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An Exact Solution of a One-Dimensional Asymmetric Exclusion Model with Open Boundaries

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A simple asymmetric exclusion model with open boundaries is solved exactly in one dimension. The exact solution is obtained by deriving a recursion relation for the steady state: if the steady state is known for all system sizes less than N, then our equation (8) gives the steady state for size N. Using this recursion, we obtain closed expressions (48) for the average occupations of all sites. The results are compared to the predictions of a mean field theory. In particular, for infinitely large systems, the effect of the boundary decays as the distance to the power $-1/2$ instead of the inverse of the distance, as predicted by the mean field theory.

KEY WORDS: Asymmetric exclusion process; steady state; phase diagram.

1. INTRODUCTION

Systems of particles with stochastic dynamics and exclusion interactions have been studied for a long time in statistical mechanics.[1] Despite their simplicity, relatively few exactly soluble cases are known,[2–7] especially when the invariant measure does not factorize.

An important class of problems deals with asymmetric exclusion processes with periodic boundary conditions. In these cases the system reaches a stationary state of constant density, and one is interested in density fluctuations and their correlations.[8–10] Recent interest in these problems is at least partly due to their close relationship to growth models,[9] whose continuum version, the KPZ equation,[10] is related in turn to the exactly soluble (in one dimension!) Burgers equation. A direct connection between

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driven lattice gas models and the noisy Bugers equation was also established.[20] On the other hand, exclusion processes in discrete space-time are related to vertex models.[6]

The purpose of this paper is to present a case which can be solved exactly: the fully asymmetric exclusion process on a segment of finite length with *open boundaries* with particles injected at one end with probability α and removed at the other end with probability β. Here, unlike the case of periodic boundary conditions, the density is *not* uniform.[11] This type of problem is related to growth models with a defect or inhomogeneity.[12-14] These models are interesting for a number of reasons. They provide examples of systems far from thermal equilibrium with long-range spatial and temporal correlations. In some cases they exhibit phase transitions in one dimension, a phenomenon which does not usually occur in systems at thermal equilibrium.

The model studied here is defined as follows. Consider a one-dimensional system of N sites. Each site i, $1 \leqslant i \leqslant N$, is either occupied by a particle ($\tau_i = 1$) or is empty ($\tau_i = 0$). This system evolves in time according to the following rule: At each time step $t \to t+1$, one chooses at random an integer $0 \leqslant i \leqslant N$ with probability $1/(N+1)$. If the integer i is between 1 and $N-1$, then the particle on site i (if there is one) jumps to site $i+1$ (if this site is empty), i.e.,

$$\begin{aligned} \tau_i(t+1) &= \tau_i(t)\,\tau_{i+1}(t) \\ \tau_{i+1}(t+1) &= \tau_{i+1}(t) + [1-\tau_{i+1}(t)]\,\tau_i(t) \end{aligned} \tag{1}$$

If the integer chosen is $i=0$, then site 1 remains occupied at time $t+1$ if it was occupied at time t, and it gets occupied with probability α if it was empty at time t. Therefore

$$\begin{aligned} \tau_1(t+1) &= 1 \qquad \text{with probability} \quad \tau_1(t) + \alpha[1-\tau_1(t)] \\ &= 0 \qquad \text{with probability} \quad (1-\alpha)[1-\tau_1(t)] \end{aligned} \tag{2}$$

Similarly, if the integer chosen is $i=N$, then site N remains empty at time $t+1$ if it was empty at time t, and it gets empty with probability β if it was occupied. So

$$\begin{aligned} \tau_N(t+1) &= 1 \qquad \text{with probability} \quad (1-\beta)\,\tau_N(t) \\ &= 0 \qquad \text{with probability} \quad 1-(1-\beta)\,\tau_N(t) \end{aligned} \tag{3}$$

In the present paper, the exact steady state is obtained for arbitrary α and β (Section 2 and the Appendix). The result is expressed as a recursion relation which relates the steady state of a system of length $N-1$ to that of a system of length N. This relation allows one to calculate the exact expression of the average occupation $\langle\tau_i\rangle_N$ of any site i of a chain of N sites. The expression of the $\langle\tau_i\rangle_N$ is derived in the present paper only for the case $\alpha=\beta=1$ (Section 3), but there is in principle no obstacle (other than a long calculation) to extend the results to arbitrary α and β. In Section 4, the results are discussed in several asymptotic limits (large N, large i). Finally, these exact results are compared with the predictions of a mean field theory in Section 5.

2. CONSTRUCTION OF THE STEADY STATE

From the stochastic dynamics defined in the introduction [Eqs. (1)–(3)], it is easy to write the equation satisfied by the steady state. If one denotes by $P_N(\tau_1,..., \tau_N)$ the steady-state probability of finding the system in configuration $\{\tau_1,..., \tau_N\}$, then the probability P_N satisfies

$$\begin{aligned}
P_N(\tau_1, \tau_2,..., \tau_N)&\\
&=\frac{1-\alpha}{N+1} P_N(\tau_1, \tau_2,..., \tau_N)\\
&\quad+\frac{\alpha}{N+1}\tau_1[P_N(0, \tau_2,..., \tau_N)+P_N(1, \tau_2,..., \tau_N)]\\
&\quad+\frac{1}{N+1}[P_N(\tau_1, \tau_2,..., \tau_N)+(\tau_2-\tau_1) P_N(1, 0, \tau_3,..., \tau_N)]\\
&\quad+\cdots\\
&\quad+\frac{1}{N+1}[P_N(\tau_1, \tau_2,..., \tau_N)+(\tau_N-\tau_{N-1}) P_N(\tau_1, \tau_2,..., \tau_{N-2}, 1, 0)]\\
&\quad+\frac{1-\beta}{N+1} P_N(\tau_1, \tau_2,..., \tau_N)\\
&\quad+\frac{\beta}{N+1}(1-\tau_N)[P_N(\tau_1,..., \tau_{N-1}, 0)+P_N(\tau_1,..., \tau_{N-1}, 1)]
\end{aligned}\tag{4}$$

which can be written as

$$\begin{aligned}
\alpha(2\tau_1-1) &P_N(0, \tau_2,..., \tau_N)\\
&+(\tau_2-\tau_1) P_N(1, 0, \tau_3,..., \tau_N)\\
&+\cdots\\
&+(\tau_N-\tau_{N-1}) P_N(\tau_1, \tau_2,..., \tau_{N-2}, 1, 0)\\
&+\beta(1-2\tau_N) P_N(\tau_1, \tau_2,..., \tau_{N-1}, 1)=0
\end{aligned}\tag{5}$$

