

An Exact Jacobian SDP Relaxation for Polynomial Optimization

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Abstract

Given polynomials $f(x), g_i(x), h_j(x)$, we study how to minimize $f(x)$ on the set

$$S = \{x \in \mathbb{R}^n : h_1(x) = \cdots = h_{m_1}(x) = 0, g_1(x) \geq 0, \dots, g_{m_2}(x) \geq 0\}.$$

Let f_{min} be the minimum of f on S . Suppose S is nonsingular and f_{min} is achievable on S , which are true generically. This paper proposes a new type semidefinite programming (SDP) relaxation which is the first one for solving this problem exactly. First, we construct new polynomials $\varphi_1, \dots, \varphi_r$, by using the Jacobian of f, h_i, g_j , such that the above problem is equivalent to

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & f(x) \\ \text{s.t.} \quad & h_i(x) = 0, \varphi_j(x) = 0, 1 \leq i \leq m_1, 1 \leq j \leq r, \\ & g_1(x)^{\nu_1} \cdots g_{m_2}(x)^{\nu_{m_2}} \geq 0, \quad \forall \nu \in \{0, 1\}^{m_2}. \end{aligned}$$

Second, we prove that for all N big enough, the standard N -th order Lasserre's SDP relaxation is exact for solving this equivalent problem, that is, its optimal value is equal to f_{min} . Some variations and examples are also shown.

Key words determinantal varieties, ideals, minors, polynomials, nonsingularity, semidefinite programming, sum of squares

AMS subject classification 14P99, 65K05, 90C22

1 Introduction

Consider the optimization problem

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & f(x) \\ \text{s.t.} \quad & h_1(x) = \cdots = h_{m_1}(x) = 0 \\ & g_1(x) \geq 0, \dots, g_{m_2}(x) \geq 0 \end{aligned} \tag{1.1}$$

where $f(x), g_i(x), h_j(x)$ are polynomial functions. When $m_1 = 0$ (resp. $m_2 = 0$), there are no equality (resp. inequality) constraints. Let S be its feasible set and f_{min} be its global minimum. We are interested in computing f_{min} . The problem is NP-hard [16].

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A standard approach for solving (1.1) is the hierarchy of semidefinite programming (SDP) relaxations proposed by Lasserre [16]. It is based on a sequence of sum of squares (SOS) type representations of polynomials that are nonnegative on S . The basic idea is, for a given integer $N > 0$ (called relaxation order), solve the SOS program

$$\begin{aligned} \max \quad & \gamma \\ \text{s.t.} \quad & f(x) - \gamma = \sum_{i=1}^{m_1} \phi_i h_i + \sum_{j=1}^{m_2} \sigma_j g_j, \\ & \deg(\phi_i h_i), \deg(\sigma_j g_j) \leq 2N \quad \forall i, j, \\ & \sigma_1, \dots, \sigma_{m_2} \text{ are SOS.} \end{aligned} \tag{1.2}$$

In the above, $g_0(x) \equiv 1$, the decision variables are the coefficients of polynomials ϕ_i and σ_j . Here a polynomial is SOS if it is a sum of squares of other polynomials. The SOS program (1.2) is equivalent to an SDP problem (see [16]). Let p_N be the optimal value of (1.2). Clearly, $p_N \leq f_{min}$ for every N . Using Putinar's Positivstellensatz [21], Lasserre proved $p_N \rightarrow f_{min}$ as $N \rightarrow \infty$, under the archimedean condition. A stronger relaxation than (1.1) would be obtained by using cross products of g_j , which is

$$\begin{aligned} \max \quad & \gamma \\ \text{s.t.} \quad & f(x) - \gamma = \sum_{i=1, \dots, m_1} \phi_i h_i + \sum_{\nu \in \{0,1\}^{m_2}} \sigma_\nu \cdot g_\nu, \\ & \deg(\phi_i h_i) \leq 2N, \deg(\sigma_\nu g_\nu) \leq 2N \quad \forall i, \nu, \\ & \sigma_\nu \text{ are all SOS.} \end{aligned} \tag{1.3}$$

In the above, $g_\nu = g_1^{\nu_1} \cdots g_{m_2}^{\nu_{m_2}}$. Let q_N be the optimal value of (1.3). When S is compact, Lasserre showed $q_N \rightarrow f_{min}$ as N goes to infinity, using Schmügen's Positivstellensatz [24]. An analysis for the convergence speed of p_N, q_N to f_{min} is given in [19, 25]. Typically, (1.2) and (1.3) are not exact for (1.1) with a finite N . Scheiderer [23] proved a very surprising result: whenever S has dimension three or higher, there always exists f such that $f(x) - f_{min}$ does not have a representation required in (1.3). Thus, we usually need to solve a big number of SDPs until convergence is met. This would be very inefficient in many applications. Furthermore, when S is not compact, typically we do not have the convergence $p_N \rightarrow f_{min}$ or $q_N \rightarrow f_{min}$. This is another difficulty. Thus, people are interested in more efficient methods for solving (1.1).

Recently, the author, Demmel and Sturmfels [18] proposed a gradient type SOS relaxation. Consider the case of (1.1) without constraints, i.e., $m_1 = m_2 = 0$. If the minimum f_{min} is achieved at a point u , then $\nabla f(u) = 0$, and the problem is equivalent to

$$\min_{x \in \mathbb{R}^n} f(x) \quad \text{s.t.} \quad \frac{\partial f}{\partial x_1} = \dots = \frac{\partial f}{\partial x_n} = 0. \tag{1.4}$$

In [18], Lasserre's relaxation is applied to solve (1.4), and it was shown that a sequence of lower bounds converging to f_{min} would be obtained. It has finite convergence if the gradient ideal, generated by the partial derivatives of $f(x)$, is radical. More recently, Demmel, the author and Powers [7] generalized the gradient SOS relaxation to solve (1.1) by using the Karush-Kuhn-Tucker (KKT) conditions of (1.1)

$$\nabla f(x) = \sum_{i=1}^{m_1} \lambda_i \nabla h_i(x) + \sum_{j=1}^{m_2} \mu_j \nabla g_j(x), \quad \mu_j g_j(x) = 0, \quad j = 1, \dots, m_2.$$

If a global minimizer of (1.1) is a KKT point, then (1.1) is equivalent to

$$\begin{aligned}
& \min_{x, \lambda, \mu} f(x) \\
& \text{s.t. } h_1(x) = \cdots = h_{m_1}(x) = 0, \\
& \quad \nabla f(x) = \sum_{i=1}^{m_1} \lambda_i \nabla h_i(x) + \sum_{j=1}^{m_2} \mu_j \nabla g_j(x), \\
& \quad \mu_j g_j(x) = 0, \quad g_j(x) \geq 0, \quad j = 1, \dots, m_2.
\end{aligned} \tag{1.5}$$

Let $\{v_N\}$ be the sequence of lower bounds for (1.5) obtained by applying Lasserre's relaxation of type (1.3). It was shown in [7] that $v_N \rightarrow f_{min}$, no matter S is compact or not. Furthermore, it holds that $v_N = f_{min}$ for a finite N when the KKT ideal is radical, but it was unknown in [7] whether this property still holds without the KKT ideal being radical. A drawback for this approach is that the involved polynomials are in (x, λ, μ) . There are totally $n + m_1 + m_2$ variables, which makes the resulting SDP very difficult to solve in practice.

Contributions This paper proposes a new type SDP relaxation for solving (1.1) via using KKT conditions but the involved polynomials are only in x . Suppose S satisfies a nonsingularity assumption (see Assumption 2.2 for its meaning) and f_{min} is achievable on S , which are true generically. We construct new polynomials $\varphi_1(x), \dots, \varphi_r(x)$, by using the minors of the Jacobian of f, h_i, g_j , such that (1.1) is equivalent to

$$\begin{aligned}
& \min_{x \in \mathbb{R}^n} f(x) \\
& \text{s.t. } h_i(x) = 0 \ (1 \leq i \leq m_1), \ \varphi_j(x) = 0 \ (1 \leq j \leq r), \\
& \quad g_1(x)^{\nu_1} \cdots g_{m_2}(x)^{\nu_{m_2}} \geq 0, \quad \forall \nu \in \{0, 1\}^{m_2}.
\end{aligned}$$

Then we prove that for all N big enough, the standard N -th order Lasserre's relaxation for the above returns the minimum f_{min} . That is, an exact SDP relaxation for (1.1) is obtained by using the Jacobian.

This paper is organized as follows. Section 2 gives the construction of this exact SDP relaxation by using Jacobian. Its exactness and genericity are proved in Section 3. Some efficient variations are proposed in Section 4. Some examples of how to apply it are shown in Section 5. Some conclusions and discussions are made in Section 6. Finally, we attach an appendix introducing some basics of algebraic geometry and real algebra that are used in the paper.

Notations The symbol \mathbb{N} (resp., \mathbb{R} , \mathbb{C}) denotes the set of nonnegative integers (resp., real numbers, complex numbers). For any $t \in \mathbb{R}$, $\lceil t \rceil$ denotes the smallest integer not smaller than t . For integer $n > 0$, $[n]$ denotes the set $\{1, \dots, n\}$, and $[n]_k$ denotes the set of subsets of $[n]$ whose cardinality is k . For a subset J of $[n]$, $|J|$ denotes its cardinality. For $x \in \mathbb{R}^n$, x_i denotes the i -th component of x , that is, $x = (x_1, \dots, x_n)$. For $\alpha \in \mathbb{N}^n$, denote $|\alpha| = \alpha_1 + \cdots + \alpha_n$. For $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{N}^n$, x^α denotes $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. The symbol $\mathbb{R}[x] = \mathbb{R}[x_1, \dots, x_n]$ (resp. $\mathbb{C}[x] = \mathbb{C}[x_1, \dots, x_n]$) denotes the ring of polynomials in (x_1, \dots, x_n) with real (resp. complex) coefficients. A polynomial is called a form if it is homogeneous. The $\mathbb{R}[x]_{\leq d}$ denotes the subspace of polynomials in $\mathbb{R}[x]$ of degrees at most d . For a general set $T \subseteq \mathbb{R}^n$, $\text{int}(T)$ denotes its interior, and ∂T denotes its boundary in standard Euclidean topology. For a symmetric matrix X , $X \succeq 0$ (resp., $X \succ 0$) means X is positive semidefinite (resp. positive definite). For $u \in \mathbb{R}^N$, $\|u\|_2$ denotes the standard Euclidean norm.

2 Construction of the exact Jacobian SDP relaxation

Let S be the feasible set of (1.1) and

$$m = \min\{m_1 + m_2, n - 1\}. \quad (2.1)$$

For convenience, we denote $h(x) = (h_1(x), \dots, h_{m_1}(x))$ and $g(x) = (g_1(x), \dots, g_{m_2}(x))$. For a subset $J = \{j_1, \dots, j_k\} \subset [m_2]$, denote

$$g_J(x) = (g_{j_1}(x), \dots, g_{j_k}(x)).$$

Let x^* be a minimizer of (1.1). If J is the active index set at x^* such that $g_J(x^*) = 0$ and the KKT conditions hold at x^* , then there exist λ_i and $\mu_j (j \in J)$ such that

$$h(x^*) = 0, \quad g_J(x^*) = 0, \quad \nabla f(x^*) = \sum_{i \in [m_1]} \lambda_i \nabla h_i(x^*) + \sum_{j \in J} \mu_j \nabla g_j(x^*).$$

The above implies the Jacobian matrix of (f, h, g_J) is singular at x^* . For a subset $J \subset [m_2]$, denote the determinantal variety of (f, h, g_J) 's Jacobian being singular by

$$G_J = \{x \in \mathbb{C}^n : \text{rank } B^J(x) \leq m_1 + |J|\}, \quad B^J(x) = [\nabla f(x) \quad \nabla h(x) \quad \nabla g_J(x)]. \quad (2.2)$$

Then, $x^* \in V(h, g_J) \cap G_J$ where $V(h, g_J) := \{x \in \mathbb{C}^n : h(x) = 0, g_J(x) = 0\}$.

