An Exact Solution to the Relativistic Equation of Motion of a Charged Particle Driven by a Linearly Polarized Electromagnetic Wave

JOHN V. SHEBALIN, MEMBER, IEEE

Abstract—**An exact analytic solution is found for a basic electromagnetic wave–charged particle interaction by solving the nonlinear equations of motion. The particle position, velocity, and corresponding time are found to be explicit functions of the total phase of the wave. Particle position and velocity are thus implicit functions of time. Applications include describing the motion of a free electron driven by an intense laser beam.**

I. Introduction

THE PROBLEM of determining the motion of a charged particle driven by an externally produced electromagnetic wave is an old one, and has been considered in a variety of contexts. In the absence of other external fields, a low-amplitude wave causes the charged particle to move, to a good approximation, with a velocity whose direction is parallel to the wave's electric field vector and whose phase lags that of the electric field by $\pi/2$. As the amplitude of the electromagnetic wave is increased, however, the wave's magnetic field begins to noticeably affect the motion, adding to the velocity a component which lies in the direction of propagation of the wave. In general, the motion is now relativistic, and the equations describing it are nonlinear. Roberts and Buchsbaum [1] considered the motion of a charged particle in the presence of a constant magnetic field and a circularly polarized electromagnetic wave which propagates in the direction of the field. They were able to transform the equations of motion into a single ordinary differential equation for the energy. They were then able to find an exact solution for the particle energy as a function of time.

Jory and Trivelpiece [2] numerically analyzed the effect of large-amplitude electromagnetic fields on charged-particle motion (neglecting radiation reaction) for a variety of interesting cases. The first and perhaps most basic case they considered was that of a charged particle, initially at rest, in the presence of a homogeneous linearly polarized plane wave. This basic case is again treated here, resulting in an exact analytic solution of the relativistic equations of motion. The purpose of this exercise is both to show that an analytic solution to this basic nonlinear problem exists and to present a general procedure which may be useful in determining the solution to similar problems.

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The author is with the National Aeronautics and Space Administration, Langley Research Center, Hampton, VA 23665.
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II. Solution of the Equations of Motion

The equations of motion for a particle of charge q, rest mass m_0, position $\boldsymbol{r} = (x, y, z)$, and velocity $\boldsymbol{v} = (v_x, v_y, v_z)$ in the presence of a linearly polarized plane wave with electric field $\boldsymbol{E}$, magnetic field $\boldsymbol{B}$, and wave vector $\boldsymbol{k}$, are

$$d[m_0 \boldsymbol{v}(1 - v^2/c^2)^{-1/2}]/dt = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \quad (1a)$$

and

$$d\boldsymbol{r}/dt = \boldsymbol{v}. \quad (1b)$$

A frame of reference can always be chosen such that the plane-wave solution to Maxwell's equations is $\boldsymbol{E} = E \cos(\Omega t - kz)\boldsymbol{e}_x$, $\boldsymbol{B} = (E/c)\cos(\Omega t - kz)\boldsymbol{e}_y$ and $\boldsymbol{k} = k\boldsymbol{e}_z$, and such that $\boldsymbol{r} = \boldsymbol{v} = 0$ at $t = 0$, and $v_y = 0$ for all t. When the time derivative of the composite function on the left side of (1a) is taken, each vector component contains time derivatives of both $U_x = v_x/c$ and $U_z = v_z/c$. These time derivatives are then separated by matrix inversion and, following the notation of Krall and Trivelpiece [3], the equations of motion (1) become

$$dU_x/d\Theta = F(1 - U_z - U_x^2)(1 - U_x^2 - U_z^2)^{1/2} \quad (2a)$$

$$dU_z/d\Theta = FU_x(1 - U_z)(1 - U_x^2 - U_z^2)^{1/2} \quad (2b)$$

$$dX/d\Theta = U_x \quad (2c)$$

$$dZ/d\Theta = U_z \quad (2d)$$

where

$$F = A\cos(\Theta - Z)$$
$$A = qE/(m_0 c^2)$$
$$\Theta = \Omega t$$
$$\Omega = kc$$
$$X = \Omega x/c$$
$$Z = \Omega z/c$$
$$U_x = v_x/c$$
$$U_z = v_z/c.$$

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It will now be shown that U_x, U_z, X, and Z are explicit functions of the phase $\varphi = \Theta - Z$. (The equation defining $Z(\Theta)$ is thus a transcendental one.)