The problem of finding the steady state amounts to solving the 2^N coupled equations (5). As explained in the Appendix, it is possible to write a recursion relation which gives $P_{N+1}(\tau_1, \tau_2,..., \tau_{N+1})$ if one knows the $P_N(\tau_1, \tau_2,..., \tau_N)$. In what follows, it will be more convenient to write recursions for weights $f_N(\tau_1, \tau_2,..., \tau_N)$ which are equal to the $P_N(\tau_1, \tau_2,..., \tau_N)$, up to a multiplicative constant. Clearly, since Eqs. (4) and (5) are linear, they are also satisfied by the f_N and the probabilities P_N will then be given by

$$P_N(\tau_1,..., \tau_N) = f_N(\tau_1,..., \tau_N)/Z_N \tag{6}$$

where

$$Z_N = \sum_{\tau_1 = 0,1} \cdots \sum_{\tau_N = 0,1} f_N(\tau_1,..., \tau_N) \tag{7}$$

We prove in the Appendix that all the $f_N(\tau_1, \tau_2,..., \tau_N)$ can be constructed by the following recursion rule:

$$\begin{aligned} f_N(\tau_1, \tau_2,..., \tau_N) & \\ = \alpha\tau_N f_{N-1}(\tau_1, \tau_2,..., \tau_{N-1}) & \\ + \alpha\beta(1-\tau_N)\,\tau_{N-1}[f_{N-1}(\tau_1, \tau_2,..., \tau_{N-2}, 1) & \\ + f_{N-1}(\tau_1, \tau_2,..., \tau_{N-2}, 0)] & \\ + \cdots & \\ + \alpha\beta(1-\tau_N)(1-\tau_{N-1}) \cdots (1-\tau_2)\,\tau_1 & \\ \times [f_{N-1}(1, \tau_2,..., \tau_{N-1}) + f_{N-1}(0, \tau_2,..., \tau_{N-1})] & \\ + \beta(1-\tau_N)(1-\tau_{N-1}) \cdots (1-\tau_1)\, f_{N-1}(\tau_1, \tau_2,..., \tau_{N-1}) & \end{aligned} \tag{8}$$

The problem is, of course, easy to solve directly for small N: for example, for $N = 1$, one can show that

$$f_1(0) = \beta \qquad \text{and} \qquad f_1(1) = \alpha \tag{9a}$$

and for $N = 2$ the choice

$$f_2(0, 0) = \beta^2; \quad f_2(1, 0) = \alpha\beta(\alpha + \beta); \quad f_2(0, 1) = \alpha\beta; \quad f_2(1, 1) = \alpha^2 \tag{9b}$$

solves the steady state equation (4) or (5). So (8) together with (9) enables one to obtain recursively the f_N for all N.

Because the recursion (8) is rather complicated, the calculation of expectations in the steady state is not immediate. In Section 3 we shall see how the average occupations $\langle\tau_i\rangle_N$ can be obtained from (8) in the case $\alpha=\beta=1$. It is, however, interesting to note that since the recursion (8) is valid for any choice of α and β, there is no difficulty, other than doing a long calculation, to extend the results of Section 3 to the more general case of arbitrary α and β.

Before discussing how the calculation of average occupations can be done from (8), it is worth noting that in the case

$$\alpha+\beta=1 \tag{10}$$

there is a simple closed expression of the f_N

$$f_N(\tau_1, \tau_2,..., \tau_N)=\alpha^T(1-\alpha)^{N-T}, \qquad T=\sum_i \tau_i \tag{11}$$

which solves (4) and (5).

This means that when (10) holds, the steady state factorizes and leads to very simple expressions of all the correlation functions,

$$\langle\tau_i\rangle=\alpha \qquad \text{and} \qquad \langle\tau_i\tau_j\cdots\tau_k\rangle=\langle\tau_i\rangle\langle\tau_j\rangle\cdots\langle\tau_k\rangle \tag{12}$$

3. THE AVERAGE OCCUPATIONS $\langle\tau_i\rangle_N$

In this section we will obtain closed expressions for the average occupation $\langle\tau_i\rangle_N$ of site i for a system of length N. From the recursion (8) on f_N, it is clear that the only quantities needed to obtain the $\langle\tau_i\rangle_N$ are the following sums:

$$Z_N=\sum_{\tau_1}\cdots\sum_{\tau_N} f_N(\tau_1,..., \tau_N) \tag{13}$$

and

$$T_{N,i}=\sum_{\tau_1}\cdots\sum_{\tau_N} \tau_i f_N(\tau_1,..., \tau_N) \tag{14}$$

Once these sums are known, the expression for $\langle\tau_i\rangle_N$ is simply given by

$$\langle\tau_i\rangle=T_{N,i}/Z_N \tag{15}$$

The difficulty in computing the Z_N and the $T_{N,i}$ from the recursion (8) is that one cannot get closed recursions for these quantities. It is necessary to compute at the same time other quantities: the $Y_N(K)$ and the $X_N(K, p)$ defined by

$$Y_N(K)=\sum_{\tau_1}\cdots\sum_{\tau_N}(1-\tau_N)(1-\tau_{N-1})\cdots(1-\tau_K) f_N(\tau_1,..., \tau_N) \tag{16}$$

and

$$X_N(K, p) = \sum_{\tau_1} \cdots \sum_{\tau_N} (1-\tau_N) \cdots (1-\tau_K)\, \tau_p f_N(\tau_1,..., \tau_N) \tag{17}$$