This motivates us to use $g_J(x) = 0$ and G_J to get tighter SDP relaxations for (1.1). To do so, a practical issue is how to get a ‘‘nice’’ description for G_J ? An obvious one is that all its maximal minors vanish. But there are totally $\binom{n}{m_1+k+1}$ such minors (if $m_1 + k + 1 \leq n$), which is huge for big n, m_1, k . Can we define G_J by a set of the smallest number of equations? Furthermore, the active index set J is usually unknown in advance. Can we get an SDP relaxation that is independent of J ?

2.1 Minimum defining equations for determinantal varieties

Let $k \leq n$ and $X = (X_{ij})$ be a $n \times k$ matrix of indeterminants X_{ij} . Define the determinantal variety

$$D_{t-1}^{n,k} = \left\{ X \in \mathbb{C}^{n \times k} : \text{rank } X < t \right\}.$$

For any index set $I = \{i_1, \dots, i_k\} \subset [n]$, denote by $\det_I(X)$ the $(i_1, \dots, i_k) \times (1, \dots, k)$ -minor of matrix X , i.e., the determinant of the submatrix of X whose row indices are i_1, \dots, i_k and column indices are $1, \dots, k$. Clearly, it holds that

$$D_{k-1}^{n,k} = \left\{ X \in \mathbb{C}^{n \times k} : \det_I(X) = 0 \quad \forall I \in [n]_k \right\}.$$

The above has $\binom{n}{k}$ defining equations of degree k . An interesting fact is that we do not need $\binom{n}{k}$ equations to define $D_{k-1}^{n,k}$. Actually, this number would be significantly smaller. There is very nice work on this issue. Bruns and Vetter [3] showed that $nk - t^2 + 1$ equations are enough to define $D_{t-1}^{n,k}$. Later, Bruns and Schwänzl [2] showed that $nk - t^2 + 1$ is the smallest number of equations for defining $D_{t-1}^{n,k}$. Typically, $nk - t^2 + 1 \ll \binom{n}{k}$ for big n and k . A general

method for constructing $nk - t^2 + 1$ defining polynomial equations for $D_{t-1}^{n,k}$ was described in Chapt. 5 of [3]. Here we briefly show how it works for $D_{k-1}^{n,k}$.

Let $\Gamma(X)$ denote the set of all k -minors of X (assume their row indices are strictly increasing). For convenience, for $i_1 < \dots < i_k$, denote by $[i_1, \dots, i_k]$ the $(i_1, \dots, i_k) \times (1, \dots, k)$ -minor of X . Define a partial ordering on $\Gamma(X)$ as follows:

$$[i_1, \dots, i_k] < [j_1, \dots, j_k] \iff i_1 \leq j_1, \dots, i_k \leq j_k, i_1 + \dots + i_k < j_1 + \dots + j_k.$$

If $I = \{i_1, \dots, i_k\}$, we also write $I = [i_1, \dots, i_k]$ as a minor in $\Gamma(X)$ for convenience. For any $I \in \Gamma(X)$, define its rank as

$$rk(I) = \max \left\{ \ell : I = I^{(\ell)} > \dots > I^{(1)}, \quad \text{every } I^{(i)} \in \Gamma(X) \right\}.$$

The maximum minor in $\Gamma(X)$ is $[n - k + 1, \dots, n]$ and has rank $nk - k^2 + 1$. For every $1 \leq \ell \leq nk - k^2 + 1$, define

$$\eta_\ell(X) = \sum_{I \in [n]_k, rk(I) = \ell} \det_I(X). \quad (2.3)$$

Lemma 2.1 (Lemma (5.9), Bruns and Vetter [3]). *It holds that*

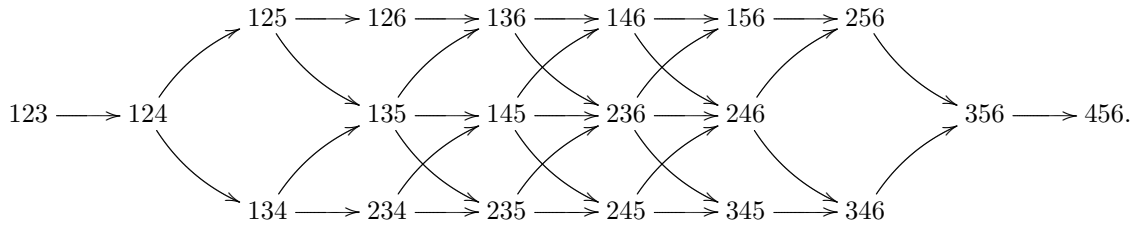
$$D_{k-1}^{n,k} = \left\{ X \in \mathbb{C}^{n \times k} : \eta_\ell(X) = 0, \ell = 1, \dots, nk - k^2 + 1 \right\}.$$

When $k = 2$, $D_1^{n,2}$ would be defined by $2n - 3$ polynomials. The biggest minor is $[n - 1, n]$ and has rank $2n - 3$. For each $\ell = 1, 2, \dots, 2n - 3$, we clearly have

$$\eta_\ell(X) = \sum_{1 \leq i_1 < i_2 \leq n: i_1 + i_2 = \ell + 2} [i_1, i_2].$$

Every 2-minor of X is a summand of some $\eta_\ell(X)$.

When $k = 3$, $D_2^{n,3}$ can be defined by $3n - 8$ polynomials of the form $\eta_\ell(X)$. For instance, when $n = 6$, the partial ordering on $\Gamma(X)$ is shown in the following diagram:



In the above, an arrow points to a bigger minor. Clearly, we have the expressions

$$\begin{aligned} \eta_1(X) &= [1, 2, 3], & \eta_2(X) &= [1, 2, 4], & \eta_3(X) &= [1, 2, 5] + [1, 3, 4], \\ \eta_4(X) &= [1, 2, 6] + [1, 3, 5] + [2, 3, 4], & \eta_5(X) &= [1, 3, 6] + [1, 4, 5] + [2, 3, 5], \\ \eta_6(X) &= [1, 4, 6] + [2, 3, 6] + [2, 4, 5], & \eta_7(X) &= [1, 5, 6] + [2, 4, 6] + [3, 4, 5], \\ \eta_8(X) &= [2, 5, 6] + [3, 4, 6], & \eta_9(X) &= [3, 5, 6], & \eta_{10}(X) &= [4, 5, 6]. \end{aligned}$$

Every above $\eta_i(X)$ has degree 3. Note that the summands $[i_1, i_2, i_3]$ from the same $\eta_i(X)$ have a constant summation $i_1 + i_2 + i_3$. Thus, for each $\ell = 1, \dots, 3n - 8$, we have

$$\eta_\ell(X) = \sum_{1 \leq i_1 < i_2 < i_3 \leq n: i_1 + i_2 + i_3 = \ell + 5} [i_1, i_2, i_3].$$

When $k > 3$ is general, $D_{k-1}^{n,k}$ can be defined by $nk - k^2 + 1$ polynomials of the form $\eta_\ell(X)$. For each $\ell = 1, 2, \dots, nk - k^2 + 1$, we similarly have the expression

$$\eta_\ell(X) = \sum_{1 \leq i_1 < \dots < i_k \leq n: i_1 + \dots + i_k = \ell + \binom{k+1}{2} - 1} [i_1, \dots, i_k].$$

2.2 The exact Jacobian SDP relaxation

For every $J = \{j_1, \dots, j_k\} \subset [m_2]$ with $k \leq m - m_1$, by applying formula (2.3), let

$$\eta_1^J, \dots, \eta_{\text{len}(J)}^J \quad \left(\text{len}(J) := n(m_1 + k + 1) - (m_1 + k + 1)^2 + 1 \right)$$

be the set of defining polynomials for the determinantal variety G_J defined in (2.2) of the Jacobian of (f, h, g_J) being singular. For each $i = 1, \dots, \text{len}(J)$, define

$$\varphi_i^J(x) = \eta_i^J \left(B^J(x) \right) \cdot \prod_{j \in J^c} g_j(x), \quad \text{where } J^c = [m_2] \setminus J. \quad (2.4)$$

Using the product $\prod_{j \in J^c} g_j(x)$ in the above is motivated by a characterization of critical points in [15]. For simplicity, list all possible φ_i^J in (2.4) sequentially as

$$\varphi_1, \varphi_2, \dots, \varphi_r, \quad \text{where } r = \sum_{J \subset [m_2], |J| \leq m - m_1} \text{len}(J). \quad (2.5)$$

Now define the variety

$$W = \{x \in \mathbb{C}^n : h_1(x) = \dots = h_{m_1}(x) = \varphi_1(x) = \dots = \varphi_r(x) = 0\}. \quad (2.6)$$

If the minimum f_{\min} of (1.1) is achieved at a KKT point, then (1.1) is equivalent to

$$\begin{aligned} & \min_{x \in \mathbb{R}^n} f(x) \\ & \text{s.t. } h_i(x) = 0 \ (1 \leq i \leq m_1), \ \varphi_j(x) = 0 \ (1 \leq j \leq r), \\ & \quad g_\nu(x) \geq 0, \ \forall \nu \in \{0, 1\}^{m_2}. \end{aligned} \quad (2.7)$$

In the above, each $g_\nu = g_1^{\nu_1} \dots g_{m_2}^{\nu_{m_2}}$. Let f^* be the minimum of (2.7). If (1.1) has a minimizer that is a KKT point, then $f_{\min} = f^*$. If (1.1) does not have a minimizer (i.e., f_{\min} is not achievable in (1.1)), or if (1.1) has one or several minimizers but none of them is a KKT point, then we might have $f_{\min} < f^*$ (e.g., for $f(x) = x_1^2 + (1 - x_1 x_2)^2$ and $S = \mathbb{R}^2$, $f_{\min} = 0 < f^* = 1$ [18]). But in any case the minimum f^* is always finite as will be shown by Theorem 2.3, while f_{\min} might not.

