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AN EXACT SOLUTION OF THE CYLINDRICAL WAVE EQUATION FOR ELECTROMAGNETIC FIELD IN FRACTIONAL DIMENSIONAL SPACE

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Abstract—This work deals with an exact solution of cylindrical wave equation for electromagnetic field in fractional dimensional space. The obtained fractional solution is a generalization of the cylindrical wave equation from integer dimensional space to a fractional dimensional space. The resulting theoretical framework can be used to study the phenomenon of electromagnetic wave propagation in any fractal media because fractal media can be described as an ordinary media in a fractional dimensional space. The classical results are recovered from fractional solution when integer dimensional space is considered.

1. INTRODUCTION

In the last few decades there has been considerable interest in the study of physical description of confinement in low dimensional systems assuming a fractional dimension of the space [1–6]. A method to replace the real confining structures with an effective space, where the measure of anisotropy or confinement is given by non-integer dimension D, was proposed in [2, 3]. Fractal structures have been studied within the fractional-dimensional space approach in [7]. The slight deviation of the value of dimension of our terrestrial locale from three has also been considered by several authors [1, 8, 9].

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Axiomatic basis for the concept of fractional space for 2-spatial coordinate space is proposed in [1] and this work was extended to n-orthogonal coordinate space in [6]. Fractional calculus [10], which is a branch of mathematics that deal with generalization of differentiations and integrations to arbitrary non integer order is used by several authors to describe many electromagnetic problems as well as fractional dimensional space [11–21].

A formulation of Schrödinger wave mechanics in D-dimensional fractional space is provided in [1]. Applications of the concept of fractional space in electromagnetic research include description of fractional multipoles in fractional space [3], study of electromagnetic field in fractional space by solving Poisson's equation in D-dimensional space with $2 < D \leq 3$ [4], study of electromagnetic fields on fractals [5] and a discussion on scattering of electromagnetic fields in fractal media [22]. A novel generalization of differential electromagnetic equations in fractional space have been presented recently in [15]. The radiations from fractal geometries have also been discussed by different authors recently [27–33].

The study of wave propagation and scattering in fractal structures is important in practical applications such as communications, remote sensing and navigation [22]. The phenomenon of wave propagation in fractal structures can be described by replacing these confining fractal structures with a D-dimensional fractional space. Thus, given this simple value of D, the real system can be modeled in a simple analytical way.

For these reasons a new solution of the wave equation in D-dimensional factional space is important. General plane wave solutions of the vector wave equation in fractional space have been presented in [14]. But the problems that exhibit cylindrical geometries are needed to be solved using cylindrical coordinate system. In this work, we present an exact solution of cylindrical wave equation in fractional space that can be used to describe the phenomenon of wave propagation in any fractal media.

In Section 2, we investigate full analytical cylindrical wave solution to the wave equation in D-dimensional fractional space, where the parameter D is used to describe the measure distribution of space. In Section 3, the solution of wave equation in integer-dimensional space is justified from the results of previous section. Finally, in Section 4, conclusions are drawn.

2. EXACT SOLUTION OF CYLINDRICAL WAVE EQUATION IN FRACTIONAL SPACE

The problems that exhibit cylindrical geometries are needed to be solved using cylindrical coordinate system. As for the case of rectangular geometries, the electric and magnetic fields of cylindrical geometry boundary-value problem must satisfy corresponding cylindrical wave equation [23]. Let us assume that the space in which fields must be solved is fractional dimensional and source-free. For source-free and lossless media, the vector wave equations for the complex electric and magnetic field intensities are given by the Helmholtz equation as follows [23].

$$\nabla^2\mathbf{E} + \beta^2\mathbf{E} = 0 \tag{1}$$

$$\nabla^2\mathbf{H} + \beta^2\mathbf{H} = 0 \tag{2}$$

where, $\beta^2 = \omega^2\mu\varepsilon$. Time dependency e^{jwt} has been suppressed throughout the discussion. Here, ∇^2 is the Laplacian operator in D-dimensional fractional space and is defined in rectangular coordinate system as follows [6].