By extension of the definition of the $Y_N(K)$ and of the $X_N(K, p)$ we shall define for convenience

$$Y_N(N+1) = Z_N \tag{18}$$

and

$$X_N(N+1, p) = T_{N,p} \tag{19}$$

Having defined the $Y_N(K)$ for $1 \leqslant K \leqslant N+1$ and the $X_N(K, p)$ for $p+1 \leqslant K \leqslant N+1$, one can obtain from (8) closed recursions for these quantities. For the Y_N one gets

$$\begin{aligned} Y_N(1) &= \beta Y_{N-1}(1) \\ Y_N(K) &= Y_N(K-1) + \alpha\beta Y_{N-1}(K) \qquad \text{for} \quad 2 \leqslant K \leqslant N \\ Y_N(N+1) &= Y_N(N) + \alpha Y_{N-1}(N) \end{aligned} \tag{20}$$

This recursion, together with the initial conditions

$$Y_1(1) = \beta \qquad \text{and} \qquad Y_1(2) = \alpha + \beta \tag{21}$$

determines all the $Y_N(K)$.

Clearly, the length of the calculation to determine all the $Y_N(K)$ increases like N^2 (instead of exponentially with N, as we would have needed had we had to calculate all the f_N).

Once the Y_N are determined, one can determine recursively all the X_N from (8):

$$\begin{aligned} X_N(p+1, p) &= \alpha\beta Y_{N-1}(p+1) && \text{for} \quad 1 \leqslant p \leqslant N-1 \\ X_N(K, p) &= X_N(K-1, p) + \alpha\beta X_{N-1}(K, p) && \text{for} \quad p+2 \leqslant K \leqslant N \\ X_N(N+1, p) &= X_N(N, p) + \alpha X_{N-1}(N, p) && \text{for} \quad 1 \leqslant p \leqslant N-1 \\ X_N(N+1, N) &= \alpha Y_{N-1}(N) \end{aligned} \tag{22}$$

This recursion, together with the initial condition

$$X_1(2, 1) = \alpha \tag{23}$$

and the knowledge of the $Y_N(K)$ obtained from (20) and (21), determine all the $X_N(K, p)$.

At this stage we see that one can obtain the exact values of all the $Y_N(K)$ and the $X_N(K, p)$ for arbitrary N, for any choice of α and β, by iterating the recursions (20) and (22) on a computer.

In order to obtain closed analytic expressions, we will limit our calculations to the case

$$\alpha = \beta = 1 \tag{24}$$

The reason for this choice is to avoid manipulation of too complicated expressions in the calculations below. However, since the recursions (20) and (22) are valid for any α and β, there is nothing to prevent extension of the results to arbitrary α and β.

A possible way to proceed with the recursions (20) and (22) is to introduce generating functions. If one defines for $p \geqslant -1$

$$L_p(\lambda) = \sum_{N=p+1}^{\infty} \lambda^N Y_N(N-p) \tag{25}$$

with the convention that $Y_0(1) = 1$, the recursion (20) on $Y_N(K)$ becomes

$$L_p(\lambda) - \lambda^{p+1} Y_{p+1}(1) = L_{p+1}(\lambda) + \lambda[L_{p-1}(\lambda) - \lambda^p Y_p(1)] \qquad \text{for} \quad p \geqslant 0 \tag{26}$$

and

$$L_{-1}(\lambda) = \lambda L_{-1}(\lambda) + 1 + L_0(\lambda) \tag{27}$$

Because the $Y_p(1)$ are easy to calculate $[Y_p(1) = 1$ for all $p]$, one immediately concludes from (26) that the general solution of (26) is

$$L_p = A_1(\lambda)\left(\frac{1-(1-4\lambda)^{1/2}}{2}\right)^{p+1} + A_2(\lambda)\left(\frac{1+(1-4\lambda)^{1/2}}{2}\right)^{p+1} \tag{28}$$

The problem now is to determine the constants $A_1(\lambda)$ and $A_2(\lambda)$. From the definition (25), it is clear that for any p the first term in the sum $L_p(\lambda)$ is proportional to λ^{p+1}. This is compatible with the result (28) only if

$$A_2(\lambda) = 0 \tag{29}$$

The expression of $A_1(\lambda)$ is then easy to find from the boundary condition (27) and one ends up with

$$L_p(\lambda) = \left(\frac{1-(1-4\lambda)^{1/2}}{2}\right)^{p+3} \frac{1}{\lambda^2} \tag{30}$$

From the definitions (18) and (25), it is clear that this allows one to obtain all the Z_N since

$$\sum_{N=0}^{\infty} Z_N \lambda^N = L_{-1}(\lambda) = \left(\frac{1-(1-4\lambda)^{1/2}}{2\lambda}\right)^2 = \frac{1-2\lambda-(1-4\lambda)^{1/2}}{2\lambda^2} \tag{31}$$

This leads to the following expression for the Z_N:

$$Z_N = \frac{(2N+2)!}{(N+1)!\,(N+2)!} \tag{32}$$

One can also treat the recursions on the $X_N(K, p)$ using generating functions. If one defines

$$M_{q,r}(\lambda) = \sum_{N=r+1}^{\infty} \lambda^N X_N(N-q, N-r) \tag{33}$$

then it follows from the recursions (22) that

$$\begin{aligned} M_{r-1,r}(\lambda) &= \lambda L_{r-2}(\lambda) - \lambda^r && \text{for} \quad r \geqslant 1 \\ M_{q,r}(\lambda) &= M_{q+1,r}(\lambda) + \lambda M_{q-1,r-1} && \text{for} \quad 0 \leqslant q \leqslant r-2 \\ M_{-1,r}(\lambda) &= M_{0,r}(\lambda) + \lambda M_{-1,r-1}(\lambda) && \text{for} \quad r \geqslant 1 \\ M_{-1,0}(\lambda) &= \lambda L_{-1}(\lambda) \end{aligned} \tag{34}$$

