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Research Article

An Exact Solution of the Second-Order Differential Equation with the Fractional/Generalised Boundary Conditions

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We analysed the initial/boundary value problem for the second-order homogeneous differential equation with constant coefficients in this paper. The second-order differential equation with respect to the fractional/generalised boundary conditions is studied. We presented particular solutions to the considered problem. Finally, a few illustrative examples are shown.

1. Introduction

The second-order differential equations provide an important mathematical tool for modelling the phenomena occurring in dynamical systems. Examples of linear or nonlinear equations appear in almost all of the natural and engineering sciences and arise in many fields of physics.

Many scientists have studied various aspects of these problems, such as physical systems described by the Duffing equation [1], noncommutative harmonic oscillators [2], oscillators in quantum physics [3], the dynamic properties of biological oscillators [4], the analysis of single and coupled low-noise microwave oscillators [5], the Mathieu oscillator [6], the relativistic oscillator [7], or the Schrodinger type oscillator [8].

Classical differential equations are defined by using the integer order derivatives. In recent years, the class of differential equations containing fractional derivatives (known as fractional differential equations) have become an important topic. There are two approaches to obtain these types of equations. The first one is to replace the integer order derivative in classical differential equations with a fractional derivative (see, e.g., [9–14]).

The second approach is a generalisation of a method known in classical and quantum mechanics, where the differential equations are obtained from conservative Lagrangian or Hamiltonian functions. These equations are known in the literature as fractional Euler-Lagrange equations, and they contain both the left and right fractional derivatives. New mechanics models for nonconservative systems, in terms of fractional derivatives, were developed by Riewe in [15, 16] and extended by Klimek [17, 18] and Agrawal [19, 20]. Since then, many authors have studied the fractional differential equations of the variational type (see [21–26]).

In contrast to the above-mentioned references, where the authors analysed the integer order differential equations with classical boundary/initial conditions or fractional differential equations with the Dirichlet or natural boundary conditions, in this paper we consider the second-order differential equation with the fractional/generalised boundary conditions.

2. Statement of the Problem

In this paper, we solve the second-order differential equation

$$D^2 y(x) \pm k^2 y(x) = 0, \quad x \in [a,b], \; k > 0 \tag{1}$$

with respect to the following fractional/generalised boundary conditions:

$$\Phi\left(D^{\alpha_1}_{a^+} y(x)\Big|_{x=b}, D^{\alpha_2}_{a^+} y(x)\Big|_{x=b}, D^{\beta_1}_{b^-} y(x)\Big|_{x=a}, D^{\beta_2}_{b^-} y(x)\Big|_{x=a} \right) = 0, \tag{2}$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2 \in [0,2]$, $\alpha_1 \neq \alpha_2$, $\beta_1 \neq \beta_2$, and the operators $D^{\alpha}_{a^+}, D^{\alpha}_{b^-}$ denote the left and right Riemann-Liouville derivatives, defined, respectively, by [27]

$$D^{\alpha}_{a^+} y(x) := \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \dfrac{d^n}{dx^n} \displaystyle\int_a^x \dfrac{y(\tau)}{(x-\tau)^{\alpha-n+1}} d\tau, & \text{for } \alpha \in \mathbb{R} \setminus \mathbb{N}_0,\ n = [\alpha]+1 \\ D^n y(x) & \text{for } \alpha = n \in \mathbb{N}_0 \end{cases}$$
$$D^{\alpha}_{b^-} y(x) := \begin{cases} \dfrac{(-1)^n}{\Gamma(n-\alpha)} \dfrac{d^n}{dx^n} \displaystyle\int_x^b \dfrac{y(\tau)}{(\tau-x)^{\alpha-n+1}} d\tau, & \text{for } \alpha \in \mathbb{R} \setminus \mathbb{N}_0,\ n = [\alpha]+1 \\ (-1)^n D^n y(x), & \text{for } \alpha = n \in \mathbb{N}_0 \end{cases} \tag{3}$$

and $D^n y(x) \equiv d^n y(x)/dx^n$.

Let us consider two particular cases of (1).

Case i. Here, we study the following differential equation:

$$D^2 y(x) + k^2 y(x) = 0. \tag{4}$$

It is well known that (4) has a general solution, for $k \neq 0$, given by

$$y(x) = C_1 \sin(kx) + C_2 \cos(kx). \tag{5}$$

The solution (5) contains two arbitrary independent constants of integration C_1 and C_2. A particular solution can be derived from the general solution by applying the set of initial or boundary conditions.

The fractional differentiation of general solution (5) (using the left-side fractional operator) gives us

$$\begin{aligned} D^{\alpha}_{a^+} y(x) &= D^{\alpha}_{a^+} \left(C_1 \sin(kx) + C_2 \cos(kx)\right) \\ &= C_1 D^{\alpha}_{a^+} \sin(kx) + C_2 D^{\alpha}_{a^+} \cos(kx) \end{aligned} \tag{6}$$

and differentiation by using the right-side operator leads to

$$\begin{aligned} D^{\alpha}_{b^-} y(x) &= D^{\alpha}_{b^-} \left(C_1 \sin(kx) + C_2 \cos(kx)\right) \\ &= C_1 D^{\alpha}_{b^-} \sin(kx) + C_2 D^{\alpha}_{b^-} \cos(kx). \end{aligned} \tag{7}$$

Now, we formulate the following properties for Riemann-Liouville derivatives of the sine and cosine functions.