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\alpha_1 - 1}{x}\frac{\partial}{\partial x} + \frac{\partial^2}{\partial y^2} + \frac{\alpha_2 - 1}{y}\frac{\partial}{\partial y} + \frac{\partial^2}{\partial z^2} + \frac{\alpha_3 - 1}{z}\frac{\partial}{\partial z} \tag{3}$$

where x, y and z are rectangular coordinates. Equation (3) uses three parameters ($0 < \alpha_1 \leq 1$, $0 < \alpha_2 \leq 1$ and $0 < \alpha_3 \leq 1$) to describe the measure distribution of space where each one is acting independently on a single coordinate and the total dimension of the system is $D = \alpha_1 + \alpha_2 + \alpha_3$. To find cylindrical wave solutions of wave equation in D-dimensional fractional space, it is likely that a cylindrical coordinate system (ρ, ϕ, z) will be used. In cylindrical coordinate system (3) becomes

$$\begin{aligned}\nabla^2 = & \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}(\alpha_1 + \alpha_2 - 1)\frac{\partial}{\partial \rho} \\ & + \frac{1}{\rho^2}\left(\frac{\partial^2}{\partial \phi^2} - \{(\alpha_1 - 1)\tan\phi + (\alpha_2 - 1)\cot\phi\}\frac{\partial}{\partial \phi}\right) \\ & + \frac{\partial^2}{\partial z^2} + \frac{\alpha_3 - 1}{z}\frac{\partial}{\partial z}\end{aligned} \tag{4}$$

Once the solution to any one of Equations (1) and (2) in fractional space is known, the solution to the other can be written by an interchange of $\mathbf{E}$ with $\mathbf{H}$ or $\mathbf{H}$ with $\mathbf{E}$ due to duality [23]. We will examine the solution for $\mathbf{E}$.

In cylindrical coordinates, a general solution for $\mathbf{E}$ can be written as

$$\mathbf{E}(\rho, \phi, z) = \hat{a}_\rho E_\rho(\rho, \phi, z) + \hat{a}_\phi E_\phi(\rho, \phi, z) + \hat{a}_z E_z(\rho, \phi, z) \tag{5}$$

Substituting (5) into (1) we can write that

$$\nabla^2(\hat{a}_\rho E_\rho + \hat{a}_\phi E_\phi + \hat{a}_z E_z) + \beta^2(\hat{a}_\rho E_\rho + \hat{a}_\phi E_\phi + \hat{a}_z E_z) = 0 \tag{6}$$

Since,

$$\nabla^2(\hat{a}_\rho E_\rho) \neq \hat{a}_\rho \nabla^2 E_\rho \tag{7}$$

$$\nabla^2(\hat{a}_\phi E_\phi) \neq \hat{a}_\phi \nabla^2 E_\phi \tag{8}$$

$$\nabla^2(\hat{a}_z E_z) = \hat{a}_z \nabla^2 E_z \tag{9}$$

So, Equation (6) cannot be reduced to simple scalar wave equations, but it can be reduced to coupled scalar partial differential equations. However for simplicity, the wave mode solution can be formed in cylindrical coordinates that must satisfy the following scalar wave equation:

$$\nabla^2\psi(\rho, \phi, z) + \beta^2\psi(\rho, \phi, z) = 0 \tag{10}$$

where, $\psi(\rho, \phi, z)$ is a scalar function that can represent a field or vector potential component. In expanded form (10) can be written as

$$\begin{aligned}&\frac{\partial^2\psi}{\partial\rho^2} + \frac{1}{\rho}(\alpha_1 + \alpha_2 - 1)\frac{\partial\psi}{\partial\rho}\\&+\frac{1}{\rho^2}\left(\frac{\partial\psi^2}{\partial\phi^2} - \{(\alpha_1 - 1)\tan\phi + (\alpha_2 - 1)\cot\phi\}\frac{\partial\psi}{\partial\phi}\right)\\&+\frac{\partial^2\psi}{\partial z^2} + \frac{\alpha_3 - 1}{z}\frac{\partial\psi}{\partial z} + \beta^2\psi = 0\end{aligned} \tag{11}$$