To proceed, one can introduce new generating functions $H_q(\mu, \lambda)$ defined by

$$H_q(\mu, \lambda) = \sum_{r=q+1}^{\infty} \mu^r M_{q,r}(\lambda) \tag{35}$$

Then the recursions (34) become

$$H_{q+1} - H_q + \lambda\mu H_{q-1} = \lambda\mu^{q+1}(\lambda L_{q-2} - L_{q-1}) \qquad \text{for} \quad q \geqslant 1 \tag{36}$$

$$H_1 - H_0 + \lambda\mu H_{-1} = \lambda\mu(\lambda-1)\, L_{-1} + \lambda\mu \tag{37}$$

$$H_{-1} = H_0 + \lambda\mu H_{-1} + \lambda L_{-1} \tag{38}$$

The general solution of (36) with the boundary conditions (37) and (38) is

$$\begin{aligned} H_q(\mu, \lambda) = {} & \frac{1}{\lambda(1-\mu)} \left(\frac{1-(1-4\lambda)^{1/2}}{2}\right)^{q+2} \mu^{q+1} \\ & + B_1(\mu, \lambda) \left(\frac{1-(1-4\lambda\mu)^{1/2}}{2}\right)^{q+1} \\ & + B_2(\mu, \lambda) \left(\frac{1+(1-4\lambda\mu)^{1/2}}{2}\right)^{q+1} \end{aligned} \tag{39}$$

As before, one can argue that

$$B_2(\mu, \lambda) = 0 \tag{40}$$

because this is the only way that the leading order in H_q is of the order μ^{q+1}. Then from the boundary conditions (37) and (38), one determines the constant $B_1(\mu, \lambda)$ and one ends up with the following final expression:

$$\begin{aligned} H_q(\mu, \lambda) = &\frac{1}{\lambda(1-\mu)} \left(\frac{1-(1-4\lambda)^{1/2}}{2} \right)^{q+2} \mu^{q+1} \\ &+ \frac{1}{(1-\mu)\,\lambda^2 \mu^2} \left(\frac{1-(1-4\lambda\mu)^{1/2}}{2} \right)^{q+3} \\ &\times \left[\mu \frac{1-(1-4\lambda)^{1/2}}{2} - 1 \right] \end{aligned} \tag{41}$$

To obtain the average occupations, it turns out that only $H_{-1}(\mu, \lambda)$ is needed [see (19), (33) and (35)] and one finally gets from (41)

$$H_{-1}(\mu, \lambda) = \frac{1}{1-\mu} \frac{1-(1-4\lambda\mu)^{1/2}}{2\lambda\mu} \left[\frac{1-(1-4\lambda)^{1/2}}{2\lambda} - \frac{1-(1-4\lambda\mu)^{1/2}}{2\lambda\mu} \right] \tag{42}$$

This expression is the generating function of all the $T_{N,i}$ [see (14), (19), (33), and (35)] since

$$H_{-1}(\mu, \lambda) = \sum_{r=0}^{\infty} \mu^r \sum_{N=r+1}^{\infty} \lambda^N T_{N, N-r} \tag{43}$$

Using expression (42) and the identity

$$\frac{1-(1-4x)^{1/2}}{2x} = \sum_{n=0}^{\infty} \frac{(2n)!}{n!\,(n+1)!} x^n \tag{44}$$

one can rewrite H_{-1} as

$$\begin{aligned} H_{-1}(\mu, \lambda) &= \frac{1}{1-\mu} \sum_{q=0}^{\infty} \frac{(2q)!}{q!\,(q+1)!} (\lambda\mu)^q \sum_{n=1}^{\infty} \frac{(2n)!}{n!\,(n+1)!} [\lambda^n - (\lambda\mu)^n] \\ &= \sum_{q=0}^{\infty} \sum_{n=1}^{\infty} \frac{(2q)!}{q!\,(q+1)!} \frac{(2n)!}{n!\,(n+1)!} \lambda^{q+n} \mu^q \sum_{m=0}^{n-1} \mu^m \\ &= \sum_{q=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=m+1}^{\infty} \frac{(2q)!}{q!\,(q+1)!} \frac{(2n)!}{n!\,(n+1)!} \lambda^{q+n} \mu^{q+m} \end{aligned} \tag{45}$$

By making the change of variables $q+m=r$ and $N=n+q$, one ends up with

$$H_{-1}(\mu, \lambda)=\sum_{r=0}^{\infty} \sum_{N=1}^{\infty} \lambda^N \mu^r \sum_{q=0}^{r} \frac{(2q)!}{q!\,(q+1)!} \frac{(2N-2q)!}{(N-q)!\,(N-q+1)!} \tag{46}$$

By comparing with (43), we see that this gives closed expressions for the $T_{N,i}$ and therefore from (15) and (32), one obtains

$$\langle \tau_{N-r} \rangle_N = \sum_{q=0}^{r} \frac{(2q)!}{q!\,(q+1)!} \frac{(2N-2q)!}{(N-q+1)!\,(N-q)!} \frac{(N+2)!\,(N+1)!}{(2N+2)!} \tag{47}$$

It is easy to prove by recursion over r that (47) can be simplified and rewritten as

$$\langle \tau_K \rangle_N = \frac{1}{2}+\frac{1}{4}\frac{(2K)!}{(K!)^2}\frac{(N!)^2}{(2N+1)!}\frac{(2N-2K+2)!}{[(N-K+1)!]^2}(N-2K+1) \tag{48}$$

This expression is the main result derived in the present paper. The initial method used to obtain it had nothing to do with the above derivation. It was first obtained by solving exactly the problem on a computer for small sizes ($N \leqslant 10$). By looking at the numerical results, it appeared that some $\langle \tau_i \rangle_N$ were simple rational numbers and (48) was first obtained by trying to find an expression which would agree with these numerical results. It is only after this formula had been guessed as a conjecture that the above method was developed to demonstrate its validity.