Property 1 (the left-sided Riemann-Liouville fractional derivatives of the sine and cosine functions). Let $\alpha \geq 0$ and $k \neq 0$. Then, the following relations hold:

$$D^{\alpha}_{a^+} \sin(kx) = \begin{cases} (x-a)^{-\alpha} \left(\cos(ka) \displaystyle\sum_{i=0}^{\infty} \dfrac{(-1)^i (k(x-a))^{2i+1}}{\Gamma(2i+2-\alpha)} + \sin(ka) \displaystyle\sum_{i=0}^{\infty} \dfrac{(-1)^i (k(x-a))^{2i}}{\Gamma(2i+1-\alpha)} \right) & \text{for } \alpha > 0 \wedge \alpha \notin \mathbb{N}_0 \\ k^n \sin\left(kx + \dfrac{n\pi}{2}\right) & \text{for } \alpha = n \in \mathbb{N}_0 \end{cases}$$
$$D^{\alpha}_{a^+} \cos(kx) = \begin{cases} (x-a)^{-\alpha} \left(\cos(ka) \displaystyle\sum_{i=0}^{\infty} \dfrac{(-1)^i (k(x-a))^{2i}}{\Gamma(2i+1-\alpha)} - \sin(ka) \displaystyle\sum_{i=0}^{\infty} \dfrac{(-1)^i (k(x-a))^{2i+1}}{\Gamma(2i+2-\alpha)} \right) & \text{for } \alpha > 0 \wedge \alpha \notin \mathbb{N}_0 \\ k^n \cos\left(kx + \dfrac{n\pi}{2}\right) & \text{for } \alpha = n \in \mathbb{N}_0. \end{cases} \tag{8}$$

Property 2 (the right-sided Riemann-Liouville fractional derivatives of the sine and cosine functions). Let $\alpha \geq 0$ and $k \neq 0$. Then, the following relations hold:

$$D^{\alpha}_{b^-} \sin(kx) = \begin{cases} (b-x)^{-\alpha} \left(\cos(kb) \displaystyle\sum_{i=0}^{\infty} \dfrac{(-1)^i (k(b-x))^{2i}}{\Gamma(2i+1-\alpha)} + \sin(kb) \displaystyle\sum_{i=0}^{\infty} \dfrac{(-1)^i (k(b-x))^{2i+1}}{\Gamma(2i+2-\alpha)} \right) & \text{for } \alpha > 0 \wedge \alpha \notin \mathbb{N}_0 \\ (-1)^n k^n \sin\left(kx + \dfrac{n\pi}{2}\right) & \text{for } \alpha = n \in \mathbb{N}_0 \end{cases}$$
$$D^{\alpha}_{b^-} \cos(kx) = \begin{cases} (b-x)^{-\alpha} \left(\cos(kb) \displaystyle\sum_{i=0}^{\infty} \dfrac{(-1)^i (k(b-x))^{2i}}{\Gamma(2i+1-\alpha)} + \sin(kb) \displaystyle\sum_{i=0}^{\infty} \dfrac{(-1)^i (k(b-x))^{2i+1}}{\Gamma(2i+2-\alpha)} \right) & \text{for } \alpha > 0 \wedge \alpha \notin \mathbb{N}_0 \\ (-1)^n k^n \cos\left(kx + \dfrac{n\pi}{2}\right) & \text{for } \alpha = n \in \mathbb{N}_0. \end{cases} \tag{9}$$

Proof (Properties 1 and 2). We use Taylor's series expansions of sine and cosine functions [28]

$$\sin(x) = \sum_{i=0}^{\infty} \frac{(-1)^i x^{2i+1}}{\Gamma(2i+2)}, \qquad \cos(x) = \sum_{i=0}^{\infty} \frac{(-1)^i x^{2i}}{\Gamma(2i+1)} \tag{10}$$

and properties of the left and right-sided fractional derivatives of power functions [27]

$$\begin{aligned} D_{a^+}^{\alpha}(x-a)^m &= \frac{\Gamma(m+1)}{\Gamma(m+1-\alpha)}(x-a)^{m-\alpha}, \quad \alpha \geq 0,\ m > -1 \\ D_{b^-}^{\alpha}(b-x)^m &= \frac{\Gamma(m+1)}{\Gamma(m+1-\alpha)}(b-x)^{m-\alpha}, \quad \alpha \geq 0,\ m > -1. \end{aligned} \tag{11}$$

Also we apply the known fundamental trigonometric identities

$$\begin{aligned} \sin(kx) &= \sin(k(x-a)+ka) \\ &= \sin(k(x-a))\cos(ka) + \cos(k(x-a))\sin(ka) \\ \cos(kx) &= \cos(k(x-a)+ka) \\ &= \cos(k(x-a))\cos(ka) - \sin(k(x-a))\sin(ka) \\ \sin(kx) &= -\sin(k(b-x)-kb) \\ &= -\sin(k(b-x))\cos(kb) + \cos(k(b-x))\sin(kb) \\ \cos(kx) &= \cos(k(b-x)-kb) \\ &= \cos(k(b-x))\cos(kb) + \sin(k(b-x))\sin(kb). \end{aligned} \tag{12}$$