To construct an SDP relaxation for (2.7), we need to define localizing moment matrices. Let $q(x)$ be a polynomial with $\deg(q) \leq 2N$. Define symmetric matrices $A_\alpha^{(N)}$ such that

$$q(x)[x]_d[x]_d^T = \sum_{\alpha \in \mathbb{N}^n: |\alpha| \leq 2N} A_\alpha^{(N)} x^\alpha, \quad \text{where } d = N - \lceil \deg(q)/2 \rceil.$$

Then the N -th order localizing moment matrix of q is defined as

$$L_q^{(N)}(y) = \sum_{\alpha \in \mathbb{N}^n: |\alpha| \leq 2N} A_\alpha^{(N)} y_\alpha.$$

Here y is a moment vector indexed by $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq 2N$. Moreover, denote

$$L_f(y) = \sum_{\alpha \in \mathbb{N}^n: |\alpha| \leq \deg(f)} f_\alpha y_\alpha \quad \text{for} \quad f(x) = \sum_{\alpha \in \mathbb{N}^n: |\alpha| \leq \deg(f)} f_\alpha x^\alpha.$$

The N -th order Lasserre's relaxation for (2.7) is the SDP

$$\begin{aligned} f_N^{(1)} := \min & \quad L_f(y) \\ \text{s.t.} & \quad L_{h_i}^{(N)}(y) = 0 \quad (1 \leq i \leq m_1), \quad L_{\varphi_j}^{(N)}(y) = 0 \quad (1 \leq j \leq r), \\ & \quad L_{g_\nu}^{(N)}(y) \succeq 0, \quad \forall \nu \in \{0, 1\}^{m_2}, \quad y_0 = 1. \end{aligned} \quad (2.8)$$

Compared to Schmüdgen type Lasserre's relaxation (1.3), the number of new constraints in (2.8) is $r = O(2^{m_2} \cdot n \cdot (m_1 + m_2))$ by (2.5). That is, r is of linear order in nm_1 for fixed m_2 , but is exponential in m_2 . So, when m_2 is small or moderately large, (2.8) is practical; but for big m_2 , (2.8) becomes more difficult to solve numerically. Now we present the dual of (2.8). Define the truncated preordering $P^{(N)}$ generated by g_j as

$$P^{(N)} = \left\{ \sum_{\nu \in \{0,1\}^{m_2}} \sigma_\nu g_\nu \mid \deg(\sigma_\nu g_\nu) \leq 2N \right\}, \quad (2.9)$$

and the truncated ideal $I^{(N)}$ generated by h_i and φ_j as

$$I^{(N)} = \left\{ \sum_{i=1}^{m_1} p_i h_i + \sum_{j=1}^r q_j \varphi_j \mid \begin{array}{l} \deg(p_i h_i) \leq 2N \quad \forall i \\ \deg(q_j \varphi_j) \leq 2N \quad \forall j \end{array} \right\}. \quad (2.10)$$

Then, as shown in Lasserre [16], the dual of (2.8) is the following SOS relaxation for (2.7):

$$\begin{aligned} f_N^{(2)} := \max & \quad \gamma \\ \text{s.t.} & \quad f(x) - \gamma \in I^{(N)} + P^{(N)}. \end{aligned} \quad (2.11)$$

Note the relaxation (2.11) is stronger than (1.3). Then, by weak duality, we have

$$f_N^{(2)} \leq f_N^{(1)} \leq f^*. \quad (2.12)$$

We are going to show that when N is big enough, (2.8) is an exact SDP relaxation for (2.7), i.e., $f_N^{(2)} = f_N^{(1)} = f^*$. For this purpose, we need the following assumption.

Assumption 2.2. (i) $m_1 \leq n$. (ii) For any $u \in S$, at most $n - m_1$ of $g_1(u), \dots, g_{m_2}(u)$ vanish. (iii) For every $J = \{j_1, \dots, j_k\} \subset [m_2]$ with $k \leq n - m_1$, the variety $V(h, g_J) = \{x \in \mathbb{C}^n : h(x) = 0, g_J(x) = 0\}$ is nonsingular (its Jacobian has full rank on $V(h, g_J)$).

Theorem 2.3. Suppose Assumption 2.2 holds. Let f^* be the minimum of (2.7). Then $f^* > -\infty$ and there exists $N^* \in \mathbb{N}$ such that $f_N^{(1)} = f_N^{(2)} = f^*$ for all $N \geq N^*$. Furthermore, if the minimum f_{min} of (1.1) is achievable, then $f_N^{(1)} = f_N^{(2)} = f_{min}$ for all $N \geq N^*$.

The proof of Theorem 2.3 is based on a new kind of Positivstellensatz using Jacobians of the objective and constraining polynomials: there exists an integer $N^* > 0$, which only depends on polynomials f, h_i, g_j , such that for all $\epsilon > 0$

$$f(x) - f^* + \epsilon \in I^{(N^*)} + P^{(N^*)}.$$

Note that the order N^* in the above is independent of ϵ . This new Positivstellensatz is given by Theorem 3.4 in the next section. Theorem 2.3 as well as this Positivstellensatz will be proved in Section 3. We would like to remark that in Theorem 2.3, for $N > N^*$, the optimal value of (2.11) might not be achievable (e.g., see Example 5.1), while the minimum of (2.8) is always achievable if (1.1) has a minimizer (e.g., $[x^*]_{2N}$ is one for any optimizer x^* of (1.1)).

When the feasible set S of (1.1) is compact, the minimum f_{min} is always achievable. Thus, Theorem 2.3 implies the following.

Corollary 2.4. Suppose Assumption 2.2 holds. If S is compact, then $f_N^{(1)} = f_N^{(2)} = f_{min}$ for all N big enough.

A practical issue in applications is how to identify whether (2.8) is exact for a given N . This would be possible by applying the flat-extension condition (FEC) [6]. Let y^* be a minimizer of (2.8). We say y^* satisfies FEC if

$$\begin{aligned} \text{rank } L_{g_0}^{(N)}(y^*) &= \text{rank } L_{g_0}^{(N-d_S)}(y^*), \quad \text{where} \\ d_S &= \max_{i \in [m_1], j \in [r], \nu \in \{0,1\}^{m_2}} \left\{ \lceil \deg(h_i)/2 \rceil, \lceil \deg(\varphi_j)/2 \rceil, \lceil \deg(g_\nu)/2 \rceil \right\}. \end{aligned}$$

Note that $g_0 \equiv 1$ and $L_{g_0}^{(N)}(y^*)$ reduces to an N -th order moment matrix. When FEC holds, (2.8) is exact for (1.1), and a finite set of global minimizers would be extracted from y^* . We refer to [13] for a numerical method on how to do this. A very nice software for solving SDP relaxations from polynomial optimization is *GloptiPoly 3* [14] which also provides routines for finding minimizers if FEC holds.

Now we discuss how general the conditions of Theorem 2.3 are. Define

$$B_d(S) = \left\{ f \in \mathbb{R}[x]_{\leq d} : \inf_{u \in S} f(u) > -\infty \right\}.$$

Clearly, $B_d(S)$ is convex and has nonempty interior. Define the projectivization of S as

$$S^{prj} = \left\{ \tilde{x} \in \mathbb{R}^{n+1} : \tilde{h}_1(\tilde{x}) = \dots = \tilde{h}_{m_1}(\tilde{x}) = 0, \quad \tilde{g}_1(\tilde{x}) \geq 0, \dots, \tilde{g}_{m_2}(\tilde{x}) \geq 0 \right\}. \quad (2.13)$$

Here \tilde{p} denotes the homogenization of p , and $\tilde{x} = (x_0, x_1, \dots, x_n)$, i.e., $\tilde{p}(\tilde{x}) = x_0^{\deg(p)} p(x/x_0)$. We say S is closed at ∞ if

$$S^{prj} \cap \{x_0 \geq 0\} = \text{closure}(S^{prj} \cap \{x_0 > 0\}).$$

Under some generic conditions, Assumption 2.2 holds and the minimum f_{min} of (1.1) is achievable. These conditions are expressed as non-vanishing of the so-called *resultants* Res or *discriminants* Δ , which are polynomial in the coefficients of f, h_i, g_j . We refer to Appendix for a short introduction about Res and Δ .

Theorem 2.5. *Let f, h_i, g_j be the polynomials in (1.1), and S be the feasible set.*

- (a) *If $m_1 > n$ and $Res(h_{i_1}, \dots, h_{i_{n+1}}) \neq 0$ for some $\{i_1, \dots, i_{n+1}\}$, then $S = \emptyset$.*
- (b) *If $m_1 \leq n$ and for every $\{j_1, \dots, j_{n-m_1+1}\} \subset [m_2]$*

$$Res(h_1, \dots, h_{m_1}, g_{j_1}, \dots, g_{j_{n-m_1+1}}) \neq 0,$$

then item (ii) of Assumption 2.2 holds.

- (c) *If $m_1 \leq n$ and for every $\{j_1, \dots, j_k\} \subset [m_2]$ with $k \leq n - m_1$*

$$\Delta(h_1, \dots, h_{m_1}, g_{j_1}, \dots, g_{j_k}) \neq 0,$$

then item (iii) of Assumption 2.2 holds.

- (d) *Suppose S is closed at ∞ and $f \in B_d(S)$. If the resultant of any n of h_i^{hom}, g_j^{hom} is nonzero (only when $m_1 + m_2 \geq n$), and for every $\{j_1, \dots, j_k\}$ with $k \leq n - m_1 - 1$*

$$\Delta(f^{hom}, h_1^{hom}, \dots, h_{m_1}^{hom}, g_{j_1}^{hom}, \dots, g_{j_k}^{hom}) \neq 0,$$

then there exists $v \in S$ such that $f_{min} = f(v)$. Here p^{hom} denotes p 's homogeneous part of the highest degree.

- (e) *If $f \in B_d(\mathbb{R}^n)$ and $\Delta(f^{hom}) \neq 0$, then the minimum of $f(x)$ in \mathbb{R}^n is achievable.*

Theorem 2.5 will be proved in Section 3. Now we consider the special case of (1.1) having no constraints. If $f_{min} > -\infty$ is achievable, then (1.1) is equivalent to (1.4). The item (e) of Theorem 2.5 tells us that this is generically true. The gradient SOS relaxation for solving (1.4) proposed in [18] is a special case of (2.11). The following is an immediate consequence of Theorem 2.3 and item (e) of Theorem 2.5.

Corollary 2.6. *If $S = \mathbb{R}^n$, $f(x)$ has minimum $f_{min} > -\infty$, and $\Delta(f^{hom}) \neq 0$, then the optimal values of (2.8) and (2.11) are equal to f_{min} if N is big enough.*

Corollary 2.6 is stronger than Theorem 10 of [18], where the exactness of gradient SOS relaxation for a finite order N is only shown when the gradient ideal is radical.

3 Proof of exactness and genericity

This section proves Theorems 2.3 and 2.5. First, we give some lemmas that are crucially used in the proofs.

Lemma 3.1. *Let K be the variety defined by the KKT conditions*

$$K = \left\{ (x, \lambda, \mu) \in \mathbb{C}^{n+m_1+m_2} \left| \begin{array}{l} \nabla f(x) = \sum_{i=1}^{m_1} \lambda_i \nabla h_i(x) + \sum_{j=1}^{m_2} \mu_j \nabla g_j(x) \\ h_i(x) = \mu_j g_j(x) = 0, \forall (i, j) \in [m_1] \times [m_2] \end{array} \right. \right\}. \quad (3.1)$$

If Assumption 2.2 holds, then $W = K_x$ where

$$K_x = \{x \in \mathbb{C}^n : (x, \lambda, \mu) \in K \text{ for some } \lambda, \mu\}.$$

Proof. First, we prove $W \subset K_x$. Choose an arbitrary $u \in W$. Let $J = \{j \in [m_2] : g_j(u) = 0\}$ and $k = |J|$. By Assumption 2.2, $m_1 + k \leq n$. Recall from (2.2) that

$$B^J(x) = [\nabla f(x) \quad \nabla h(x) \quad \nabla g_J(x)].$$

Case $m_1 + k = n$ By Assumption 2.2, the matrix $H(u) = [\nabla h(u) \quad \nabla g_J(u)]$ is nonsingular. Note that $H(u)$ is now a square matrix. So, $H(u)$ is invertible, and there exist λ_i and $\mu_j (j \in J)$ such that

$$\nabla f(u) = \sum_{i=1}^{m_1} \lambda_i \nabla h_i(u) + \sum_{j \in J} \mu_j \nabla g_j(u). \quad (3.2)$$

Define $\mu_j = 0$ for $j \notin J$, then we have $u \in K_x$.