It is worth noting that in the whole problem (when $\alpha=\beta=1$), there is a particle–hole symmetry which implies that

$$\langle \tau_K \rangle_N = 1 - \langle \tau_{N+1-K} \rangle_N \tag{49}$$

This relation, which could have been guessed from the very beginning, is of course satisfied by (48).

4. SOME LIMITING CASES

From the exact expression (48), it is easy to obtain the expression for $\langle \tau_K \rangle_N$ in some asymptotic limits corresponding to large systems.[15,16]

First let us consider the limit $N \to \infty$ with fixed K. One gets

$$\langle \tau_K \rangle_\infty = \frac{1}{2}+\frac{(2K)!}{(K!)^2}\frac{1}{2^{2K+1}} \tag{50}$$

Then if K becomes large, keeping $N \to \infty$ first, one obtains

$$\langle \tau_K \rangle_\infty - \frac{1}{2} \simeq \frac{1}{2\sqrt{\pi}} K^{-1/2} \tag{51}$$

This $K^{-1/2}$ convergence is consistent with a conjecture that was verified by simulations[11] and with what was found recently in a model of growth with an inhomogeneity.[14]

Another limit of interest is when N and K are large, with $|N-K| \gg 1$ and $K \gg 1$. Then, using Stirling's formula, one gets

$$\langle \tau_K \rangle_N - \frac{1}{2} \simeq \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{N}} \frac{1+N-2K}{[4K(N-K)]^{1/2}} \tag{52}$$

which becomes, in the scaling limit $K = Nx$,

$$\langle \tau_K \rangle_N - \frac{1}{2} \simeq \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{N}} \frac{1-2x}{2[x(1-x)]^{1/2}} \tag{53}$$

5. COMPARISON TO THE MEAN FIELD PREDICTIONS

In this section we see how a mean field theory can be developed for the model defined in the introduction. To do so, it is useful to first obtain some exact relations which relate correlation functions in the steady state.

The easiest relation one can obtain is by requiring that in the steady state the average occupation $\langle \tau_i(t) \rangle$ remains unchanged.[17] Consider a site i $(2 \leqslant i \leqslant N-2)$ at time t with an occupation number $\tau_i(t)$. Then the occupation $\tau_i(t+1)$ at time $t+1$ is given by

$$\begin{aligned} \tau_i(t+1) &= \tau_i(t) && \text{with probability} \quad 1 - \frac{2}{N+1} \\ &= \tau_i(t) + [1-\tau_i(t)]\, \tau_{i-1}(t) && \text{with probability} \quad \frac{1}{N+1} \\ &= \tau_i(t)\, \tau_{i+1}(t) && \text{with probability} \quad \frac{1}{N+1} \end{aligned} \tag{54}$$

These three possibilities correspond respectively to the cases where the site updated is different from $i-1$ and i, equal to $i-1$, or equal to i.

Averaging (54), one gets

$$\begin{aligned} \langle \tau_i(t+1) \rangle = \langle \tau_i(t) \rangle + \frac{1}{N+1} [&\langle \tau_i(t)\, \tau_{i+1}(t) \rangle - \langle \tau_i(t)\, \tau_{i-1}(t) \rangle \\ &+ \langle \tau_{i-1}(t) \rangle - \langle \tau_i(t) \rangle] \end{aligned} \tag{55}$$

In the steady state the expectations are time independent and one obtains

$$\langle \tau_i \rangle - \langle \tau_{i+i} \tau_i \rangle = \langle \tau_{i-1} \rangle - \langle \tau_i \tau_{i-1} \rangle \tag{56}$$

This could have been guessed, since it expresses the conservation of the flux of particles.

By similar reasoning one can treat the special cases of sites $i = 1$ and $i = N$, and one gets

$$\begin{aligned} \langle \tau_1 \rangle - \langle \tau_1 \tau_2 \rangle &= \alpha(1 - \langle \tau_1 \rangle) \\ \beta \langle \tau_N \rangle &= \langle \tau_{N-1} \rangle - \langle \tau_N \tau_{N-1} \rangle \end{aligned} \tag{57}$$

The steady-state equations (56) and (57) are exact. They are, however, not very useful to calculate the $\langle \tau_i \rangle$ because the $\langle \tau_i \rangle$ are related to higher correlations $(\langle \tau_i \tau_{i+1} \rangle)$ which themselves are related to other correlations.

An approximation scheme very often used in statistical mechanics is the mean field theory, in which the effect of correlations is neglected, i.e., correlations like $\langle \tau_i \tau_j \rangle$ are replaced by $\langle \tau_i \rangle \langle \tau_j \rangle$. If we denote by t_i the value of $\langle \tau_i \rangle$ in the mean field theory, one gets from (56) and (57) the following equations for the t_i:

$$t_i - t_i t_{i+1} = t_{i-1} - t_{i-1} t_i \tag{58a}$$

$$t_1 - t_1 t_2 = \alpha(1 - t_1) \tag{58b}$$

$$\beta t_N = t_{N-1} - t_N t_{N-1} \tag{58c}$$

The solution of these N equations (with N unknowns) determines for any finite N the average occupations t_i.

These equations are nonlinear and therefore are at first sight difficult to solve. There is, however, a simple way of looking at Eqs. (58). We can rewrite (58a) as a recursion

$$t_{i+1} = 1 - \frac{C}{t_i} \tag{59}$$

C is a constant (to be determined later). In fact C is the current of particles in the chain. This recursion is shown in Figs. 1a–1c. For $C < 1/4$ there are two fixed points

$$t_\pm = \tfrac{1}{2}[1 \pm (1 - 4C)^{1/2}] \tag{60}$$

t_+ is stable and t_- is unstable.