Then

$$D_{a^+}^{\alpha}\sin(kx) = \cos(ka)\, D_{a^+}^{\alpha}\sin(k(x-a)) + \sin(ka)\, D_{a^+}^{\alpha}\cos(k(x-a)) \tag{13}$$

$$D_{a^+}^{\alpha}\cos(kx) = \cos(ka)\, D_{a^+}^{\alpha}\cos(k(x-a)) - \sin(ka)\, D_{a^+}^{\alpha}\sin(k(x-a)) \tag{14}$$

$$D_{b^-}^{\alpha}\sin(kx) = -\cos(kb)\, D_{b^-}^{\alpha}\sin(k(b-x)) + \sin(kb)\, D_{b^-}^{\alpha}\cos(k(b-x)) \tag{15}$$

$$D_{b^-}^{\alpha}\cos(kx) = \cos(kb)\, D_{b^-}^{\alpha}\cos(k(b-x)) + \sin(kb)\, D_{b^-}^{\alpha}\sin(k(b-x)), \tag{16}$$

where

$$\begin{aligned} D_{a^+}^{\alpha}&\sin(k(x-a)) \\ &= D_{a^+}^{\alpha}\left(\sum_{i=0}^{\infty}\frac{(-1)^i}{\Gamma(2i+2)}(k(x-a))^{2i+1}\right) \\ &= \sum_{i=0}^{\infty}\frac{(-1)^i k^{2i+1}}{\Gamma(2i+2)} D_{a^+}^{\alpha}(x-a)^{2i+1} \\ &= \sum_{i=0}^{\infty}\frac{(-1)^i k^{2i+1}}{\Gamma(2i+2-\alpha)}(x-a)^{2i+1-\alpha} \\ &= (x-a)^{-\alpha}\sum_{i=0}^{\infty}\frac{(-1)^i (k(x-a))^{2i+1}}{\Gamma(2i+2-\alpha)} \end{aligned} \tag{17}$$

and in a similar way we obtain

$$D_{a^+}^{\alpha}\cos(k(x-a)) = (x-a)^{-\alpha}\sum_{i=0}^{\infty}\frac{(-1)^i (k(x-a))^{2i}}{\Gamma(2i+1-\alpha)} \tag{18}$$

$$D_{b^-}^{\alpha}\sin(k(b-x)) = (b-x)^{-\alpha}\sum_{i=0}^{\infty}\frac{(-1)^i (k(b-x))^{2i+1}}{\Gamma(2i+2-\alpha)} \tag{19}$$

$$D_{b^-}^{\alpha}\cos(k(b-x)) = (b-x)^{-\alpha}\sum_{i=0}^{\infty}\frac{(-1)^i (k(b-x))^{2i}}{\Gamma(2i+1-\alpha)}. \tag{20}$$

Finally, putting (17)–(20) into (13)–(16), we obtain the formulas in Properties 1 and 2. □

Remark 3. Note that the infinite series included in formulas (17)–(20) can be expressed by formulas containing the Mittag-Leffler function. This observation leads us to the following expressions:

$$\begin{aligned} D_{a^+}^{\alpha}&\sin(k(x-a)) = k(x-a)^{1-\alpha}E_{2,2-\alpha}\left(-k^2(x-a)^2\right) \\ D_{a^+}^{\alpha}&\cos(k(x-a)) = (x-a)^{-\alpha}E_{2,1-\alpha}\left(-k^2(x-a)^2\right) \\ D_{b^-}^{\alpha}&\sin(k(b-x)) = k(b-x)^{1-\alpha}E_{2,2-\alpha}\left(-k^2(b-x)^2\right) \\ D_{b^-}^{\alpha}&\cos(k(b-x)) = (b-x)^{-\alpha}E_{2,1-\alpha}\left(-k^2(b-x)^2\right), \end{aligned} \tag{21}$$

where $E_{p,q}$ denotes the Mittag-Leffler function defined in [27]

$$E_{p,q}(x) = \sum_{i=0}^{\infty}\frac{x^i}{\Gamma(pi+q)}, \quad p,q \in \mathbb{R},\ p > 0,\ x \in \mathbb{R}. \tag{22}$$

The above-mentioned notations can be useful in case when one uses the built-in function (the Mittag-Leffler function) in a mathematical software.

Case ii. The second problem has the following form:

$$D^2 y(x) - k^2 y(x) = 0. \quad (23)$$

In this case the general solution of (23), for $k \neq 0$, is given by

$$y(x) = C_1 \sinh(kx) + C_2 \cosh(kx). \quad (24)$$

The fractional differentiation of solution (24) (using the left-side operator) gives

$$\begin{aligned} D^{\alpha}_{a^+} y(x) &= D^{\alpha}_{a^+} \left(C_1 \sinh(kx) + C_2 \cosh(kx)\right) \\ &= C_1 D^{\alpha}_{a^+} \sinh(kx) + C_2 D^{\alpha}_{a^+} \cosh(kx) \end{aligned} \quad (25)$$

and for the right-side derivative we have

$$\begin{aligned} D^{\alpha}_{b^-} y(x) &= D^{\alpha}_{b^-} \left(C_1 \sinh(kx) + C_2 \cosh(kx)\right) \\ &= C_1 D^{\alpha}_{b^-} \sinh(kx) + C_2 D^{\alpha}_{b^-} \cosh(kx). \end{aligned} \quad (26)$$

Next, we formulate the following properties for Riemann-Liouville derivatives of the hyperbolic sine and hyperbolic cosine functions.