Case $m_1 + k < n$ By the construction of polynomials $\varphi_i(x)$ in (2.5), some of them are

$$\varphi_i^J(x) := \eta_i(B^J(x)) \cdot \prod_{j \in J^c} g_j(x), \quad i = 1, \dots, n(m_1 + k + 1) - (m_1 + k + 1)^2 + 1.$$

So the equations $\varphi_i(u) = 0$ imply every above $\varphi_i^J(u) = 0$ (see its definition in (2.4)). Hence $B^J(u)$ is singular. By Assumption 2.2, the matrix $H(u)$ is nonsingular. So there exist λ_i and $\mu_j (j \in J)$ satisfying (3.2). Define $\mu_j = 0$ for $j \notin J$, then we also have $u \in K_x$.

Second, we prove $K_x \subset W$. Choose an arbitrary $u \in K_x$ with $(u, \lambda, \mu) \in K$. Let $I = \{j \in [m_2] : g_j(u) = 0\}$. If $I = \emptyset$, then $\mu = 0$, and $[\nabla f(u) \quad \nabla h(u)]$ and $B^J(u)$ are both singular, which implies all $\varphi_i(u) = 0$ and $u \in W$. If $I \neq \emptyset$, write $I = \{i_1, \dots, i_t\}$. Let $J = \{j_1, \dots, j_k\} \subset [m_2]$ be an arbitrary index set with $m_1 + k \leq m$.

Case $I \not\subseteq J$ At least one $j \in J^c$ belongs to I . By choice of I , we know from (2.4)

$$\varphi_i^J(u) = \eta_i(B^J(u)) \cdot \prod_{j \in J^c} g_j(u) = 0.$$

Case $I \subseteq J$ Then $\mu_j = 0$ for all $j \in J^c$. By definition of K , the matrix $B^J(u)$ must be singular. All polynomials $\varphi_i^J(x)$ vanish at u by their construction.

Combining the above two cases, we know all $\varphi_i^J(x)$ vanish at u , that is, $\varphi_1(u) = \dots = \varphi_r(u) = 0$. So $u \in W$. \square

Lemma 3.2. *Suppose Assumption 2.2 holds. Let W be defined in (2.6), and $T = \{x \in \mathbb{R}^n : g_j(x) \geq 0, j = 1, \dots, m_2\}$. Then there exist disjoint subvarieties W_0, W_1, \dots, W_r of W and distinct $v_1, \dots, v_r \in \mathbb{R}$ such that*

$$W = W_0 \cup W_1 \cup \dots \cup W_r, \quad W_0 \cap T = \emptyset, \quad W_i \cap T \neq \emptyset, i = 1, \dots, r,$$

and $f(x)$ is constantly equal to v_i on W_i for $i = 1, \dots, r$.

Proof. Let $K = K_1 \cup \dots \cup K_r$ be a decomposition of irreducible varieties. Then $f(x)$ is equaling a constant v_i on each K_i , as shown by Lemma 3.3 in [7]. By grouping all K_i on which v_i are same into a single variety, we can assume all v_i are distinct. Let \widehat{W}_i be the projection of K_i into x -space, then by Lemma 3.1 we get

$$W = \widehat{W}_1 \cup \dots \cup \widehat{W}_r.$$

Let $W_i = \text{Zar}(\widehat{W}_i)$. Applying Zariski closure in the above gives

$$W = \text{Zar}(W) = W_1 \cup \dots \cup W_r.$$

Note that $f(x)$ still achieves a constant value on each W_i . Group all W_j for which $W_j \cap T = \emptyset$ into a single variety W_0 (if every $W_j \cap T \neq \emptyset$ we set $W_0 = \emptyset$). For convenience, we still write the resulting decomposition as $W = W_0 \cup W_1 \cup \dots \cup W_r$. Clearly, $W_0 \cap T = \emptyset$, and for $i > 0$ the values v_i are real and distinct (because $\emptyset \neq W_i \cap T \subset \mathbb{R}^n$ and $f(x)$ has real coefficients). Since $f(x)$ achieves distinct values on different W_i , we know W_i must be disjoint from each other. Therefore, we get a desired decomposition for W . \square

Lemma 3.3. *Let I_0, I_1, \dots, I_k be ideals of $\mathbb{R}[x]$ such that $V(I_i) \cap V(I_j) = \emptyset$ for distinct i, j , and $I = I_0 \cap I_1 \cap \dots \cap I_k$. Then there exist $a_0, a_1, \dots, a_k \in \mathbb{R}[x]$ satisfying*

$$a_0^2 + \dots + a_k^2 - 1 \in I, \quad a_i \in \bigcap_{i \neq j \in \{0, \dots, k\}} I_j.$$

Proof. We prove by induction. When $k = 1$, by Theorem A.2, there exist $p \in I_0, q \in I_1$ such that $p + q = 1$. Then $a_0 = p, a_1 = q$ satisfy the lemma.

Suppose the lemma is true for $k = t$. We prove it is also true for $k = t + 1$. Let $J = I_0 \cap \dots \cap I_t$. By induction, there exist $b_0, \dots, b_t \in \mathbb{R}[x]$ such that

$$b_0^2 + \dots + b_t^2 - 1 \in J, \quad b_i \in \bigcap_{i \neq j \in \{0, \dots, t\}} I_j, \quad i = 0, \dots, t.$$

Since $V(I_{t+1})$ is disjoint from $V(J) = V(I_0) \cup \dots \cup V(I_t)$, by Theorem A.2, there exist $p \in I_{t+1}$ and $q \in J$ such that $p + q = 1$. Let $a_i = b_i p$ for $i = 0, \dots, t$ and $a_{t+1} = q$. Then

$$a_i \in \bigcap_{i \neq j \in \{0, \dots, t+1\}} I_j, \quad i = 0, \dots, t+1.$$

Since $(p + q)^2 = 1$, $I = I_{t+1} \cap J$, we have $pq \in I$, $(b_0^2 + \dots + b_t^2 - 1)p^2 \in I$, and

$$a_0^2 + a_1^2 + \dots + a_{t+1}^2 - 1 = (b_0^2 + \dots + b_t^2 - 1)p^2 - 2pq \in I,$$

which completes the proof. \square

Theorem 3.4. *Suppose Assumption 2.2 holds and let f^* be the minimum of (2.7). Then $f^* > -\infty$ and there exists $N^* \in \mathbb{N}$ such that for all $\epsilon > 0$*

$$f(x) - f^* + \epsilon \in I^{(N^*)} + P^{(N^*)}. \quad (3.3)$$

Proof. Note that the feasible set of (2.7) is contained in the variety W defined by (2.6). Decompose W as in Lemma 3.2. Thus, $f(x)$ achieves finitely many values on W and

$$f^* = \min\{v_1, \dots, v_r\} > -\infty.$$

So, we can generally assume $f^* = 0$. Reorder W_i such that $v_1 > v_2 > \dots > v_r = 0$. The ideal

$$I_W = \langle h_1, \dots, h_{m_1}, \varphi_1, \dots, \varphi_r \rangle \quad (3.4)$$

has a primary decomposition (see Sturmfels [28, Chapter 5])

$$I_W = E_0 \cap E_1 \cap \dots \cap E_r$$

such that each ideal $E_i \subset \mathbb{R}[x]$ has variety $W_i = V(E_i)$.

When $i = 0$, we have $V_{\mathbb{R}}(E_0) \cap T = \emptyset$ (T is defined in Lemma 3.2). By Theorem A.3, there exist SOS polynomials τ_ν satisfying

$$-1 \equiv \sum_{\nu \in \{0,1\}^{m_2}} \tau_\nu \cdot g_\nu(x) \pmod{E_0}.$$

Thus, from $f = \frac{1}{4}(f+1)^2 - \frac{1}{4}(f-1)^2$, we have

$$\begin{aligned} f &\equiv \frac{1}{4} \left\{ (f+1)^2 + (f-1)^2 \sum_{\nu \in \{0,1\}^{m_2}} \tau_\nu \cdot g_\nu \right\} \pmod{E_0} \\ &\equiv \sum_{\nu \in \{0,1\}^{m_2}} \hat{\tau}_\nu \cdot g_\nu \pmod{E_0} \end{aligned}$$

for certain SOS polynomials $\hat{\tau}_\nu$. Let

$$\sigma_0 = \epsilon + \sum_{\nu \in \{0,1\}^{m_2}} \hat{\tau}_\nu \cdot g_\nu.$$

Clearly, if $N_0 > 0$ is big enough, then $\sigma_0 \in P^{(N_0)}$ for all $\epsilon > 0$. Let $q_0 = f + \epsilon - \sigma_0 \in E_0$, which is independent of ϵ .

For each $i = 1, \dots, r-1$, $v_i > 0$ and $v_i^{-1}f(x) - 1$ vanishes on W_i . By Theorem A.1, there exists $k_i > 0$ such that $(v_i^{-1}f(x) - 1)^{k_i} \in E_i$. Thus, it holds that

$$s_i(x) := \sqrt{v_i} \left(1 + (v_i^{-1}f(x) - 1) \right)^{1/2} \equiv \sqrt{v_i} \sum_{j=0}^{k_i-1} \binom{1/2}{j} (v_i^{-1}f(x) - 1)^j \pmod{E_i}.$$

Let $\sigma_i = s_i(x)^2 + \epsilon$, and $q_i = f + \epsilon - \sigma_i \in E_i$, which is also independent of $\epsilon > 0$.

When $i = r$, $v_r = 0$ and $f(x)$ vanishes on W_r . By Theorem A.1, there exists $k_r > 0$ such that $f(x)^{k_r} \in E_r$. Thus we obtain that

$$s_r(x) := \sqrt{\epsilon} (1 + \epsilon^{-1} f(x))^{1/2} \equiv \sqrt{\epsilon} \sum_{j=0}^{k_r-1} \binom{1/2}{j} \epsilon^{-j} f(x)^j \pmod{E_r}.$$

Let $\sigma_r = s_r(x)^2$, and $q_r = f + \epsilon - \sigma_r \in E_r$. Clearly, we have

$$q_r(x) = \sum_{j=0}^{k_r-2} c_j(\epsilon) f(x)^{k_r+j}$$

for some real scalars $c_j(\epsilon)$. Note each $f(x)^{k_r+j} \in E_r$.