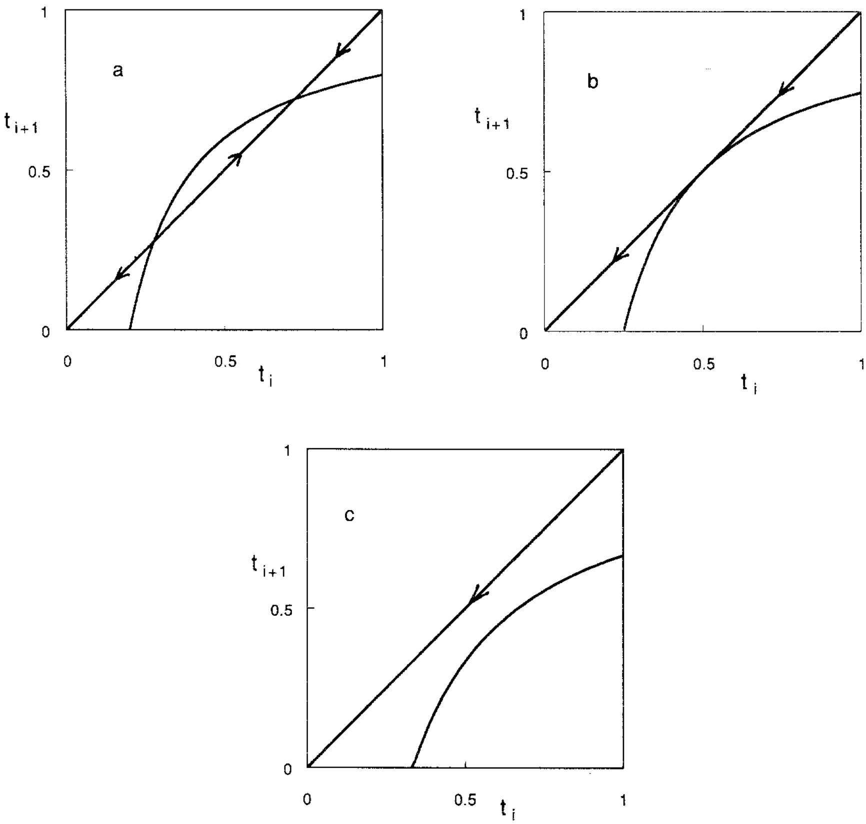

Fig. 1. Graphical representation of the mean-field recursion relation (59) for (a) $C < 1/4$, (b) $C = 1/4$, and (c) $C > 1/4$.

When $C = 1/4$ there is only one marginal fixed point, and there are no real-valued fixed points when $C > 1/4$. The recursion (59) can be solved (using the fact that t_{i+1} is a homographic function of t_i),

$$t_i = \frac{-t_+ t_- (t_+^{i-1} - t_-^{i-1}) + (t_+^i - t_-^i)\, t_1}{-t_+ t_- (t_+^{i-2} - t_-^{i-2}) + (t_+^{i-1} - t_-^{i-1})\, t_1} \tag{61}$$

Clearly t_i depends on C and on t_1; for $i = N$ this relation can be cast as

$$t_N = f(C, t_1) \tag{62}$$

This solution holds for any α, β; these parameters enter through the boundary conditions (58b), (58c), which, together with (62), determine,

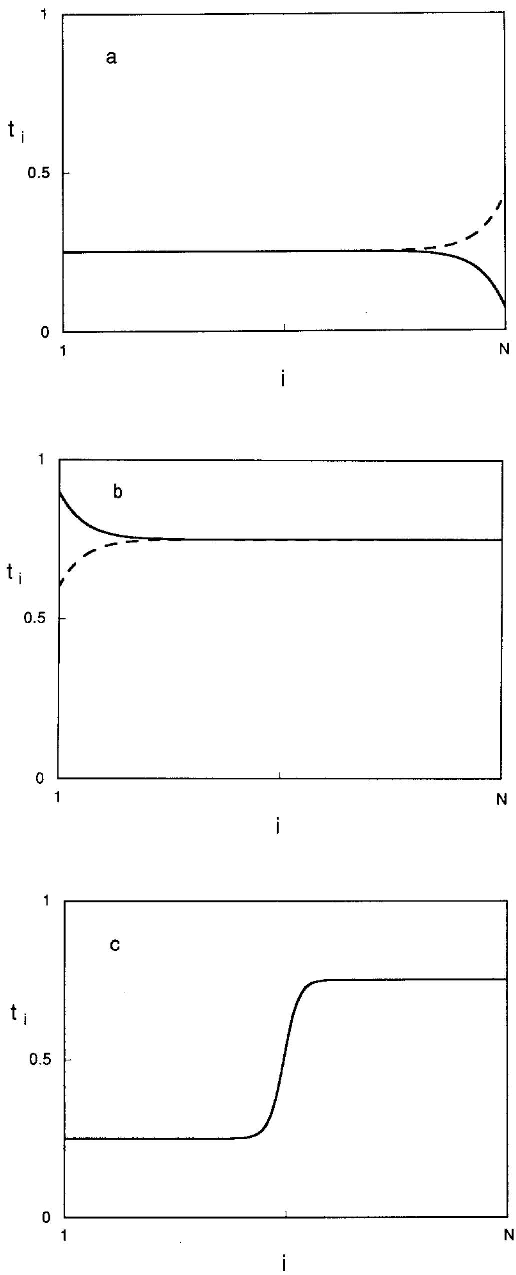

Fig. 2. Schematic mean-field density profiles in (a) the low-density phase, (b) the high-density phase, (c) on the coexistence line, and (d) in the maximal current phase.

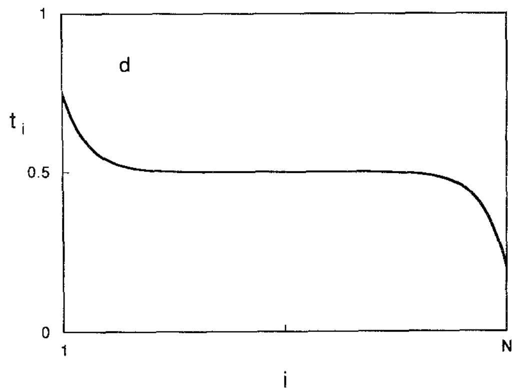

Fig. 2. *(Continued)*

t_1, t_N, and C. We now proceed to show graphically the different kinds of solutions that can be obtained. These correspond to different phases of the system.