Property 4 (the left-sided Riemann-Liouville fractional derivatives of the hyperbolic sine and hyperbolic cosine functions). Let $\alpha \geq 0$ and $k \neq 0$. Then, the following relations hold:

$$\begin{aligned} D^{\alpha}_{a^+} \sinh(kx) &= \begin{cases} (x-a)^{-\alpha} \left(\cosh(ka) \displaystyle\sum_{i=0}^{\infty} \frac{(k(x-a))^{2i+1}}{\Gamma(2i+2-\alpha)} + \sinh(ka) \sum_{i=0}^{\infty} \frac{(k(x-a))^{2i}}{\Gamma(2i+1-\alpha)} \right) & \text{for } \alpha > 0 \wedge \alpha \notin \mathbb{N}_0 \\ k^n \begin{cases} \cosh(kx) & \text{for } n = 1, 3, 5, \ldots \\ \sinh(kx) & \text{for } n = 0, 2, 4, \ldots \end{cases} & \text{for } \alpha = n \in \mathbb{N}_0. \end{cases} \\ D^{\alpha}_{a^+} \cosh(kx) &= \begin{cases} (x-a)^{-\alpha} \left(\cosh(ka) \displaystyle\sum_{i=0}^{\infty} \frac{(k(x-a))^{2i}}{\Gamma(2i+1-\alpha)} + \sinh(ka) \sum_{i=0}^{\infty} \frac{(k(x-a))^{2i+1}}{\Gamma(2i+2-\alpha)} \right) & \text{for } \alpha > 0 \wedge \alpha \notin \mathbb{N}_0 \\ k^n \begin{cases} \sinh(kx) & \text{for } n = 1, 3, 5, \ldots \\ \cosh(kx) & \text{for } n = 0, 2, 4, \ldots \end{cases} & \text{for } \alpha = n \in \mathbb{N}_0 \end{cases} \end{aligned} \quad (27)$$

Property 5 (the right-sided Riemann-Liouville fractional derivatives of the hyperbolic sine and hyperbolic cosine functions). Let $\alpha \geq 0$ and $k \neq 0$. Then, the following relations hold:

$$\begin{aligned} D^{\alpha}_{b^-} \sinh(kx) &= \begin{cases} (b-x)^{-\alpha} \left(-\cosh(kb) \displaystyle\sum_{i=0}^{\infty} \frac{(k(b-x))^{2i+1}}{\Gamma(2i+2-\alpha)} + \sinh(kb) \sum_{i=0}^{\infty} \frac{(k(b-x))^{2i}}{\Gamma(2i+1-\alpha)} \right) & \text{for } \alpha > 0 \wedge \alpha \notin \mathbb{N}_0 \\ (-1)^n k^n \begin{cases} \cosh(kx) & \text{for } n = 1, 3, 5, \ldots \\ \sinh(kx) & \text{for } n = 0, 2, 4, \ldots \end{cases} & \text{for } \alpha = n \in \mathbb{N}_0 \end{cases} \\ D^{\alpha}_{b^-} \cosh(kx) &= \begin{cases} (b-x)^{-\alpha} \left(\cosh(kb) \displaystyle\sum_{i=0}^{\infty} \frac{(k(b-x))^{2i}}{\Gamma(2i+1-\alpha)} - \sinh(kb) \sum_{i=0}^{\infty} \frac{(k(b-x))^{2i+1}}{\Gamma(2i+2-\alpha)} \right) & \text{for } \alpha > 0 \wedge \alpha \notin \mathbb{N}_0 \\ (-1)^n k^n \begin{cases} \sinh(kx) & \text{for } n = 1, 3, 5, \ldots \\ \cosh(kx) & \text{for } n = 0, 2, 4, \ldots \end{cases} & \text{for } \alpha = n \in \mathbb{N}_0. \end{cases} \end{aligned} \quad (28)$$

Proof (Properties 4 and 5). Here we apply Taylor's series expansions of the hyperbolic sine and cosine functions [28]

$$\begin{aligned} \sinh(x) &= \sum_{i=0}^{\infty} \frac{x^{2i+1}}{\Gamma(2i+2)}, \\ \cosh(x) &= \sum_{i=0}^{\infty} \frac{x^{2i}}{\Gamma(2i+1)} \end{aligned} \quad (29)$$

and the trigonometric identities

$$\begin{aligned} \sinh(kx) &= \sinh(k(x-a) + ka) \\ &= \sinh(k(x-a)) \cosh(ka) \\ &\quad + \cosh(k(x-a)) \sinh(ka) \\ \cosh(kx) &= \cosh(k(x-a) + ka) \\ &= \cosh(k(x-a)) \cosh(ka) \\ &\quad + \sinh(k(x-a)) \sinh(ka) \end{aligned}$$

$$\begin{aligned}
\sinh(kx) &= -\sinh(k(b-x)-kb) \\
&= -\sinh(k(b-x))\cosh(kb) \\
&\quad + \cosh(k(b-x))\sinh(kb) \\
\cosh(kx) &= \cosh(k(b-x)-kb) \\
&= \cosh(k(b-x))\cosh(kb) \\
&\quad - \sinh(k(b-x))\sinh(kb).
\end{aligned} \tag{30}$$