Applying Lemma 3.3 to E_0, E_1, \dots, E_r , we can find $a_0, \dots, a_r \in \mathbb{R}[x]$ satisfying

$$a_0^2 + \dots + a_r^2 - 1 \in I_W, \quad a_i \in \bigcap_{i \neq j \in \{0,1,\dots,r\}} E_j.$$

Let $\sigma = \sigma_0 a_0^2 + \sigma_1 a_1^2 + \dots + \sigma_r a_r^2$, then

$$f(x) + \epsilon - \sigma = \sum_{i=0}^r (f + \epsilon - \sigma_i) a_i^2 + (f + \epsilon)(1 - a_0^2 - \dots - a_r^2).$$

Since $q_i = f + \epsilon - \sigma_i \in E_i$, it holds that

$$(f + \epsilon - \sigma_i) a_i^2 \in \bigcap_{j=0}^r E_j = I_W.$$

For each $0 \leq i < r$, q_i is independent of ϵ . There exists $N_1 > 0$ such that for all $\epsilon > 0$

$$(f + \epsilon - \sigma_i) a_i^2 \in I^{(N_1)}, \quad i = 0, 1, \dots, r-1.$$

For $i = r$, $q_r = f + \epsilon - \sigma_r$ depends on ϵ . By the choice of q_r , it holds that

$$(f + \epsilon - \sigma_r) a_r^2 = \sum_{j=0}^{k_r-2} c_j(\epsilon) f^{k_r+j} a_r^2.$$

Note each $f^{k_r+j} a_r^2 \in I_W$, since $f^{k_r+j} \in E_r$. So, there exists $N_2 > 0$ such that for all $\epsilon > 0$

$$(f + \epsilon - \sigma_r) a_r^2 \in I^{(N_2)}.$$

Since $1 - a_1^2 - \dots - a_r^2 \in I_W$, there also exists $N_3 > 0$ such that for all $\epsilon > 0$

$$(f + \epsilon)(1 - a_1^2 - \dots - a_r^2) \in I^{(N_3)}.$$

Combining the above, we know if N^* is big enough, then $f(x) + \epsilon - \sigma \in I^{(N^*)}$ for all $\epsilon > 0$. From the constructions of σ_i and a_i , we know their degrees are independent of ϵ . So, $\sigma \in P^{(N^*)}$ for all $\epsilon > 0$ if N^* is big enough, which completes the proof. \square

Theorem 3.4 is a kind of Positivstellensatz of representing $f(x) - f^* + \epsilon$, which is positive on S for all $\epsilon > 0$, by the preordering generated by g_j modulo the ideal I_W in (3.4) of variety W . Usually, we can not conclude $f(x) - f^* \in I^{(N^*)} + P^{(N^*)}$ by setting $\epsilon = 0$, because the coefficients of the representing polynomials of $f(x) - f^* + \epsilon$ in $I^{(N^*)} + P^{(N^*)}$ go to infinity as $\epsilon \rightarrow 0$ (see $s_r(x)$ in the proof). It is possible that $f(x) - f^* \notin I^{(N)} + P^{(N)}$ for every $N > 0$. Such a counterexample is Example 5.1. However, Theorem 3.4 shows that the degree bound N^* required for representing $f(x) - f^* + \epsilon$ is independent of ϵ . This is a crucial property justifying the exactness of the SDP relaxation (2.8). Now we present its proof below.

Proof of Theorem 2.3 By Theorem 3.4, we know $f^* > -\infty$ and there exists $N^* \in \mathbb{N}$ such that for all $\epsilon > 0$

$$f(x) - (f^* - \epsilon) \in I^{(N^*)} + P^{(N^*)}.$$

Since $f_{N^*}^{(1)}, f_{N^*}^{(2)}$ are the optimal values of (2.8) and (2.11) respectively, we know

$$f^* - \epsilon \leq f_{N^*}^{(2)} \leq f_{N^*}^{(1)} \leq f^*.$$

Because $\epsilon > 0$ is arbitrary, the above implies $f_{N^*}^{(1)} = f_{N^*}^{(2)} = f^*$. Since the sequence $\{f_N^{(2)}\}$ is monotonically increasing and every $f_N^{(2)} \leq f_N^{(1)} \leq f^*$ by (2.12), we get $f_N^{(1)} = f_N^{(2)} = f^*$ for all $N \geq N^*$. If the minimum f_{min} of (1.1) is achievable, then there exists $x^* \in S$ such that $f_{min} = f(x^*)$. By Assumption 2.2, we must have $x^* \in W$. So x^* is feasible for (2.7), and $f^* = f_{min}$. Thus, we also have $f_N^{(1)} = f_N^{(2)} = f_{min}$ for all $N \geq N^*$. \square

Last we prove Theorem 2.5 by using the properties of resultants and discriminants described in Appendix.

Proof of Theorem 2.5 (a) If $Res(h_{i_1}, \dots, h_{i_{n+1}}) \neq 0$, then the polynomial system

$$h_{i_1}(x) = \dots = h_{i_{n+1}}(x) = 0$$

does not have a complex solution. Hence, $V(h) = \emptyset$ and consequently $S = \emptyset$.

(b) For a contradiction, suppose $n - m_1 + 1$ of g_j vanish at $u \in S$, say, $g_{j_1}, \dots, g_{j_{n-m_1+1}}$. Then the polynomial system

$$h_1(x) = \dots = h_{m_1}(x) = g_{j_1}(x) = \dots = g_{j_{n-m_1+1}}(x) = 0$$

has a solution, which contradicts $Res(h_1, \dots, h_{m_1}, g_{j_1}, \dots, g_{j_{n-m_1+1}}) \neq 0$.

(c) For every $J = \{j_1, \dots, j_k\} \subset [m_2]$ with $k \leq n - m_1$, if

$$\Delta(h_1, \dots, h_{m_1}, g_{j_1}, \dots, g_{j_k}) \neq 0,$$

then the polynomial system

$$h_1(x) = \dots = h_{m_1}(x) = g_{j_1}(x) = \dots = g_{j_k}(x) = 0$$

has no singular solution, i.e., the variety $V(h, g_J)$ is smooth.

(d) Let $f_0(x) = f(x) - f_{min}$. Then f_0 lies on the boundary of the set

$$P_d(S) = \left\{ p \in B_d(S) : p(x) \geq 0 \forall x \in S \right\}.$$

Since S is closed at ∞ , by Prop. 6.1 of [20], $f_0 \in \partial P_d(S)$ implies

$$0 = \min_{\tilde{x} \in S^{prj}, \|\tilde{x}\|_2=1, x_0 \geq 0} \tilde{f}_0(\tilde{x}).$$

Let $\tilde{u} = (u_0, u_1, \dots, u_n) \neq 0$ be a minimizer of the above, which must exist because the feasible set is compact. We claim that $u_0 \neq 0$. Otherwise, suppose $u_0 = 0$. Then $u = (u_1, \dots, u_n) \neq 0$ is a minimizer of

$$\begin{aligned} 0 = \min \quad & f^{hom}(x) \\ \text{s.t.} \quad & h_1^{hom}(x) = \dots = h_{m_1}^{hom}(x) = 0, \\ & g_1^{hom}(x) \geq 0, \dots, g_{m_2}^{hom}(x) \geq 0. \end{aligned}$$

Let $j_1, \dots, j_k \in [m_2]$ be the indices of active constraints. By Fritz-John optimality condition (see Sec. 3.3.5 in [1]), there exists $(\lambda_0, \lambda_1, \dots, \lambda_{m_1}, \mu_1, \dots, \mu_k) \neq 0$ satisfying

$$\begin{aligned} \lambda_0 \nabla f^{hom}(u) + \sum_{i=1}^{m_1} \lambda_i \nabla h_i^{hom}(u) + \dots + \sum_{\ell=1}^k \mu_\ell \nabla g_{j_\ell}^{hom}(u) &= 0, \\ f^{hom}(u) = h_1^{hom}(u) = \dots = h_{m_1}^{hom}(u) = g_{j_1}^{hom}(u) = \dots = g_{j_k}^{hom}(u) &= 0. \end{aligned}$$

Thus, the homogeneous polynomial system

$$f^{hom}(x) = h_1^{hom}(x) = \dots = h_{m_1}^{hom}(x) = g_{j_1}^{hom}(x) = \dots = g_{j_k}^{hom}(x) = 0$$

has a nonzero singular solution. Since the resultant of any n of h_i^{hom}, g_j^{hom} is nonzero, we must have $m_1 + k \leq n - 1$. So the discriminant

$$\Delta(f^{hom}, h_1^{hom}, \dots, h_{m_1}^{hom}, g_{j_1}^{hom}, \dots, g_{j_k}^{hom})$$

is defined and must vanish, which is a contradiction. So $u_0 \neq 0$. Let $v = u/u_0$, then $\tilde{u} \in S^{prj}$ implies $v \in S$ and $f(v) - f_{min} = u_0^{-d} \tilde{f}_0(\tilde{u}) = 0$.

Clearly, (e) is true since it is a special case of (d). \square

4 Some variations

This section presents some variations of the exact SDP relaxation (2.8) and its dual (2.11).

4.1 A refined version based on all maximal minors

An SDP relaxation tighter than (2.8) would be obtained by using all the maximal minors to define the determinantal variety G_J in (2.2), while the number of equations would be significantly larger. For every $J = \{j_1, \dots, j_k\} \subset [m_2]$ with $m_1 + k \leq m$, let $\tau_1^J, \dots, \tau_\ell^J$ be all the maximal minors of $B^J(x)$ defined in (2.2). Then define new polynomials

$$\psi_i^J := \tau_i^J \cdot \prod_{j \in J^c} g_j, \quad i = 1, \dots, \ell. \quad (4.1)$$

List all such possible ψ_i^J as

$$\psi_1, \psi_2, \dots, \psi_t, \quad \text{where} \quad t = \sum_{J \subset [m_2], |J| \leq m - m_1} \binom{n}{|J| + m_1 + 1}.$$

Like (2.7), we formulate (1.1) equivalently as

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & f(x) \\ \text{s.t.} \quad & h_i(x) = \psi_j(x) = 0, \quad i \in [m_1], j \in [t], \\ & g_\nu(x) \geq 0, \quad \forall \nu \in \{0, 1\}^{m_2}. \end{aligned} \quad (4.2)$$

The standard N -th order Lasserre's relaxation for the above is

$$\begin{aligned} \min \quad & L_f(y) \\ \text{s.t.} \quad & L_{h_i}^{(N)}(y) = 0, L_{\psi_j}^{(N)}(y) = 0, \quad i \in [m_1], j \in [t], \\ & L_{g_\nu}^{(N)}(y) \succeq 0, \quad \forall \nu \in \{0, 1\}^{m_2}, \quad y_0 = 1. \end{aligned} \quad (4.3)$$

Note that every φ_i^J in (2.4) is a sum of polynomials like ψ_i^J in (4.1). So the equations $L_{\psi_j}^{(N)}(y) = 0$ in (4.3) implies $L_{\varphi_j}^{(N)}(y) = 0$ in (2.8). Hence, (4.3) is stronger than (2.8). Its dual is an SOS program like (2.11). Theorem 2.3 then implies the following.