Low-Density Phase. $t_1 = t_- + 0^{\pm}$; $t_N < t_+$. In this phase t_1 is set infinitesimally close to the unstable fixed point t_-. For many iterations t_i stays there, and deviates from t_- only when $i \simeq N$, either up or down, depending on t_1. The resulting density profile is shown on Fig. 2a.

The conditions $t_1 = t_-$ and $t_N < t_+$ are consistent with Eqs. (58b), (58c), and (62) provided

$$\alpha \leqslant 1/2, \qquad \beta > \alpha \tag{63}$$

In this regime we have the solution

$$t_1 = \alpha, \qquad t_N = \frac{\alpha(1-\alpha)}{\beta} \\ C = \alpha(1-\alpha) \tag{64}$$

High-Density Phase. $t_N = t_+ + 0^{\pm}$, $t_1 > t_-$. In this phase t_1 is set in the domain of attraction of the stable fixed point. Hence t_i relaxes toward t_+, which is (almost) reached by the time we get to t_N. The corresponding density profile is shown on Fig. 2b. This phase occurs when

$$\beta \leqslant 1/2, \qquad \beta < \alpha \tag{65}$$

and the solution is characterized by

$$t_N = 1-\beta, \qquad t_1 = 1 - \frac{\beta(1-\beta)}{\alpha} \\ C = \beta(1-\beta) \tag{66}$$

Coexistence Line. $t_1 = t_- + 0^+$, $t_N = t_+ + 0^-$. The high- and low-density phases described above coexist when the recursion starts at t_1 infinitesimally above t_- and iterates to the stable fixed point t_+. The solution, shown in Fig. 2c, contains a front or "domain wall." Such a solution occurs when

$$\alpha = \beta < 1/2 \tag{67}$$

As usual in coexistence regions, the position of this wall depends on the manner in which the limits $\alpha \to \beta$ and $N \to \infty$ are taken. On the coexistence line we have

$$\begin{gathered} t_1 = \alpha, \qquad t_N = 1 - \alpha \\ C = \alpha(1 - \alpha) \end{gathered} \tag{68}$$

Maximal Current Phase. $t_1 \geqslant 1/2$, $t_N \leqslant 1/2$. In this phase the system attains the maximal current it can support,

$$C = 1/4 \tag{69}$$

throughout the phase. The recursion is described by Fig. 1b; actually for any finite N one has the situation of Fig. 1c, with that of Fig. 1b approached in the $N \to \infty$ limit. The exact nature of this limit will be discussed below. The resulting density profile is shown in Fig. 2d.

Except near $i = 1$ and $i = N$, the density is near 1/2. This kind of solution exists when

$$\alpha \geqslant 1/2 \qquad \text{and} \qquad \beta \geqslant 1/2 \tag{70}$$

and is characterized by

$$t_1 = 1 - \frac{1}{4\alpha}, \qquad t_N = \frac{1}{4\beta} \tag{71}$$

The transition to this phase, along the $\beta = 1$ line, was recently studied by Krug.[11]

Phase Diagram. Collecting the results of Eqs. (63), (65), (67), and (70), we obtain the phase diagram shown in Fig. 3. The low- and high-density phases are separated by a first-order transition line. Each of these phases undergoes continuous transitions to the phase at maximal current.

The Density Profile in the $N \to \infty$ Limit. We return now to the maximal current phase, and obtain expressions for the density profile in the

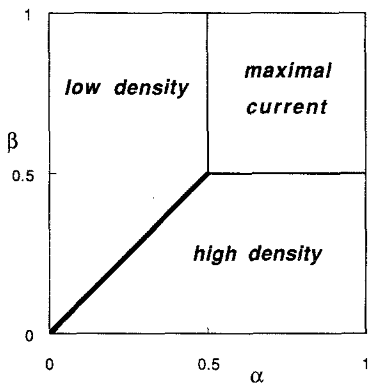

Fig. 3. Mean-field phase diagram. The high- and low-density phases coexist on the heavy line.

$N \to \infty$ limit. These (mean field) expressions can be compared with their exact analogs given in Section 4. For simplicity we restrict the discussion to the case

$$\alpha = \beta = 1 \tag{72}$$

In this case the boundary conditions imply

$$t_1 = 1 - C, \qquad t_N = C, \qquad C = \tfrac{1}{4} + O^+ \tag{73}$$

The density profile, as given by (61) in this case, takes the form

$$t_i = \frac{1}{2} + \frac{1}{2}(t_+ - t_-)\frac{t_+^i + t_-^i}{t_+^{i+1} - t_-^{i+1}} \tag{74}$$

Note that for $C > 1/4$ the fixed points $t_\pm$ are complex. One can now compare the result (74) with those of Section 4.

First, if i is fixed and $N \to \infty$, one finds that $C \to 1/4$ and that t_i is given by

$$t_i = \frac{i+2}{2(i+1)} \tag{75}$$

This implies that the mean field prediction[11] is

$$t_K - \frac{1}{2} = \langle \tau_K \rangle_\infty - \frac{1}{2} \simeq \frac{1}{2K} \tag{76}$$

instead of the $K^{-1/2}$ decay found in the exact solution (51).

Another prediction[11] of the mean field theory which follows from (74) is that as $N \to \infty$,

$$C - \frac{1}{4} = \frac{\pi^2}{4N^2} \tag{77}$$

and

$$\langle \tau_K \rangle - \frac{1}{2} \simeq \frac{\pi}{2N} \frac{\cos(\pi x)}{\sin(\pi x)} \tag{78}$$

for $K = Nx$, to be compared with (53). We see that the mean field prediction is a $1/N$ correction instead of the $1/\sqrt{N}$ correction obtained in the exact solution (53).

6. CONCLUSION

The main results of the present paper are the recursion relation (8), which gives the steady state for systems of arbitrary size N, and the exact expression (48) of the average occupations in the steady state.

Since the recursion relation (8) is valid for arbitrary α and β, it is certainly possible to generalize some of the results of this paper to this more general case. From the knowledge of the steady state one should be able to calculate also the correlation functions, in addition to the average densities.