Equation (11) is separable using method of separation of variables. We consider

$$\psi(\rho, \phi, z) = f(\rho)g(\phi)h(z) \tag{12}$$

the resulting ordinary differential equations are obtained as follows:

$$\left[\rho^2\frac{d^2}{d\rho^2} + \rho(\alpha_1 + \alpha_2 - 1)\frac{d}{d\rho} + (\beta_\rho\rho)^2 - m^2\right] f(\rho) = 0 \tag{13}$$

$$\left[\frac{d^2}{d\phi^2} + \{(\alpha_1 - 1)\tan\phi + (\alpha_2 - 1)\cot\phi\}\frac{d}{d\phi} - m^2\right] g(\phi) = 0 \tag{14}$$

$$\left[\frac{d^2}{dz^2} + \frac{\alpha_3 - 1}{z}\frac{d}{dz} + \beta_z^2\right] h(z) = 0 \tag{15}$$

where, m is a constant (integer usually). In addition,

$$\beta_\rho^2 + \beta_z^2 = \beta^2 \tag{16}$$

Equation (16) is referred to as constraint equation. In addition β_ρ, β_z are known as wave constants in the ρ, z directions, respectively, which will be determined using boundary conditions.

Now, Equations (13) through (15) are needed to be solved for $f(\rho)$, $g(\phi)$ and $h(z)$, respectively. We choose to work first with $f(\rho)$. Equation (13) can be written as:

$$\left[\rho^2 \frac{d^2}{d\rho^2} + a\rho \frac{d}{d\rho} + (b\rho^\ell + c)\right] f(\rho) = 0 \tag{17}$$

where, $a = \alpha_1 + \alpha_2 - 1$, $b = \beta_\rho^2$, $c = -m^2$, $\ell = 2$. Equation (15) is closely related to Bessel's equation and its solutions is given as [24]:

$$f(\rho) = \rho^{\frac{1-a}{2}} \left[C_1 J_v\left(\frac{2}{\ell}\sqrt{b}\rho^{\frac{\ell}{2}}\right) + C_2 Y_v\left(\frac{2}{\ell}\sqrt{b}\rho^{\frac{\ell}{2}}\right)\right] \tag{18}$$

where, $v = \frac{1}{\ell}\sqrt{(1-a)^2 - 4c}$.

Using (18), the final solution of (13) is given by

$$f_1(\rho) = \rho^{1-\frac{\alpha_1+\alpha_2}{2}} \left[C_1 J_v(\beta_\rho \rho) + C_2 Y_v(\beta_\rho \rho)\right] \tag{19}$$

or

$$f_2(\rho) = \rho^{1-\frac{\alpha_1+\alpha_2}{2}} \left[D_1 H_v^{(1)}(\beta_\rho \rho) + D_2 H_v^{(2)}(\beta_\rho \rho)\right] \tag{20}$$

where, $v = \frac{1}{2}\sqrt{(2-\alpha_1-\alpha_2)^2 + 4m^2}$. In (19) $J_v(\beta_\rho \rho)$ is referred to as Bessel function of the first kind of order v and $Y_v(\beta r)$ as the Bessel function of the second kind of order v. They are used to represent standing waves. In (20) $H_v^{(1)}(\beta r)$ is referred to as Hankel function of the first kind of order v and $H_v^{(2)}(\beta_\rho \rho)$ as the Hankel function of the second kind of order v, and are used to represent traveling waves.

Now, we find the solution of Equation (14) for $g(\phi)$. Equation (14) can be reduced to following Gaussian hypergeometric equation after proper mathematical steps under substitution $\xi = \sin^2(\phi)$ [24]:

$$\xi(1-\xi)\frac{d^2 g(\phi)}{d\xi^2} + \{(A+B+1)\xi - C\}\frac{dg(\phi)}{d\xi} + ABg(\phi) = 0 \tag{21}$$

where,

$$A + B + 1 = \frac{1}{2}(2 - \alpha_2 + \alpha_1) \tag{22}$$

$$AB = -\frac{m^2}{4} \tag{23}$$

$$C = \frac{1}{2}(2 - \alpha_2) \tag{24}$$

solution to Equation (21) is given as [24]:

$$g(\phi) = C_3 F(A, B, C;\, \xi) + C_4 \xi^{1-C} F(A-C+1, B-C+1, 2-C;\, \xi) \quad (25)$$

where,

$$F(A, B, C;\, \xi) = 1 + \sum_{k=1}^{\infty} \frac{(A)_k (B)_k}{(C)_k} \frac{\xi^k}{k!} \quad (26)$$

with,

$$(A)_k = A(A+1)\dots(A+k+1) \quad (27)$$

$F(A, B, C;\, \xi)$ is known as Gaussian hypergeometric function, and A, B, C are known from (22) through (24).