Corollary 4.1. *Suppose Assumption 2.2 is true, and the minimum f_{\min} of (1.1) is achievable. If N is big enough, then the optimal value of (4.3) is equal to f_{\min} .*

4.2 A Putinar type variation without using cross products of g_j

If the minimum f_{\min} of (1.1) is achieved at a KKT point, then (1.1) is equivalent to

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & f(x) \\ \text{s.t.} \quad & h_i(x) = \varphi_j(x) = 0, \quad i \in [m_1], j \in [r], \\ & g_1(x) \geq 0, \dots, g_{m_2}(x) \geq 0. \end{aligned} \quad (4.4)$$

The standard N -th order Lasserre's relaxation for (4.4) is

$$\begin{aligned} \min \quad & L_f(y) \\ \text{s.t.} \quad & L_{h_i}^{(N)}(y) = 0, L_{\varphi_j}^{(N)}(y) = 0, \quad i \in [m_1], j \in [r], \\ & L_{g_i}^{(N)}(y) \succeq 0, \quad i = 0, 1, \dots, m_2, \quad y_0 = 1. \end{aligned} \quad (4.5)$$

The difference between (4.5) and (2.8) is that the cross products of $g_j(x)$ are not used in (4.5), which makes the number of resulting LMIs much smaller. Similar to $P^{(N)}$, define the truncated quadratic module $M^{(N)}$ generated by g_i as

$$M^{(N)} = \left\{ \sum_{i=0}^{m_2} \sigma_i g_i \mid \deg(\sigma_i g_i) \leq 2N \right\}. \quad (4.6)$$

The dual of (4.5) would be shown to be the following SOS relaxation for (4.4):

$$\begin{aligned} \max \quad & \gamma \\ \text{s.t.} \quad & f(x) - \gamma \in I^{(N)} + M^{(N)}. \end{aligned} \quad (4.7)$$

Clearly, for the same N , (4.7) is stronger than the standard Lasserre's relaxation (1.2). To prove (4.5) and (4.7) are exact for some N , we need the archimedean condition (AC) for S , i.e., there exist $R > 0$, $\phi_1, \dots, \phi_{m_1} \in \mathbb{R}[x]$ and SOS $s_0, \dots, s_{m_2} \in \mathbb{R}[x]$ such that

$$R - \|x\|_2^2 = \sum_{i=1}^{m_1} \phi_i h_i + \sum_{j=0}^{m_2} s_j g_j.$$

Theorem 4.2. *Suppose Assumption 2.2 and the archimedean condition hold. If N is big enough, then the optimal values of (4.5) and (4.7) are equal to f_{min} .*

To prove Theorem 4.2, we need the following.

Theorem 4.3. *Suppose Assumption 2.2 and the archimedean condition hold. Let f^* be the optimal value of (4.4). Then there exists an integer $N^* > 0$ such that for every $\epsilon > 0$*

$$f(x) - f^* + \epsilon \in I^{(N^*)} + M^{(N^*)}. \quad (4.8)$$

Proof. The proof is almost same as for Theorem 3.4. We follow the same approach used there. The only difference occurs for the case $i = 0$ and $V_{\mathbb{R}}(E_0) \cap T = \emptyset$. By Theorem A.3, there exist SOS polynomials η_ν satisfying

$$-2 \equiv \sum_{\nu \in \{0,1\}^{m_2}} \eta_\nu \cdot g_\nu \quad \text{mod } E_0.$$

Clearly, each $\frac{1}{2^{m_2}} + \eta_\nu \cdot g_1^{\nu_1} \cdots g_{m_2}^{\nu_{m_2}}$ is positive on S . Since AC holds, by Putinar's Positivstellensatz (Theorem A.4), there exist SOS polynomials $\theta_{\nu,i}$ such that

$$\frac{1}{2^{m_2}} + \eta_\nu \cdot g_\nu = \sum_{i=0}^{m_2} \theta_{\nu,i} g_i \quad \text{mod } \langle h_1, \dots, h_{m_1} \rangle.$$

Hence, it holds that

$$\begin{aligned} -1 &\equiv \sum_{\nu \in \{0,1\}^{m_2}} \left(\frac{1}{2^{m_2}} + \eta_\nu \cdot g_\nu \right) \quad \text{mod } \langle h_1, \dots, h_{m_1} \rangle + E_0 \\ &\equiv \sum_{i=0}^{m_2} \left(\sum_{\nu \in \{0,1\}^{m_2}} \theta_{\nu,i} \right) g_i \quad \text{mod } E_0. \end{aligned}$$

The second equivalence above is due to the relation

$$\langle h_1, \dots, h_{m_1} \rangle \subset I_W \subset E_0.$$

Letting $\tau_i = \sum_{\nu \in \{0,1\}^{m_2}} \theta_{\nu,i}$, which is clearly SOS, we get

$$-1 \equiv \tau_0 + \tau_1 g_1 + \cdots + \tau_{m_2} g_{m_2} \quad \text{mod } E_0.$$

The rest of the proof is almost same as for Theorem 3.4. □

Proof of Theorem 4.2 For convenience, still let $f_N^{(1)}, f_N^{(2)}$ be the optimal values of (4.5) and (4.7) respectively. From Theorem 4.3, there exists an integer N^* such that for all $\epsilon > 0$

$$f(x) - (f^* - \epsilon) \in I^{(N^*)} + M^{(N^*)}.$$

Like in the proof of Theorem 2.3, we can similarly prove $f_N^{(1)} = f_N^{(2)} = f^*$ for all $N \geq N^*$. Since AC holds, the set S must be compact. So the minimum f_{min} of (1.1) must be achievable. By Assumption 2.2, we know $f^* = f_{min}$, and the proof is complete. □

4.3 A simplified version for inactive constraints

Suppose in (1.1) we are only interested in a minimizer making all the inequality constraints inactive. Consider the problem

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & f(x) \\ \text{s.t.} \quad & h_1(x) = \cdots = h_{m_1}(x) = 0, \\ & g_1(x) > 0, \dots, g_{m_2}(x) > 0. \end{aligned} \tag{4.9}$$

Let u be a minimizer of (4.9). If $V(h)$ is smooth at u , there exist λ_i such that

$$\nabla f(u) = \lambda_1 \nabla h_1(u) + \cdots + \lambda_{m_1} \nabla h_{m_1}(u).$$

Thus, u belongs to the determinantal variety

$$G_h = \{x : \text{rank} [\nabla f(x) \quad \nabla h(x)] \leq m_1\}.$$

If $m_1 < n$, let ϕ_1, \dots, ϕ_s be a minimum set of defining polynomials for G_h by using formula (2.3). If $m_1 = n$, then $G_h = \mathbb{R}^n$ and we do not need these polynomials; set $s = 0$, and $[s]$ is empty. Then, (4.9) is equivalent to

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & f(x) \\ \text{s.t.} \quad & h_i(x) = \phi_j(x) = 0, \quad i \in [m_1], j \in [s], \\ & g_1(x) > 0, \dots, g_{m_2}(x) > 0. \end{aligned} \tag{4.10}$$

The difference between (4.10) and (2.7) is that the number of new equations in (4.10) is $s = O(nm_1)$, which is much smaller than r in (2.7). So, (4.10) is preferable to (2.7) when the inequality constraints are all inactive. The N -th order Lasserre's relaxation for (4.10) is

$$\begin{aligned} \min \quad & L_f(y) \\ \text{s.t.} \quad & L_{h_i}^{(N)}(y) = 0, L_{\phi_j}^{(N)}(y) = 0, \quad i \in [m_1], j \in [s], \\ & L_{g_j}^{(N)}(y) \succeq 0, \quad j = 1, \dots, m_2, \quad y_0 = 1. \end{aligned} \tag{4.11}$$

A tighter version than the above using cross products of g_j is

$$\begin{aligned} \min \quad & L_f(y) \\ \text{s.t.} \quad & L_{h_i}^{(N)}(y) = 0, L_{\phi_j}^{(N)}(y) = 0, \quad i \in [m_1], j \in [s], \\ & L_{g_\nu}^{(N)}(y) \succeq 0, \quad \forall \nu \in \{0, 1\}^{m_2}, \quad y_0 = 1. \end{aligned} \tag{4.12}$$

Define the truncated ideal $J^{(N)}$ generated by $h_i(x)$ and ϕ_j as

$$J^{(N)} = \left\{ \sum_{i=1}^{m_1} p_i h_i + \sum_{j=1}^s q_j \phi_j \mid \begin{array}{l} \deg(p_i h_i) \leq 2N \quad \forall i \\ \deg(q_j \phi_j) \leq 2N \quad \forall j \end{array} \right\}.$$

The dual of (4.11) is the SOS relaxation

$$\begin{aligned} \max \quad & \gamma \\ \text{s.t.} \quad & f(x) - \gamma \in J^{(N)} + M^{(N)}. \end{aligned} \tag{4.13}$$

The dual of (4.12) is the SOS relaxation

$$\begin{aligned} \max \quad & \gamma \\ \text{s.t.} \quad & f(x) - \gamma \in J^{(N)} + P^{(N)}. \end{aligned} \tag{4.14}$$

The exactness of the above relaxations is summarized as follows.

Theorem 4.4. *Suppose the variety $V(h)$ is nonsingular and the minimum f_{min} of (4.9) is achieved at some feasible u with every $g_j(u) > 0$. If N is big enough, then the optimal values of (4.12) and (4.14) are equal to f_{min} . If, in addition, the archimedean condition holds for S , the optimal values of (4.11) and (4.13) are also equal to f_{min} for N big enough.*

Proof. The proof is almost same as for Theorems 2.3 and 4.2. We can first prove a decomposition result like Lemma 3.2, and then prove there exists $N^* > 0$ such that for all $\epsilon > 0$ (like in Theorem 3.4)

$$f(x) - f_{min} + \epsilon \in J^{(N^*)} + P^{(N^*)}.$$

Furthermore, if AC holds, we can similarly prove there exists $N^* > 0$ such that for all $\epsilon > 0$ (like in Theorem. 4.3)

$$f(x) - f_{min} + \epsilon \in J^{(N^*)} + M^{(N^*)}.$$

The rest of the proof is almost same as for Theorems 2.3 and 4.2. Due to its repeating, we omit the details here for cleanness of the paper. \square

5 Examples

This section presents some examples on how to apply the SDP relaxation (2.8) and its dual (2.11) to solve polynomial optimization problems. The software *GloptiPoly 3* [14] is used to solve (2.8) and (2.11).

First, consider unconstrained polynomial optimization. Then the resulting SOS relaxation (2.11) is reduced to the gradient SOS relaxation in [18], which is a special case of (2.11).