Then

$$\begin{aligned}
D_{a^+}^{\alpha}\sinh(kx) &= \cosh(ka)\, D_{a^+}^{\alpha}\sinh(k(x-a)) \\
&\quad + \sinh(ka)\, D_{a^+}^{\alpha}\cosh(k(x-a)) \\
D_{a^+}^{\alpha}\cosh(kx) &= \cosh(ka)\, D_{a^+}^{\alpha}\cosh(k(x-a)) \\
&\quad + \sinh(ka)\, D_{a^+}^{\alpha}\sinh(k(x-a)) \\
D_{b^-}^{\alpha}\sinh(kx) &= -\cosh(kb)\, D_{b^-}^{\alpha}\sinh(k(b-x)) \\
&\quad + \sinh(kb)\, D_{b^-}^{\alpha}\cosh(k(b-x)) \\
D_{b^-}^{\alpha}\cosh(kx) &= \cosh(kb)\, D_{b^-}^{\alpha}\cosh(k(b-x)) \\
&\quad - \sinh(kb)\, D_{b^-}^{\alpha}\sinh(k(b-x)),
\end{aligned} \tag{31}$$

where

$$\begin{aligned}
D_{a^+}^{\alpha}\sinh(k(x-a)) &= (x-a)^{-\alpha}\sum_{i=0}^{\infty}\frac{(k(x-a))^{2i+1}}{\Gamma(2i+2-\alpha)} \\
D_{a^+}^{\alpha}\cosh(k(x-a)) &= (x-a)^{-\alpha}\sum_{i=0}^{\infty}\frac{(k(x-a))^{2i}}{\Gamma(2i+1-\alpha)} \\
D_{b^-}^{\alpha}\sinh(k(b-x)) &= (b-x)^{-\alpha}\sum_{i=0}^{\infty}\frac{(k(b-x))^{2i+1}}{\Gamma(2i+2-\alpha)} \\
D_{b^-}^{\alpha}\cosh(k(b-x)) &= (b-x)^{-\alpha}\sum_{i=0}^{\infty}\frac{(k(b-x))^{2i}}{\Gamma(2i+1-\alpha)}.
\end{aligned} \tag{32}$$

Finally, putting (32) into (31), we obtain the formulas in Properties 4 and 5. □

Remark 6. In formulas (32), the infinite series can be also expressed by using the Mittag-Leffler function, and we obtain

$$\begin{aligned}
&D_{a^+}^{\alpha}\sinh(k(x-a)) \\
&\quad = k(x-a)^{1-\alpha}E_{2,2-\alpha}\left(k^2(x-a)^2\right) \\
&D_{a^+}^{\alpha}\cosh(k(x-a)) = (x-a)^{-\alpha}E_{2,1-\alpha}\left(k^2(x-a)^2\right) \\
&D_{b^-}^{\alpha}\sinh(k(b-x)) \\
&\quad = k(b-x)^{1-\alpha}E_{2,2-\alpha}\left(k^2(b-x)^2\right) \\
&D_{b^-}^{\alpha}\cosh(k(b-x)) = (b-x)^{-\alpha}E_{2,1-\alpha}\left(k^2(b-x)^2\right).
\end{aligned} \tag{33}$$

3. Examples of the Determination of Particular Solutions

The boundary conditions, written in the general form (2), can be used in many combinations. Now, we show three selected examples. Other combinations of the particular boundary conditions can be easily adopted by the reader (in a similar way).

Example 7. Equation (4) with the following boundary conditions given on both sides of the domain:

$$D_{b^-}^{\beta_1}y(x)\Big|_{x=a} = L_1 \wedge D_{a^+}^{\alpha_1}y(x)\Big|_{x=b} = L_2. \tag{34}$$

We substitute the general solution (5) into (34) and we have

$$\begin{aligned}
C_1 D_{b^-}^{\beta_1}\sin(kx)\Big|_{x=a} + C_2 D_{b^-}^{\beta_1}\cos(kx)\Big|_{x=a} &= L_1 \\
C_1 D_{a^+}^{\alpha_1}\sin(kx)\Big|_{x=b} + C_2 D_{a^+}^{\alpha_1}\cos(kx)\Big|_{x=b} &= L_2.
\end{aligned} \tag{35}$$

The independent constants of integration C_1 and C_2 can be determined from the solution of the following system of equations:

$$\begin{aligned}
\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} &= \begin{bmatrix} D_{b^-}^{\beta_1}\sin(kx)\Big|_{x=a} & D_{b^-}^{\beta_1}\cos(kx)\Big|_{x=a} \\ D_{a^+}^{\alpha_1}\sin(kx)\Big|_{x=b} & D_{a^+}^{\alpha_1}\cos(kx)\Big|_{x=b} \end{bmatrix}^{-1} \\
&\quad \cdot \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}.
\end{aligned} \tag{36}$$