Now, we find the solution of Equation (15) for $h(z)$. Equation (15) can be written as:

$$\left[z\frac{d^2}{dz^2} + e\frac{d}{dz} + \beta_z^2 z\right] h(z) = 0 \quad (28)$$

where, $e = \alpha_3 - 1$. Equation (28) is reducible to Bessel's equation under substitution $h = z^n \zeta$ as follows:

$$\left[z^2\frac{d^2}{dz^2} + z\frac{d}{dz} + (\beta_z^2 z^2 - n^2)\right] \zeta(z) = 0, \quad n = \frac{|1-e|}{2} \quad (29)$$

The solution of Bessel's equation in (29) is given as [24]

$$\zeta(z) = C_5 J_n(\beta_z z) + C_6 Y_n(\beta_z z) \quad (30)$$

where, $J_n(\beta_z z)$ is referred to as Bessel function of the first kind of order n, $Y_n(\beta_z z)$ as the Bessel function of the second kind of order n. Finally the solution of (15) becomes

$$h(z) = z^n \left[C_5 J_n(\beta_z z) + C_6 Y_n(\beta_z z)\right], \quad n = 1 - \frac{\alpha_3}{2} \quad (31)$$

The appropriate solution forms of $f(\rho)$, $g(\phi)$ and $h(z)$ depend upon the problem. From (12), (19), (25) and (31), a typical solution for $\psi(r, \theta, \phi)$ to represent the fields within a cylindrical geometry may take the form

$$\begin{aligned}\psi(\rho, \phi, z) = &\left[\rho^{1-\frac{\alpha_1+\alpha_2}{2}} \{C_1 J_v(\beta_\rho \rho) + C_2 Y_v(\beta_\rho \rho)\}\right] \times [\{C_3 F(A, B, C;\, \xi) \\ &+ C_4 \xi^{1-C} F(A-C+1, B-C+1, 2-C;\, \xi)\}] \\ &\times [z^n \{C_5 J_n(\beta_z z) + C_6 Y_n(\beta_z z)\}] \quad (32)\end{aligned}$$

where, $\xi = \sin^2(\phi)$ and C_1 through C_6 are constant coefficients. Equation (32) provides a general solution to cylindrical wave equation in fractional space. This solution can be used to study the phenomenon of electromagnetic wave propagation in any non-integer dimensional space.

3. DISCUSSION ON CYLINDRICAL WAVE SOLUTION IN FRACTIONAL SPACE

Equation (32) is the generalization of the concept of wave propagation from integer dimensional space to the non-integer dimensional space. As a special case, for three-dimensional space, this problem reduces to classical wave propagation concept; i.e., as a special case, if we set $\alpha_1 = \alpha_2 = \alpha_3 = 1$ in Equations (19), (25) and (31), we get cylindrical wave solution in integer dimensional space. For $\alpha_1 = \alpha_2 = 1$ Equations (19) and (20) provide

$$f_1(\rho) = C_1 J_m(\beta_\rho \rho) + C_2 Y_m(\beta_\rho \rho) \tag{33}$$

and

$$f_2(\rho) = D_1 H_m^{(1)}(\beta_\rho \rho) + D_2 H_m^{(2)}(\beta_\rho \rho) \tag{34}$$

Similarly, if we set $\alpha_1 = \alpha_2 = 1$ in Equations (22) and (24), we get $A = -B = \frac{m}{2}$, $C = \frac{1}{2}$. Now, considering following special forms of Gaussian hypergeometric function [25]:

$$F\left(\lambda, -\lambda, \frac{1}{2}; \sin^2 \nu\right) = \cos(2\lambda\nu) \tag{35}$$

$$F\left(\lambda, 1-\lambda, \frac{3}{2}; \sin^2 \nu\right) = \frac{\sin[(2\lambda - 1)\nu]}{(2\lambda - 1)\sin(\nu)} \tag{36}$$

Equation (25) can be reduced to

$$g(\phi) = C_3 \cos(m\phi) + C_4 \sin(m\phi) \tag{37}$$

In a similar way, if we set $\alpha_3 = 1$ in (32) then $n = \frac{1}{2}$ and it gives

$$h(z) = z^{\frac{1}{2}} \left[C_5 J_{\frac{1}{2}}(\beta_z z) + C_6 Y_{\frac{1}{2}}(\beta_z z)\right] \tag{38}$$

Using Bessel functions of fractional order [26]:

$$J_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \sin(z) \tag{39}$$

$$Y_{\frac{1}{2}}(z) = -\sqrt{\frac{2}{\pi z}} \cos(z) \tag{40}$$

Equation (13) can be reduced to

$$h(z) = C_5' \sin(\beta_z z) + C_6' \cos(\beta_z z) \tag{41}$$

where, $C_i' = C_i \sqrt{\frac{2}{\pi \beta_z}}$, $i = 5, 6$.