Example 5.1. Consider the optimization problem

$$\min_{x \in \mathbb{R}^3} x_1^8 + x_2^8 + x_3^8 + x_1^4 x_2^2 + x_1^2 x_2^4 + x_3^6 - 3x_1^2 x_2^2 x_3^2.$$

This example was studied in [18]. Its global minimum is zero. The SDP relaxation (2.8) and its dual (2.11) for this problem are equivalent to gradient SOS relaxations in [18] (there are no constraints). We apply (2.8) of order $N = 4$, and get a lower bound $-9.7 \cdot 10^{-9}$. The resulting SDP (2.8) has a single block of size 35×35 . The minimizer $(0, 0)$ is extracted. In [18], it was shown that $f(x)$ is not SOS modulo its gradient ideal I_{grad} . But for every $\epsilon > 0$, $f(x) + \epsilon \equiv s_\epsilon(x)$ modulo I_{grad} for some SOS polynomial $s_\epsilon(x)$, whose degree is independent of ϵ (see equation (10) of [18]). But its coefficients go to infinity as $\epsilon \rightarrow 0$. This shows that the optimal value of (2.11) might not be achievable. On the other hand, if (1.1) has a minimizer (say, x^*) that is a KKT point (then $x^* \in W$), then its dual problem (2.8) always achieves its optimal value f_{min} for N big enough, e.g., $[x^*]_{2N}$ is a minimizer. Thus, for this example, the minimum of the dual (2.8) is achievable. \square

Second, consider polynomial optimization having only equality constraints:

$$\min_{x \in \mathbb{R}^n} f(x) \quad \text{s.t.} \quad h_1(x) = \cdots = h_m(x) = 0. \quad (5.1)$$

When $V(h)$ is nonsingular, its equivalent version (2.7) reduces to

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & f(x) \\ \text{s.t.} \quad & h_1(x) = \cdots = h_m(x) = 0, \\ & \sum_{\substack{I \in [n]_{m+1} \\ \text{sum}(I) = \ell}} \det_I F(x) = 0, \quad \ell = \binom{m+2}{2}, \dots, (n - \frac{m}{2})(m+1). \end{aligned} \quad (5.2)$$

In the above $\text{sum}(I)$ denotes the summation of the indices in I , $F(x) = [\nabla f(x) \quad \nabla h(x)]$, and $\det_I F(x)$ denotes the maximal minor of $F(x)$ whose row indices are in I . When $m \geq n$, there are no minor equations in (5.2).

Example 5.2. Consider the optimization problem

$$\begin{aligned} \min_{x \in \mathbb{R}^3} \quad & x_1^6 + x_2^6 + x_3^6 + 3x_1^2x_2^2x_3^2 - (x_1^2(x_2^4 + x_3^4) + x_2^2(x_3^4 + x_1^4) + x_3^2(x_1^4 + x_2^4)) \\ \text{s.t.} \quad & x_1 + x_2 + x_3 - 1 = 0. \end{aligned}$$

The objective is the Robinson polynomial, which is nonnegative everywhere but not SOS [22]. So the minimum $f_{\min} = 0$. We apply SDP relaxation (2.8) of order $N = 4$, and get a lower bound $-4.4600 \cdot 10^{-9}$. The resulting SDP (2.8) has a single block of size 35×35 . The minimizer $(1/3, 1/3, 1/3)$ is also extracted. Applying Lasserre's relaxation (1.2) of orders $N = 3, 4, 5, 6, 7$, we get lower bounds respectively

$$-0.0582, \quad -0.0479, \quad -0.0194, \quad -0.0053, \quad -4.8358 \cdot 10^{-5}.$$

We can see that (1.2) is weaker than (2.8). It is not clear whether the sequence of lower bounds returned by (1.2) converges to zero or not, because it is not guaranteed when the feasible set is noncompact, which is the case in this example. The objective $f(x)$ here is not SOS modulo the constraint. Otherwise, suppose there exist polynomials $\sigma(x)$ being SOS and $\phi(x)$ such that

$$f(x) = \sigma(x) + \phi(x)(x_1 + x_2 + x_3 - 1).$$

In the above, replacing every x_i by $x_i/(x_1 + x_2 + x_3)$ gives

$$f(x) = (x_1 + x_2 + x_3)^6 \sigma(x/(x_1 + x_2 + x_3)).$$

So, there exist polynomials $p_1, \dots, p_k, q_1, \dots, q_\ell$ such that

$$f(x) = p_1^2 + \cdots + p_k^2 + \frac{q_1^2}{(x_1 + x_2 + x_3)^2} + \cdots + \frac{q_\ell^2}{(x_1 + x_2 + x_3)^{2\ell}}.$$

Since the objective $f(x)$ does not have any pole, every q_i must vanish on the plane $x_1 + x_2 + x_3 = 0$. Thus $q_i = (x_1 + x_2 + x_3)^i w_i$ for some polynomials w_i . Hence, we get

$$f(x) = p_1^2 + \cdots + p_r^2 + w_1^2 + \cdots + w_\ell^2$$

is SOS, which is a contradiction. \square

Third, consider polynomial optimization having only a single inequality constraint.

$$\min_{x \in \mathbb{R}^n} f(x) \quad \text{s.t.} \quad g(x) \geq 0. \quad (5.3)$$

Its equivalent problem (2.7) becomes

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & f(x) \\ \text{s.t.} \quad & g(x) \frac{\partial f(x)}{\partial x_i} = 0, \quad i = 1, \dots, n, \quad g(x) \geq 0, \\ & \sum_{i+j=\ell} \left(\frac{\partial f(x)}{\partial x_i} \frac{\partial g(x)}{\partial x_j} - \frac{\partial f(x)}{\partial x_j} \frac{\partial g(x)}{\partial x_i} \right) = 0, \quad \ell = 3, \dots, 2n-1. \end{aligned} \quad (5.4)$$

There are totally $3(n-1)$ new equality constraints.

Example 5.3. Consider the optimization problem

$$\begin{aligned} \min_{x \in \mathbb{R}^3} \quad & x_1^4 x_2^2 + x_1^2 x_2^4 + x_3^6 - 3x_1^2 x_2^2 x_3^2 \\ \text{s.t.} \quad & x_1^2 + x_2^2 + x_3^2 \leq 1. \end{aligned}$$

The objective is the Motzkin polynomial which is nonnegative everywhere but not SOS [22]. So its minimum $f_{min} = 0$. We apply SDP relaxation (2.8) of order $N = 4$, and get a lower bound $-1.6948 \cdot 10^{-8}$. The resulting SDP (2.8) has two blocks of sizes 35×35 and 20×20 . The minimizer $(0, 0, 0)$ is also extracted. Now we apply Lasserre's relaxation (1.2). For orders $N = 4, 5, 6, 7, 8$, (1.2) returns the lower bounds respectively

$$-2.0331 \cdot 10^{-4}, \quad -2.9222 \cdot 10^{-5}, \quad -8.2600 \cdot 10^{-6}, \quad -4.2565 \cdot 10^{-6}, \quad -2.3465 \cdot 10^{-6}.$$

Clearly, (1.2) is weaker than (2.8). The sequence of lower bounds given by (1.2) converges to zero for this example, because the archimedean condition holds. The feasible set has nonempty interior. Hence, for every N , there is no duality gap between (1.2) and its dual, and (1.2) has an optimizer. The objective does not belong to the preordering generated by the ball condition. This fact was kindly pointed out to the author by Claus Scheiderer (implied by his proof of Prop. 6.1 in [23], since the objective is a nonnegative but non-SOS form vanishing at origin). Therefore, for every N , the optimal value of (1.2) as well as its dual is strictly smaller than the minimum f_{min} . \square

Example 5.4. Consider Example 5.3 but the constraint is the outside of the unit ball:

$$\begin{aligned} \min_{x \in \mathbb{R}^3} \quad & x_1^4 x_2^2 + x_1^2 x_2^4 + x_3^6 - 3x_1^2 x_2^2 x_3^2 \\ \text{s.t.} \quad & x_1^2 + x_2^2 + x_3^2 \geq 1. \end{aligned}$$

Its minimum is still 0. We apply SDP relaxation (2.8) of order $N = 4$, and get a lower bound $1.7633 \cdot 10^{-9}$ (its sign is not correct due to numerical issues). The resulting SDP (2.8) has two blocks of sizes 35×35 and 20×20 . Now we compare it with Lasserre's relaxation (1.2). When $N = 4$, (1.2) is not feasible. When $N = 5, 6, 7, 8$, (1.2) returns the following lower bounds respectively

$$-4.8567 \cdot 10^5, \quad -98.4862, \quad -0.7079, \quad -0.0277.$$

So we can see (1.2) is much weaker than (2.8). Again, it is not clear whether the sequence of lower bounds given by (1.2) converges to zero or not, because it is only guaranteed when the feasible set is compact, which is not the case in this example. \square

Last, we show some general examples.

Example 5.5. Consider the following polynomial optimization

$$\begin{aligned} \min_{x \in \mathbb{R}^2} \quad & x_1^2 + x_2^2 \\ \text{s.t.} \quad & x_2^2 - 1 \geq 0, \\ & x_1^2 - Mx_1x_2 - 1 \geq 0, \\ & x_1^2 + Mx_1x_2 - 1 \geq 0. \end{aligned}$$

This problem was studied in [7, 12]. Its global minimum is $2 + \frac{1}{2}M(M + \sqrt{M^2 + 4})$. Let $M = 5$ here. Applying (2.8) of order $N = 4$, we get a lower bound 27.9629 which equals the global minimum, and four global minimizers $(\pm 5.1926, \pm 1.0000)$. The resulting SDP (2.8) has eight blocks whose matrix lengths are 15, 10, 10, 10, 6, 6, 6, 3 respectively. However, if we apply the Lasserre's relaxation either (1.2) or (1.3), the best lower bound we would obtain is 2, no matter how big the relaxation order N is (see Example 4.5 of [7]). The reason for this is that its feasible set is noncompact, while Lasserre's relaxations (1.2) or (1.3) are only guaranteed to converge for compact sets. \square

Example 5.6. Consider the polynomial optimization

$$\begin{aligned} \min_{x \in \mathbb{R}^3} \quad & x_1^4x_2^2 + x_2^4x_3^2 + x_3^4x_1^2 - 3x_1^2x_2^2x_3^2 \\ \text{s.t.} \quad & 1 - x_1^2 \geq 0, 1 - x_2^2 \geq 0, 1 - x_3^2 \geq 0. \end{aligned}$$

The objective is a nonnegative form being non-SOS [22, Sec. 4c]. Its minimum $f_{min} = 0$. We apply SDP relaxation (2.8) of order $N = 6$, and get a lower bound $-9.0752 \cdot 10^{-9}$. A minimizer $(0, 0, 0)$ is also extracted. The resulting SDP (2.8) has eight blocks whose matrix lengths are 84, 56, 56, 56, 35, 35, 35, 20 respectively. Now we apply Lasserre's relaxation of type (1.3). Let f_N^{smg} be the optimal value of (1.3) for an order N , and f_N^{mom} be the optimal value of its dual optimization problem (an analogue of (2.8) by using moments and localizing matrices, see Lasserre [16]). Since the feasible set here has nonempty interior, the dual optimization problem of (1.3) has an interior point, so (1.3) always has an optimizer and there is no duality gap, i.e., $f_N^{smg} = f_N^{mom}$ for every order N (cf. [16]). For $N = 6, 7, 8$, (1.3) returns the lower bounds f_N^{smg} respectively

$$-3.5619 \cdot 10^{-5}, \quad -1.0406 \cdot 10^{-5}, \quad -7.6934 \cdot 10^{-6}.$$

The sequence of lower bounds given by (1.3) converges to zero, since the feasible set is compact. However, the objective does not belong to the preordering generated by the constraints, which is implied by the proof of Prop. 6.1 of [23] (the objective is a nonnegative but non-SOS form vanishing at origin). Because (1.3) has an optimizer for every order N , this implies that $f_N^{smg} = f_N^{mom} < f_{min} = 0$ for every N . Thus, the relaxation (1.3) and its dual could not be exact for any order N . \square

6 Some conclusions and discussions

This paper proposes an exact type SDP relaxation (2.8) and its dual (2.11) for polynomial optimization (1.1) by using the Jacobian of its defining polynomials. Under some generic

conditions, we showed that the minimum of (1.1) would be found by solving the SDP (2.8) for a finite relaxation order.