The analytical solution of (36) is of the form

$$\begin{aligned}
C_1 &= \frac{L_1 D_{a^+}^{\alpha_1}\cos(kx)\Big|_{x=b} - L_2 D_{b^-}^{\beta_1}\cos(kx)\Big|_{x=a}}{D_{b^-}^{\beta_1}\sin(kx)\Big|_{x=a}\cdot D_{a^+}^{\alpha_1}\cos(kx)\Big|_{x=b} - D_{a^+}^{\alpha_1}\sin(kx)\Big|_{x=b}\cdot D_{b^-}^{\beta_1}\cos(kx)\Big|_{x=a}} \\
C_2 &= \frac{-L_1 D_{a^+}^{\alpha_1}\sin(kx)\Big|_{x=b} + L_2 D_{b^-}^{\beta_1}\sin(kx)\Big|_{x=a}}{D_{b^-}^{\beta_1}\sin(kx)\Big|_{x=a}\cdot D_{a^+}^{\alpha_1}\cos(kx)\Big|_{x=b} - D_{a^+}^{\alpha_1}\sin(kx)\Big|_{x=b}\cdot D_{b^-}^{\beta_1}\cos(kx)\Big|_{x=a}}.
\end{aligned} \tag{37}$$

Example 8. Equation (23) with the conditions given on the left side of the domain ($x = a$): this case corresponds to the initial value problem

$$D_{b^-}^{\beta_1} y(x)\Big|_{x=a} = L_1 \wedge D_{b^-}^{\beta_2} y(x)\Big|_{x=a} = L_2, \quad \beta_1 \neq \beta_2. \quad (38)$$

When we put the general solution (24) into (38), then we obtain the following linear system of equations:

$$\begin{aligned} &C_1 D_{b^-}^{\beta_1} \sinh(kx)\Big|_{x=a} + C_2 D_{b^-}^{\beta_1} \cosh(kx)\Big|_{x=a} = L_1 \\ &C_1 D_{b^-}^{\beta_2} \sinh(kx)\Big|_{x=a} + C_2 D_{b^-}^{\beta_2} \cosh(kx)\Big|_{x=a} = L_2 \end{aligned} \quad (39)$$

that can be also written in the matrix form

$$\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} D_{b^-}^{\beta_1} \sinh(kx)\Big|_{x=a} & D_{b^-}^{\beta_1} \cosh(kx)\Big|_{x=a} \\ D_{b^-}^{\beta_2} \sinh(kx)\Big|_{x=b} & D_{b^-}^{\beta_2} \cosh(kx)\Big|_{x=a} \end{bmatrix}^{-1} \cdot \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}. \quad (40)$$

Example 9. Equation (4) with the following set of boundary conditions:

$$\begin{aligned} &\mu_1 D_{b^-}^{\beta_1} y(x)\Big|_{x=a} + \mu_2 D_{b^-}^{\beta_2} y(x)\Big|_{x=a} \\ &\quad = L_1 \wedge D_{a^+}^{\alpha_1} y(x)\Big|_{x=b} = L_2, \\ &\qquad \mu_1, \mu_2 \in \mathbb{R}, \quad |\mu_1| + |\mu_2| > 0. \end{aligned} \quad (41)$$

This case corresponds to the generalisation of the Robin boundary condition given at $x = a$. Also here we put general solution (5) into (41) and we obtain

$$\begin{aligned} &\mu_1 \left(C_1 D_{b^-}^{\beta_1} \sin(kx)\Big|_{x=a} + C_2 D_{b^-}^{\beta_1} \cos(kx)\Big|_{x=a} \right) \\ &\quad + \mu_2 \left(C_1 D_{b^-}^{\beta_2} \sin(kx)\Big|_{x=a} + C_2 D_{b^-}^{\beta_2} \cos(kx)\Big|_{x=a} \right) \\ &\quad = L_1 \\ &C_1 D_{a^+}^{\alpha_1} \sin(kx)\Big|_{x=b} + C_2 D_{a^+}^{\alpha_1} \cos(kx)\Big|_{x=b} = L_2 \end{aligned} \quad (42)$$

or

$$\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} \mu_1 D_{b^-}^{\beta_1} \sin(kx)\Big|_{x=a} + \mu_2 D_{b^-}^{\beta_2} \sin(kx)\Big|_{x=a} & \mu_1 D_{b^-}^{\beta_1} \cos(kx)\Big|_{x=a} + \mu_2 D_{b^-}^{\beta_2} \cos(kx)\Big|_{x=a} \\ D_{a^+}^{\alpha_1} \sin(kx)\Big|_{x=b} & D_{a^+}^{\alpha_1} \cos(kx)\Big|_{x=b} \end{bmatrix}^{-1} \cdot \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \quad (43)$$

from which we can easily determine the constants of integration C_1 and C_2.

Remark 10. One can note that the fractional boundary conditions $D_{b^-}^{\beta} y(x)|_{x=a}$ and $D_{a^+}^{\alpha} y(x)|_{x=b}$ for integral values of parameters α and β take the classical forms of boundary conditions; this means $D_{b^-}^{n} y(x)|_{x=a} \equiv (-1)^n y^{(n)}(a)$ and $D_{a^+}^{n} y(x)|_{x=b} \equiv y^{(n)}(b)$. In particular, it should be noted that the difference occurs, among others, in the boundary condition $D_{b^-}^{1} y(x)|_{x=a} \equiv -y'(a)$ and this form should be taken into account.

4. Example of Solutions

On the basis of the proposed method, to find the particular solutions to the considered equations (4) and (23), we calculated constants of integration C_1 and C_2 occurring in the general solutions that satisfy the sets of the given initial or boundary conditions (various combinations). In Figures 1–3, numerous examples of solutions have been presented.