From (12), (33), (37) and (41), a typical solution in three dimensional space (a special case of fractional space) for $\psi(\rho,\phi,z)$ to represent the fields within a cylindrical geometry will take the form

$$\psi(\rho,\phi,z) = [C_1 J_m(\beta_\rho\rho) + C_2 Y_m(\beta_\rho\rho)] \times [C_3\cos(m\phi) + C_4\sin(m\phi)] \times [C'_5\sin(\beta_z z) + C'_6\cos(\beta_z z)] \tag{42}$$

which is comparable to the cylindrical wave solutions of the wave equation in integer dimensional space obtained by Balanis [23].

As an example, the fields inside a circular waveguide filled with fractal media of dimension D can be obtained by assuming a D-dimensional fractional space inside the circular waveguide. Within such circular waveguide of radius a (see Figure 1), standing waves are created in the radial (ρ) direction, periodic waves in the ϕ-direction, and traveling waves in the z-direction.

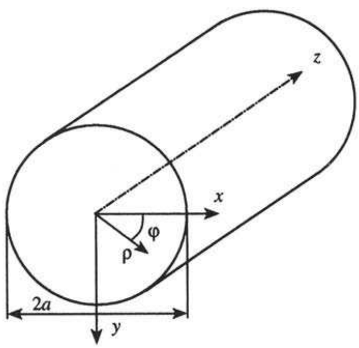

Figure 1. Cylindrical waveguide of circular cross section.

For the fields to be finite at $\rho = 0$ where $Yv(\beta_\rho\rho)$ possesses a singularity, (32) reduces to

$$\begin{aligned}\psi_1(\rho,\phi,z) &= \left[\rho^{1-\frac{\alpha_1+\alpha_2}{2}}\{C_1 J_v(\beta_\rho\rho)\}\right] \times [\{C_3 F(A,B,C;\xi) \\ &\quad + C_4\xi^{1-C} F(A-C+1, B-C+1, 2-C;\xi)\}] \\ &\quad \times \left[z^n\left\{C_5 H_n^{(2)}(\beta_z z) + C_6 H_n^{(1)}(\beta_z z)\right\}\right]\end{aligned} \tag{43}$$

To represent the fields in the region outside the cylinder, where three dimensional space is assumed because there is no fractal media outside the cylinder, a typical solution for $\psi(\rho,\phi,z)$ would take the form

$$\begin{aligned}\psi_2(\rho,\phi,z) &= \left[C_2 H_m^{(2)}(\beta_\rho\rho)\right] \times [C_3\cos(m\phi) + C_4\sin(m\phi)] \\ &\quad \times [C'_5\sin(\beta_z z) + C'_6\cos(\beta_z z)]\end{aligned} \tag{44}$$

In the region outside the cylinder, outward traveling waves are formed, in contract to standing waves inside the cylinder. In this way, the general cylindrical wave solution in fractional space can be used to study the wave propagation in the cylindrical geometries containing fractal media.

Now, as another example we assume that a cylindrical wave exists in a fractional space due to some infinite line source. Since the source do not vary with z, the fields will not vary with z but will propagate away from the source in ρ-direction. Also for simplicity, we choose to visualize only the radial amplitude variations of scalar field ψ in

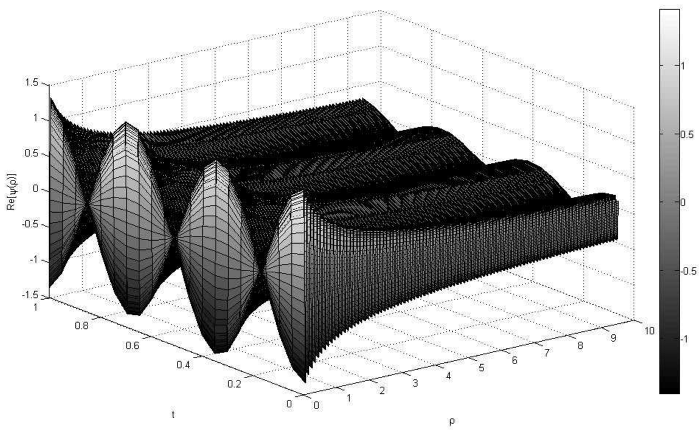

Figure 2. Cylindrical wave propagation in Euclidean space ($D = 3$).