A drawback of the proposed relaxation (2.8) and its dual (2.11) is that there are totally $O(2^{m_2} \cdot n \cdot (m_1 + m_2))$ new constraints. This would make the computation very difficult to implement if either m_1 or m_2 is big. Thus, this method is more interesting theoretically. However, this paper discovers an important fact: it is possible to solve the polynomial optimization (1.1) exactly by a single SDP relaxation, which was not known in the prior existing literature. Currently, it is not clear for the author whether the number of newly introduced constraints would be significantly dropped while the exactness is still wanted. This is an interesting future research topic. On the other hand, the relaxation (2.8) is not too bad in applications. For problems that have only a few constraints, the method would also be efficiently implemented. For instance, in all the examples of Section 5, they are all solved successfully, and the advantages of the method over the prior existing ones are very clear. Thus, this method would also be computationally attractive in such applications.

The results of this paper improve the earlier work [7, 18], where the exactness of gradient or KKT type SOS relaxations for a finite relaxation order is only proved when the gradient or KKT ideal is radical. There are other conditions like boundary hessian condition (BHC) guaranteeing this property, like in [15, 17]. In [17], Marshall showed that the gradient SOS relaxation is also exact for a finite relaxation order by assuming BHC, in unconstrained optimization. In this paper, the exactness of (2.8) and (2.11) for a finite N is proved without the conditions like radicalness or BHC. The only assumptions required are nonsingularity of S and the minimum f_{min} being achievable (the earlier related work also requires this), but they are generically true as shown by Theorem 2.5.

We would like to point out that the KKT type SOS relaxation proposed in [7] using Lagrange multipliers is also exact for a finite order, no matter the KKT ideal is radical or not. This would be proved in a similar way as we did in Section 3. First, we can get a similar decomposition for the KKT variety like Lemma 3.2. Second, we can prove a similar representation for $f(x) - f^* + \epsilon$ like in Theorem 3.4, with degree bounds independent of ϵ . Based on these two steps, we can similarly prove its exactness for a finite relaxation order. Since the proof is almost a repeating of Section 3, we omit it for cleanness of the paper.

The proof of the exactness of (2.8) provides a representation of polynomials that are positive on S through using the preordering of S and the Jacobian of all the involved polynomials. A nice property of this representation is that the degrees of the representing polynomials are independent of the minimum value. This is presented by Theorem 3.4. A similar representation result using the quadratic module of S is given by Theorem 4.3.

An issue that is not addressed by the paper is that the feasible set S has singularities. If a global minimizer x^* of (1.1) is singular on S , then the KKT condition might no longer hold, and $x^* \notin W$. In this case, the original optimization (1.1) is not equivalent to (2.7), and the SDP relaxation (2.8) might not give a correct lower bound for f_{min} . It is not clear how to handle singularities generally in an efficient way.

Another issue that is not addressed by the paper is the minimum f_{min} of (1.1) is not achievable, which happens only if S is noncompact. For instance, when $S = \mathbb{R}^2$, the polynomial $x_1^2 + (x_1x_2 - 1)^2$ has minimum 0 but it is not achievable. If applying the relaxation (2.8) for this instance, we would not get a correct lower bound. Generally, this case will not happen, as shown by items (d), (e) of Theorem 2.5. In unconstrained optimization, when

f_{min} is not achievable, excellent approaches are proposed in [9, 10, 26]. It is an interesting future work to generalize them to constrained optimization.

An important question is for what concrete relaxation order N^* the SDP relaxation (2.8) is exact for solving (1.1). No good estimates for N^* in Theorem 2.3 are available currently. Since the original problem (1.1) is NP-hard, any such estimates would be very bad if they exist. This is another interesting future work.

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A Some basics in algebraic geometry and real algebra

In this appendix, we give a short review on basic algebraic geometry and real algebra. More details would be found in the books [4, 11].

An ideal I of $\mathbb{R}[x]$ is a subset such that $I \cdot \mathbb{R}[x] \subseteq I$ and $I + I \subseteq I$. Given polynomials $p_1, \dots, p_m \in \mathbb{R}[x]$, $\langle p_1, \dots, p_m \rangle$ denotes the smallest ideal containing every p_i , which is the set $p_1\mathbb{R}[x] + \dots + p_m\mathbb{R}[x]$. The ideals in $\mathbb{C}[x]$ are defined similarly. An algebraic variety is a subset of \mathbb{C}^n that are common complex zeros of polynomials in an ideal. Let I be an ideal of $\mathbb{R}[x]$. Define

$$\begin{aligned} V(I) &= \{x \in \mathbb{C}^n : p(x) = 0 \quad \forall p \in I\}, \\ V_{\mathbb{R}}(I) &= \{x \in \mathbb{R}^n : p(x) = 0 \quad \forall p \in I\}. \end{aligned}$$

The $V(I)$ is called an algebraic variety or just a variety, and $V_{\mathbb{R}}(I)$ is called a real algebraic variety or just a real variety. Every subset $T \subset \mathbb{C}^n$ is contained in a variety in \mathbb{C}^n . The smallest one containing T is called the *Zariski* closure of T , and is denoted by $Zar(T)$. In the Zariski topology on \mathbb{C}^n , the varieties are called closed sets, and the complements of varieties are called open sets. A variety V is irreducible if there exist no proper subvarieties V_1, V_2 of V such that $V = V_1 \cup V_2$. Every variety is a finite union of irreducible varieties.

Theorem A.1 (Hilbert's Strong Nullstellensatz). *Let $I \subset \mathbb{R}[x]$ be an ideal. If $p \in \mathbb{R}[x]$ vanishes on $V(I)$, then $p^k \in I$ for some integer $k > 0$.*

If an ideal I has empty variety $V(I)$, then $1 \in I$. This is precisely the Hilbert's weak Nullstellensatz.

Theorem A.2 (Hilbert's Weak Nullstellensatz). *Let $I \subset \mathbb{R}[x]$ be an ideal. If $V(I) = \emptyset$, then $1 \in I$.*

Now we consider I to be an ideal generated by polynomials having real coefficients. Let T be a basic closed semialgebraic set. There is a certificate for $V_{\mathbb{R}}(I) \cap T = \emptyset$. This is the so-called Positivstellensatz.

Theorem A.3 (Positivstellensatz, [27]). *Let $I \subset \mathbb{R}[x]$ be an ideal, and $T = \{x \in \mathbb{R}^n : g_1(x) \geq 0, \dots, g_r(x) \geq 0\}$ be defined by real polynomials g_i . If $V_{\mathbb{R}}(I) \cap T = \emptyset$, then there exist SOS polynomials σ_ν such that*

$$-1 \equiv \sum_{\nu \in \{0,1\}^r} \sigma_\nu \cdot g_1^{\nu_1} \cdots g_r^{\nu_r} \quad \text{mod } I.$$

Theorem A.4 (Putinar's Positivstellensatz, [21]). *Let I be an ideal of $\mathbb{R}[x]$ and $T = \{x \in \mathbb{R}^n : g_1(x) \geq 0, \dots, g_r(x) \geq 0\}$ be defined by real polynomials g_i . Suppose there exist $R > 0$ and SOS polynomials $s_0(x), \dots, s_m(x)$ such that (the archimedean condition holds)*

$$R - \|x\|_2^2 \equiv s_0(x) + s_1(x)g_1(x) + \dots + s_m(x)g_m(x) \pmod{I}.$$

If a polynomial $f(x)$ is positive on $V_{\mathbb{R}}(I) \cap T$, then there exist SOS polynomials σ_i such that

$$f(x) \equiv \sigma_0(x) + \sigma_1(x)g_1(x) + \dots + \sigma_m(x)g_m(x) \pmod{I}.$$

In the following, we review some elementary background about resultants and discriminants. More details would be found in [5, 8, 28].

Let f_1, \dots, f_n be homogeneous polynomials in $x = (x_1, \dots, x_n)$. The resultant $Res(f_1, \dots, f_n)$ is a polynomial in the coefficients of f_1, \dots, f_n satisfying

$$Res(f_1, \dots, f_n) = 0 \iff \exists 0 \neq u \in \mathbb{C}^n, f_1(u) = \dots = f_n(u) = 0.$$

The resultant $Res(f_1, \dots, f_n)$ is homogeneous, irreducible and has integer coefficients. When $f(x)$ is a single homogeneous polynomial, its discriminant is defined to be

$$\Delta(f) = Res\left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right).$$

Thus, we have the relation

$$\Delta(f) = 0 \iff \exists 0 \neq u \in \mathbb{C}^n, \nabla f(u) = 0.$$

The discriminants and resultants are also defined for inhomogeneous polynomials. Let f_0, f_1, \dots, f_n be general polynomials in $x = (x_1, \dots, x_n)$. Their resultant $Res(f_0, f_1, \dots, f_n)$ is then defined to be $Res(\tilde{f}_0(\tilde{x}), \tilde{f}_1(\tilde{x}), \dots, \tilde{f}_n(\tilde{x}))$, where each $\tilde{f}_i(\tilde{x}) = x_0^{\deg(f_i)} f_i(x/x_0)$ is the homogenization of $f_i(x)$. Clearly, if the polynomial system

$$f_0(x) = f_1(x) = \dots = f_n(x) = 0$$

has a solution in \mathbb{C}^n , then the homogeneous system

$$\tilde{f}_0(\tilde{x}) = \tilde{f}_1(\tilde{x}) = \dots = \tilde{f}_n(\tilde{x}) = 0$$

has a nonzero solution in \mathbb{C}^{n+1} , and hence $Res(f_0, f_1, \dots, f_n) = 0$. The reverse is not always true, because the latter homogeneous system might have a solution at infinity $x_0 = 0$. If $f(x)$ is a single nonhomogeneous polynomial, its discriminant is defined similarly as $\Delta(\tilde{f})$.

The discriminants are also defined for several polynomials. More details are in [20, Sec. 3]. Let $f_1(\tilde{x}), \dots, f_m(\tilde{x})$ be forms in $x = (x_1, \dots, x_n)$ of degrees d_1, \dots, d_m respectively, and $m \leq n - 1$. Suppose at least one $d_i > 1$. The discriminant for f_1, \dots, f_m , denoted by $\Delta(f_1, \dots, f_m)$, is a polynomial in the coefficients of f_i such that

$$\Delta(f_1, \dots, f_m) = 0$$

if and only if the polynomial system

$$f_1(x) = \cdots = f_m(x) = 0$$

has a solution $u \neq 0$ such that the matrix $[\nabla f_1(u) \ \cdots \ \nabla f_m(u)]$ does not have full rank. When $m = 1$, $\Delta(f_1, \dots, f_m)$ reduces to the standard discriminant of a single polynomial.

When f_1, \dots, f_m are nonhomogeneous polynomials in $x = (x_1, \dots, x_n)$ and $m \leq n$, the discriminant $\Delta(f_1, \dots, f_m)$ is then defined to be $\Delta(\tilde{f}_1(\tilde{x}), \dots, \tilde{f}_m(\tilde{x}))$, where each $\tilde{f}_i(\tilde{x})$ is the homogenization of $f_i(x)$.

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