5. Conclusions

The initial/boundary value problem for the second-order homogeneous differential equations with constant coefficients has been considered. The general solutions to these equations are widely known and involve arbitrary constants. Our aim was to find the particular solutions to this problem which satisfy the generalised boundary conditions. Such boundary conditions complement the set of classic boundary conditions (including the Dirichlet, Neumann, and Robin types) by including the fractional ones.

The use of the fractional boundary conditions in the considered initial/boundary value problem required the fractional differentiation of the general solutions. We derived the formulas for the left- and right-sided Riemann-Liouville fractional derivatives of the sine, cosine, hyperbolic sine, and hyperbolic cosine functions that occur in the general solutions. On this basis, the integration constants in these solutions were determined analytically.

On the plots, one can observe that the obtained results for the fractional boundary conditions are located between the solutions to the considered problem with respect to the classical (integer order) boundary conditions. Such behaviour of the particular solutions gives new possibilities in physical phenomena modelling, like the harmonic oscillator modelling, among others. In the future, we plan to apply this approach to seek solutions to other types of the initial/boundary value problems, in particular to the four-order problems.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

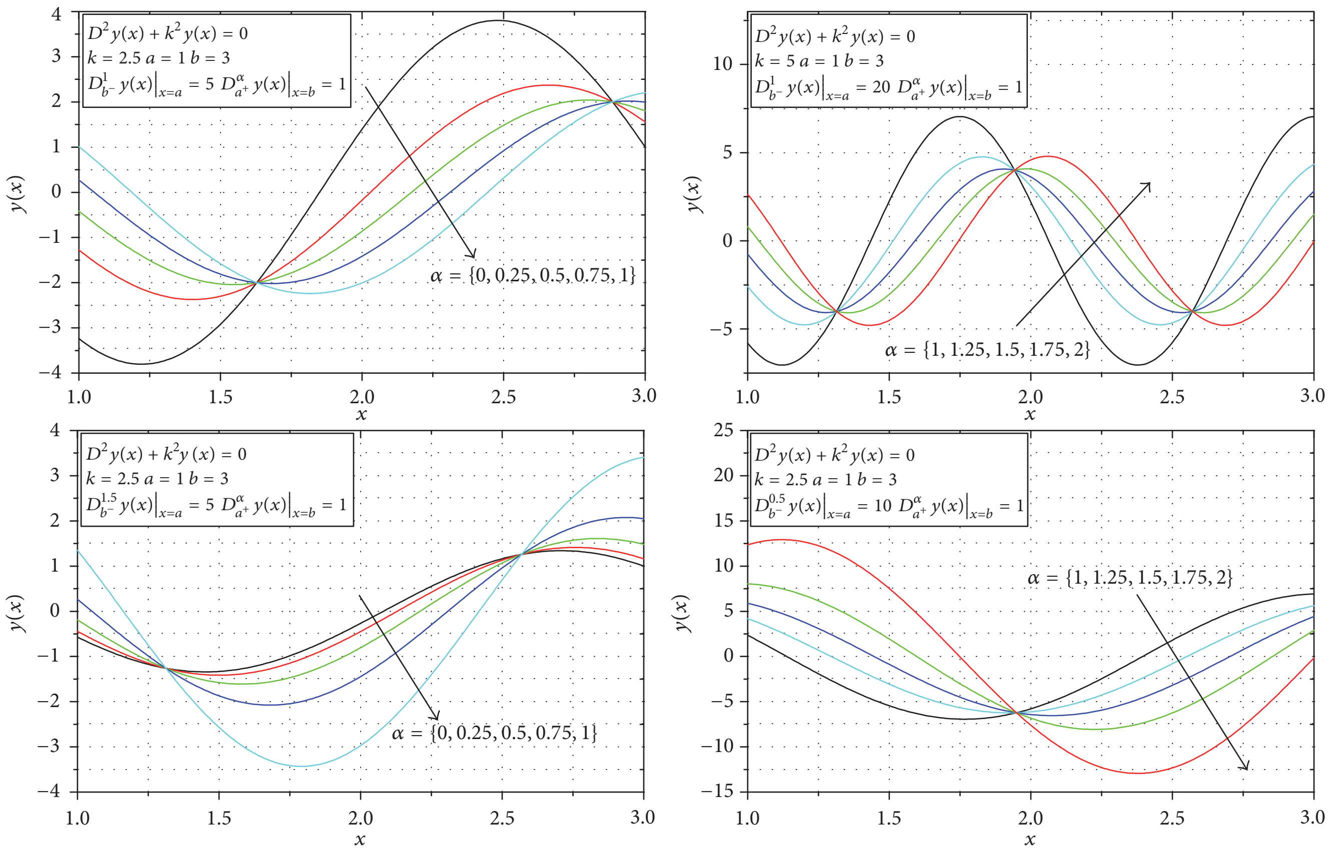

FIGURE 1: The particular solutions of (4) for selected boundary conditions (see details in the legends).