The phase diagram shown in Fig. 3 was obtained by a mean field calculation. Since the steady state is known exactly for arbitrary α and β, it is certainly possible to obtain from (8) the true phase diagram to check that it remains identical to the mean field phase diagram as conjectured by Krug.[21]

By exploiting the results in terms of the corresponding interface growth model,[12–14] one should be able to obtain explicit expressions describing the fluctuating properties of a growing interface.

Looking at the above results, one can wonder whether exact results could be obtained for the partially asymmetric case (where particles have a nonzero probability of jumping to their left), or to other geometries. A case of interest would be the ring geometry with one special bond having a different hopping rate.[18] At the moment, a direct generalization of our results to these other cases seems difficult. One can try, however, to follow an approach similar to ours, by solving exactly the steady-state equations on a computer for small $N \leqslant 10$ (easy part), guessing the general result from these finite-N results (difficult part), and then, it is hoped, finding a proof (tedious part).

Another possibility would be to try Bethe ansatz techniques,[8,19] which were used recently to calculate the gap for the ring geometry.

APPENDIX

The goal of this Appendix is to prove that recursion (8) together with the initial condition (9) gives the steady state for all N. To prove the recursion (8), we assume that it gives the solution of the steady-state equations for all systems of size up to $N-1$ and show that f_N built from (8) is also a solution of (5). To do so we need to distinguish three cases:

Case 1. $\tau_N = 1$. In this case, according to (8), f_N is given by

$$f_N(\tau_1, \tau_2, ..., \tau_N) = \alpha f_{N-1}(\tau_1, \tau_2, ..., \tau_{N-1}) \tag{A1}$$

By using the expression (A1) in (5) and the fact that f_{N-1} is a solution of (5), one finds that the condition for f_N to solve (5) is

$$\begin{aligned}(1-\tau_{N-1})\, & f_N(\tau_1, ..., \tau_{N-2}, 1, 0) \\ & - \beta f_N(\tau_1, \tau_2, ..., \tau_{N-1}, 1) \\ & = \alpha\beta(1-2\tau_{N-1})\, f_{N-1}(\tau_1, \tau_2, ..., \tau_{N-2}, 1)\end{aligned} \tag{A2}$$

which is always satisfied, as f_N is given by recursion (8).

Case 2. $\tau_N = \tau_{N-1} = \cdots \tau_{K+1} = 0$ *and* $\tau_K = 1$ *for* $1 \leqslant K \leqslant N-1$. Using Eq. (8), one finds

$$\begin{aligned} f_N(\tau_1, & \tau_2, ..., \tau_{K-1}, 1, 0, ..., 0) \\ & = \alpha\beta[f_{N-1}(\tau_1, \tau_2, ..., \tau_{K-1}, 1, 0, ..., 0) \\ & \quad + f_{N-1}(\tau_1, \tau_2, ..., \tau_{K-1}, 0, 0, ..., 0)]\end{aligned} \tag{A3}$$

For f_N to satisfy (5), one needs to prove that

$$\begin{aligned}(1-\tau_{K-1})\, & f_N(\tau_1, \tau_2, ..., \tau_{K-2}, 1, 0, ..., 0) \\ & - f_N(\tau_1, \tau_2, ..., \tau_{K-2}, \tau_{K-1}, 1, 0, ..., 0) \\ & + \beta f_N(\tau_1, \tau_2, ..., \tau_{K-1}, 1, 0, ..., 0, 1) \\ = \alpha\beta[& (1-\tau_{K-1})\, f_{N-1}(\tau_1, \tau_2, ..., \tau_{K-2}, 1, 0, ..., 0) \\ & - f_{N-1}(\tau_1, ..., \tau_{K-2}, \tau_{K-1}, 1, 0, ..., 0) \\ & + \beta f_{N-1}(\tau_1, ..., \tau_{K-2}, \tau_{K-1}, 1, 0, ..., 0, 1) \\ & - \tau_{K-1} f_{N-1}(\tau_1, ..., \tau_{K-2}, 1, 0, ..., 0) \\ & + \beta f_{N-1}(\tau_1, \tau_2, ..., \tau_{K-2}, \tau_{K-1}, 0, ..., 0, 1)]\end{aligned} \tag{A4}$$

By replacing f_N by its expression (8) and (A4), one gets that (A4) is satisfied if

$$\begin{aligned}(1-\tau_{K-1})\,&f_{N-1}(\tau_1,..., \tau_{K-2}, 0, 0,..., 0)\\ &-f_{N-1}(\tau_1, \tau_2,..., \tau_{K-1}, 0,..., 0)\\ =&-f_{N-1}(\tau_1,..., \tau_{K-2}, \tau_{K-1}, 1, 0,..., 0)\\ &-\tau_{K-1} f_{N-1}(\tau_1,..., \tau_{K-2}, 1, 0,..., 0)\\ &+\beta f_{N-1}(\tau_1,..., \tau_{K-2}, \tau_{K-1}, 1, 0,..., 0, 1)\\ &+\beta f_{N-1}(\tau_1, \tau_2,..., \tau_{K-1}, 0,..., 0, 1)\end{aligned} \tag{A5}$$

Using once more the recursion (8) to replace f_{N-1} by its expression in terms of f_{N-2}, one completes the proof of (A5).

Case 3. $\tau_N=\tau_{N-1}=\cdots=\tau_1=0$. In this case,

$$f_N(0, 0,..., 0)=\beta f_{N-1}(0,..., 0) \tag{A6}$$

For f_N to satisfy (5) in this case, one needs that

$$-\alpha f_N(0,..., 0)+\beta f_N(0, 0,..., 0, 1)=0 \tag{A7}$$

which is a clear consequence of (A6) and of the fact that

$$f_N(0,..., 0, 1)=\alpha f_{N-1}(0, 0,..., 0) \tag{A8}$$

This completes the proof that the recursion (8) gives all the steady states for arbitrary N.

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