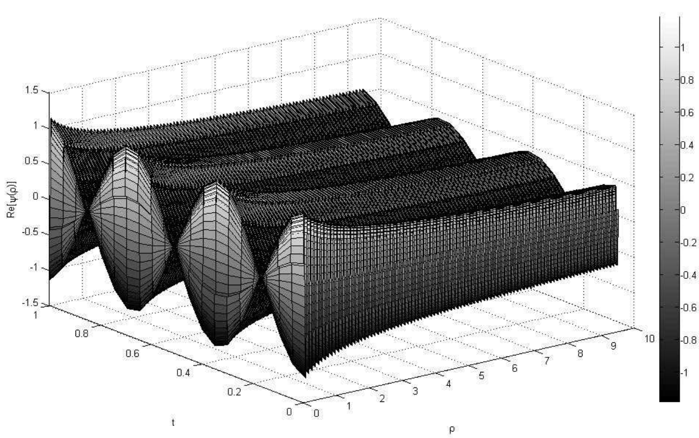

Figure 3. Cylindrical wave propagation in fractional space ($D = 2.5$).

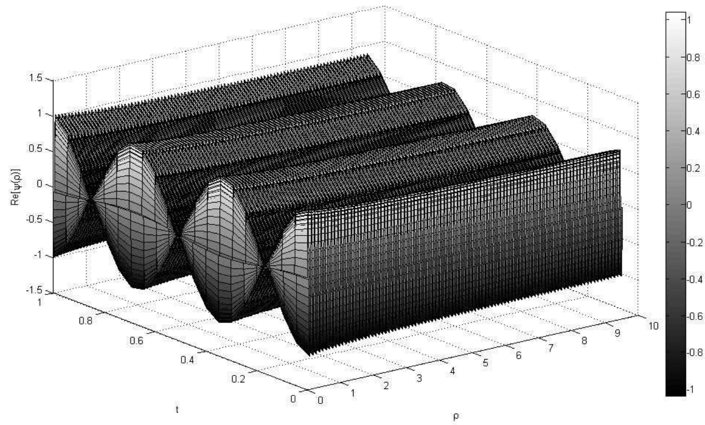

Figure 4. Cylindrical wave propagation in fractional space ($D = 2.1$).

fractional space which is given by (32) as:

$$\psi(\rho) = A\rho^{1-\frac{\alpha_1+\alpha_2}{2}} H_v^{(2)}(\beta_\rho \rho) \tag{45}$$

Also, if we choose a single parameter for non-integer dimension D where $2 < D \leq 3$, i.e, we take $\alpha_2 = \alpha_3 = 1$ so $D = \alpha_1 + 2$. In this case (45) becomes

$$\psi(\rho) = A\rho^{\frac{3-D}{2}} H_v^{(2)}(\beta_\rho \rho) \tag{46}$$

In (46), using asymptotic expansions of Hankel functions [25] for $\rho \to \infty$, we see that the amplitude variations of field ψ are related with radial distance ρ as

$$\psi(\rho) \propto \rho^{1-\frac{D}{2}} \tag{47}$$

From (47),
for $D = 3$, $\psi(\rho) \propto \frac{1}{\sqrt{\rho}}$,
for $D = 2.5$, $\psi(\rho) \propto \frac{1}{\rho^{0.25}}$,
for $D = 3$, $\psi(\rho) \propto \frac{1}{\rho^{0.05}}$.

Assuming a time dependency e^{jwt}, the radial amplitude variations of scalar field ψ are shown for different values of dimension D in Figure 2 through Figure 4. It is seen that the amplitude of cylindrical wave propagating in higher dimensional space decays rapidly.

4. CONCLUSION

An exact solution of cylindrical wave equation for electromagnetic field in D-dimensional fractional space is presented. The obtained exact solution of cylindrical wave equation is a generalization of classical integer-dimensional solution to a non-integer dimensional space. For all investigated cases when D is an integer dimension, the classical results are recovered. The investigated solution provides a basis for the application of the concept of fractional space to the wave propagation phenomenon in fractal media.

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