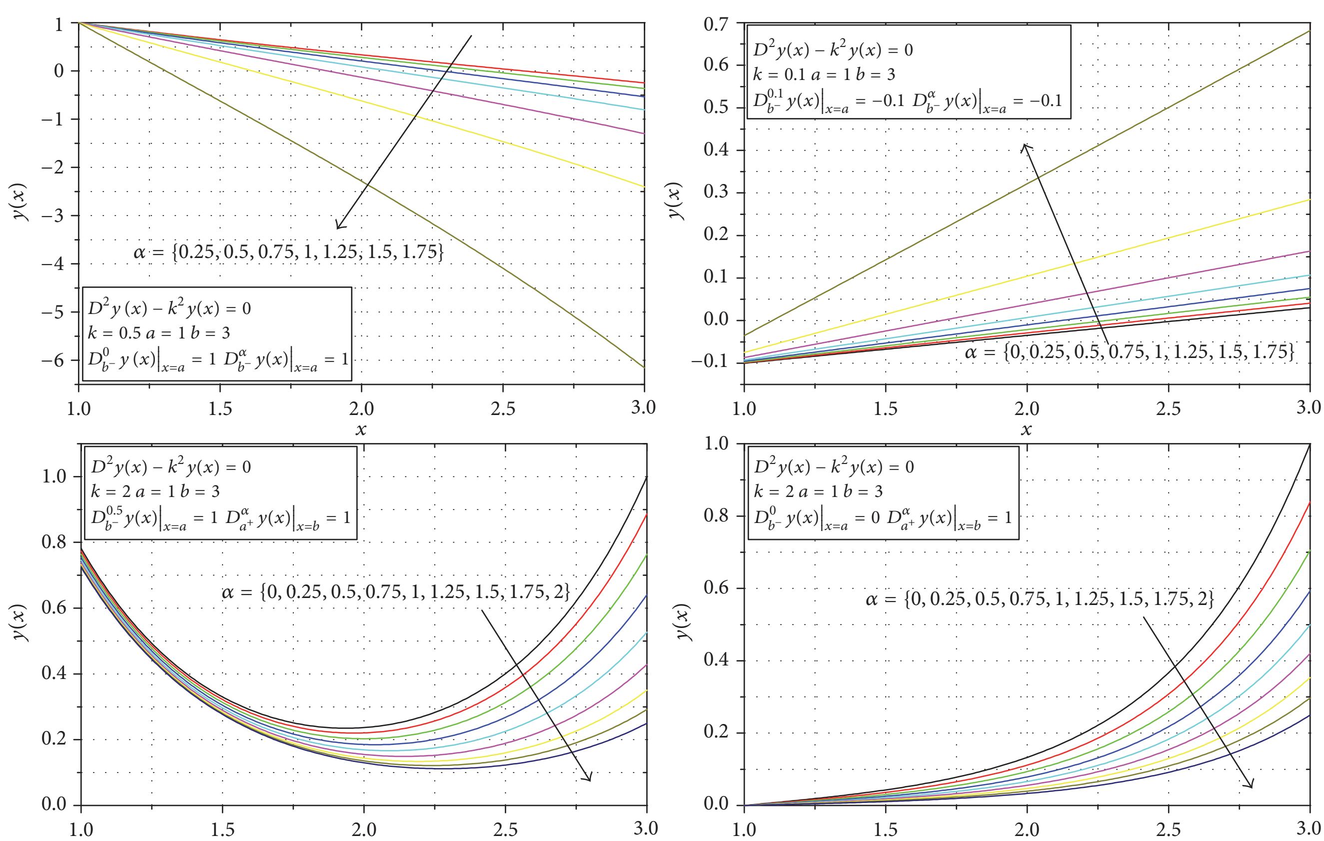

FIGURE 2: The particular solutions of (23) for selected boundary conditions (see details in the legends).

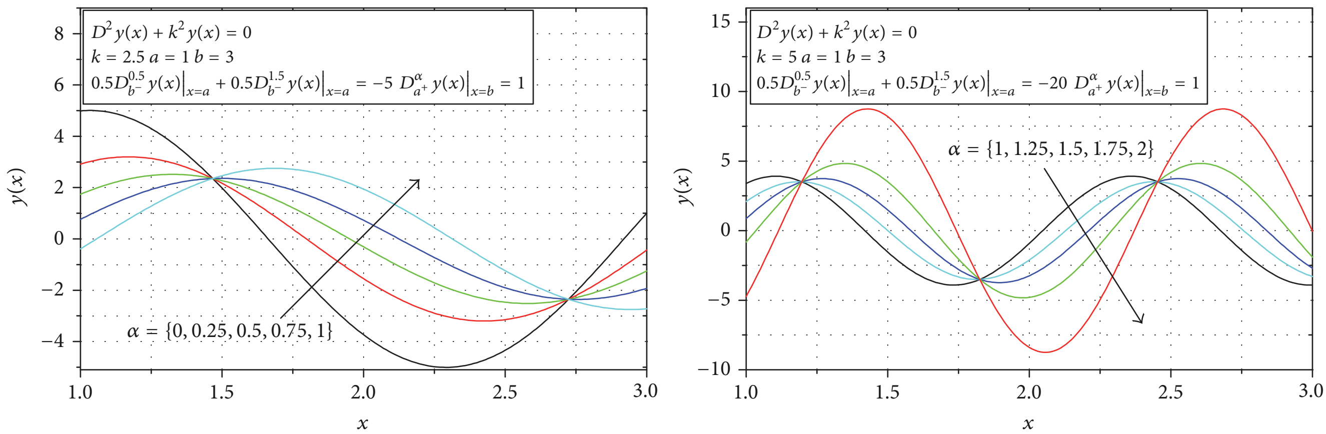

FIGURE 3: The particular solutions of (4) for selected boundary conditions (see details in the legends).

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