g > 0$, Λ can be calculated directly using Eq. (6), but for $g = 0$ the argument has a singularity at $\theta = 1$. Integrating by parts, the following equation is obtained:

$$\Lambda(\theta, \theta_s, r, C) = \frac{2[\theta^5 - 5\theta_s^4\theta + r(\theta^2 - 2\theta_s\theta) + C]^{1/2}}{5(\theta^4 - \theta_s^4) + 2r(\theta - \theta_s)} + \frac{2C^{1/2}}{5\theta_s^4 + 2r\theta_s}$$
$$+ 4\int_0^\theta \frac{[\theta^5 - 5\theta_s^4\theta + r(\theta^2 - 2\theta_s\theta) + C]^{1/2}(10\theta^3 + r)\,d\theta}{[5(\theta^4 - \theta_s^4) + 2r(\theta - \theta_s)]^2}. \tag{11}$$

Note that Λ is finite in the entire range of interest except at $\theta = 1$, $\theta_s = 1$, and $g = 0$. The same procedure can be used to evaluate Λ as Φ.

The minimum mass fin. As an example of the use of the solution, consider a fin with minimum mass, constant area, and constant base temperature, T_L, transferring Q amount of heat. For this case the following equation is readily obtained:

$$Q = -[2\sigma EkPA/5]^{1/2}T_L^{5/2}[1 - 5\theta_s^4\lambda_L^4 + r\lambda_L^3(1 - 2\theta_s\lambda_L) + C\lambda_L^5]^{1/2} \tag{12}$$

therefore,

$$A = \frac{5Q^2}{2\sigma EkPT_L^5[1 - 5\theta_s^4\lambda_L^4 + r\lambda_L^3(1 - 2\theta_s\lambda_L) + C\lambda_L^5]} \tag{13}$$

also

$$L = [5kA/2\sigma EPT_L^3]^{1/2}\theta_L^{3/2}[\Phi(1, \theta_s, r, C) - \Phi(\lambda_L, \theta_s, r, C)]. \tag{14}$$

For a rectangular fin of thickness 2δ, $P = 2$, and $A = 2\delta$,

$$\delta = \frac{5Q^2}{8\sigma EkT_L^5[1 - 5\theta_s^4\lambda_L^4 + r\lambda_L^3(1 - 2\theta_s\lambda_L) + C\lambda_L^5]} \tag{15}$$

and

$$L = [5k\,\delta/2\sigma EPT_L^3]^{1/2}\theta_L^{3/2}[\Phi(1, \theta_s, r, C) - \Phi(\lambda_L, \theta_s, r, C)]. \tag{16}$$

Equation (17) is obtained by combining Eqs. (15) and (16).

$$V = 2\,\delta L = \frac{25Q^3}{16\sigma^2E^2kT_L^9}\,\frac{[\Phi(1, \theta_s, r, C) - \Phi(\lambda_L, \theta_s, r, C)]}{\lambda_L^{3/2}[1 - 5\theta_s^4\lambda_L^4 + r\lambda_L^3(1 - 2\theta_s\lambda_L) + C\lambda_L^5]^{3/2}}. \tag{17}$$

For minimum V, the expression

$$\frac{[\Phi(1, \theta_s, r, C) - \Phi(\lambda_L, \theta_s, r, C)]}{\lambda_L^{3/2}[1 - 5\theta_s^4\lambda_L^4 + r\lambda_L^3(1 - 2\theta_s\lambda_L) + C\lambda_L^5]^{3/2}}$$

should be a minimum. Differentiating and equating to zero gives the following equation:

$$\Phi(1, \theta_s, r, C) - \Phi(\lambda_L, \theta_s, r, C) = -\frac{2}{3}\,\frac{\lambda_L^{3/2}[1 - 5\theta_s^4\lambda_L^4 + r\lambda_L^3(1 - 2\theta_s\lambda_L) + C\lambda_L^5]^{1/2}}{[1 - 25\theta_s^4\lambda_L^4 + 2r\lambda_L^3(2 - 5\theta_s\lambda_L) + 6C\lambda_L^5]}. \tag{18}$$

The solution of Eq. (18) gives the condition required for a minimum mass rectangular fin.

For a circular fin of diameter D, $A = \pi D^2/4$ and $P = \pi D$; this leads to the expression

$$V = \frac{1}{16}\left(\frac{10^4 Q^5}{\pi^2 \sigma^4 E^4 k T_L^{17}}\right)^{1/3} \frac{[\Phi(1, \theta_s, r, C) - \Phi(\lambda_L, \theta_s, r, C)]}{\lambda_L^{3/2}[1 - 5\theta_s^4\lambda_L^4 + r\lambda_L^3(1 - 2\theta_s\lambda_L) + C\lambda_L]^{5/6}}. \tag{19}$$

Differentiating and equating to zero, the condition for minimum V is found to be

$$[\Phi(1, \theta_s, r, C) - \Phi(\lambda_L, \theta_s, r, C)] = -\frac{\lambda_L^{3/2}[1 - 5\theta_s^4\lambda_L^4 + r\lambda_L^3(1 - 2\theta_s\lambda_L) + C\lambda_L^5]^{1/2}}{\frac{3}{2} - \frac{145}{6}\theta_s^4\lambda_L^4 + r\lambda_L^3(4 - \frac{29}{3}\theta_s\lambda_L) + \frac{17}{3}C\lambda_L^5}. \tag{20}$$

The solution of Eq. (20) gives the requirements for minimum fin geometry.

Conclusion. An exact general solution is presented for the nonlinear differential equation that describes one-dimensional heat transfer by radiation and convection between a constant area fin and a surrounding space at an equivalent sink temperature. The fin can be either dissipating or receiving heat. The solution results in a number of functions that can be presented in tabulated form, graphed form, or both. The use of the solution is illustrated by considering an optimization problem.

Definition of Symbols

Symbol	Definition
A	Area, ft^2
C	Constant
C'	Constant
D	Diameter, ft
E	Emissivity factor, dimensionless
g	Constant
h	Convection heat transfer coefficient, Btu/hr ft^2 °R
k	Thermal conductivity, Btu/hr ft °R
L	Length, ft
P	Perimeter, ft
Q	Rate of heat transfer, Btu/hr
r	Constant, $5h/2\sigma E T_0^3$
V	Volume, ft^3
T	Temperature, °R
x	Distance, ft
σ	Stefan–Boltzmann constant, 0.1714×10^{-8} Btu/hr ft^2 °R^4
θ	Dimensionless variable, T/T_0
λ	$1/\theta$
δ	Half thickness of rectangular fin, ft
Φ	Function
Λ	Function
	Subscripts
L	Condition at a distance L from origin
0	Condition at origin
s	Condition of surroundings

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