First, a simple differential relation between U_x and U_z can be found by dividing (2a) by (2b) and rearranging the result to give

$$U_x(1 - U_z)\,dU_x = (1 - U_z - U_x^2)\,dU_z. \quad (3)$$

Upon making the substitutions $V = U_x^2$ and $W = 1 - U_z$, a simple equation arises:

$$dV/dW = 2(V/W - 1). \quad (4)$$

Assuming the solution to (4) is a power series leads to $V = 2W + \alpha W^2$; the constant α is found from the initial conditions ($U_x = U_z = 0$, at $t = 0$) to be $\alpha = -2$. Thus, the relation between U_x and U_z is

$$U_x^2 = 2U_z(1 - U_z) \quad (5)$$

which immediately reveals that the maximum absolute value U_x can attain is $2^{-1/2}$, which occurs whenever $U_z = \frac{1}{2}$.

Using (5), (2a) and (2b) become

$$dU_x/d\Theta = F(1 - 2U_z)(1 - U_z)^2 \quad (6a)$$

and

$$dU_z/d\Theta = FU_x(1 - U_z)^2. \quad (6b)$$

In terms of $\varphi = \Theta - Z$, $d/d\Theta = (1 - U_z)\,d/d\varphi$ and (2a)–(2d) can now be written as

$$dU_x/d\varphi = A\cos\varphi(1 - U_z)(1 - 2U_z) \quad (7a)$$

$$dU_z/d\varphi = A\cos\varphi(1 - U_z)U_x \quad (7b)$$

$$dX/d\varphi = U_x/(1 - U_z) \quad (7c)$$

$$dZ/d\varphi = U_z/(1 - U_z). \quad (7d)$$

In looking at (7a)–(7d), it is clear that the dynamic variables U_x and U_z are symmetric about $\varphi = n\pi/2$ ($n = 1, 3, 5, \cdots$) and U_z and X are symmetric about $\varphi = m\pi$, while U_x is antisymmetric ($m = 1, 2, 3, \cdots$). Therefore, in order to understand the behavior of U_x and U_z (and X and Z) for all φ (i.e., for all time), it suffices to understand the behavior of U_x and U_z for $0 \le \varphi \le \pi/2$.

Now it is useful to define another variable G: $G \equiv \sin\varphi$ with $0 \le G \le 1$. Since $A\,dG/d\varphi = F$, (7b) can be written

$$dU_z/dG = AU_x(1 - U_z). \quad (8)$$

For this range of G, U_x will have the same sign as A, which we will assume to be positive, for the sake of definiteness; U_x can thus be determined as the positive square root of the right side of (5). Then, (8) takes the form

$$dU_z/dG = A(2U_z)^{1/2}(1 - U_z)^{3/2}. \quad (9)$$

This equation is easily integrable; the solution which satisfies the initial conditions is $U_z = A^2G^2/(2 + A^2G^2)$. Using this expression for U_z, along with (5), (7c), and (7d), yields the following set:

$$U_x = 2AG/(2 + A^2G^2) \quad (10a)$$

$$dX/d\varphi = AG \quad (10b)$$

$$U_z = A^2G^2/(2 + A^2G^2) \quad (10c)$$

$$dZ/d\varphi = A^2G^2/2. \quad (10d)$$

The integrations required to determine X and Z in the set (10) are easily performed; upon replacing G by $\sin\varphi$, the solution to the stated problem is achieved:

$$U_x = 2A\sin\varphi/(2 + A^2\sin^2\varphi) \quad (11a)$$

$$U_z = A^2\sin^2\varphi/(2 + A^2\sin^2\varphi) \quad (11b)$$

$$X = A(1 - \cos\varphi) \quad (11c)$$

$$Z = A^2(2\varphi - \sin 2\varphi)/8. \quad (11d)$$

Since $\varphi = \Theta - Z$, (11d) is a transcendental equation which implicitly defines $Z(\Theta)$. Although the numerical solution of (11d) for $Z(\Theta)$ is straightforward, an alternative (and perhaps more natural) approach is to completely parameterize the solution in terms of the phase φ (and amplitude A) by determining $\Theta(\varphi)$:

$$\Theta = (1 + A^2/4)\varphi - (A^2/8)\sin 2\varphi. \quad (12)$$

III. Discussion

Now that the solution to the equations of motion in terms of the parameters φ and A exists, some general observations can be made. A complete cycle in the motion occurs when U_x, U_z, and X return to their initial values; this happens when φ changes by 2π. The dimensionless period T can then be determined from (12): $T = 2\pi(1 + A^2/4)$. During a complete period, Z (and, of course, Θ) increase monotonically with φ, and the charged particle moves a net distance L_z in the z direction of $L_z = \pi A^2/2$.

For φ between 0 and 2π, U_x has differing numbers of maxima and minima, depending on the value of A, as can be seen in Fig. 1. If $0 < A \le 2^{1/2}$, U_x has a maximum at $\varphi = \pi/2$ and a minimum at $\varphi = 3\pi/2$; the value of U_x at these minima and maxima can be found from (11a). However, if $A > 2^{1/2}$, then the maxima at $\varphi = \pi/2$ becomes a local minimum and the minimum at $\varphi = 3\pi/2$ becomes a local maximum; in addition, absolute maxima occur in the range $0 < \varphi < \pi$, and absolute minima in the range $\pi < \varphi < 2\pi$ at those values of φ which satisfy $\sin\varphi = 2^{1/2}/A$. (The absolute value of U_x at these absolute extrema is $2^{-1/2}$, which is the largest absolute value U_x can have.) For φ between 0 and 2π, U_z has maxima at $\varphi = \pi/2$ and $\varphi = 3\pi/2$, and minima at $\varphi = 0, \pi, 2\pi$, for any $A > 0$; the absolute value of U_z at its extrema tends to unity as A increases without bound, as can be seen in Fig. 2. (Graphs of U_x and U_z for other values of A are shown elsewhere [2], [3].)

Using (5) and (11b), it follows by substitution that the total energy $E_{\text{tot}} = m_0c^2(1 - v^2/c^2)^{-1/2}$ of the charged

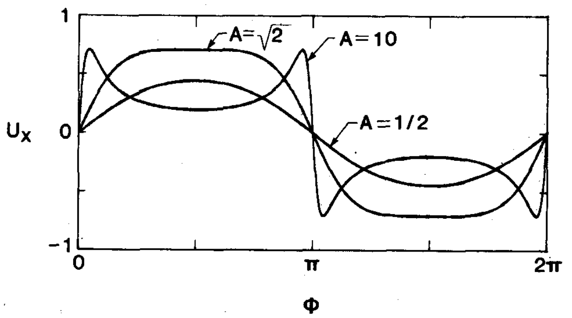

Fig. 1. Transverse velocity U_x as a function of phase φ and amplitude A.

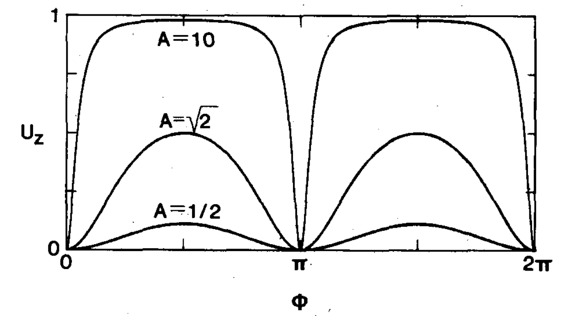

Fig. 2. Longitudinal velocity U_z as a function of phase φ and amplitude A.

particle is

$$E_{\text{tot}} = m_0 c^2/(1 - U_z) = m_0 c^2(1 + \tfrac{1}{2}A^2 \sin^2 \varphi). \tag{13}$$

The kinetic energy E_{kin} is

$$E_{\text{kin}} \equiv E_{\text{tot}} - m_0 c^2 = \tfrac{1}{2} m_0 c^2 A^2 \sin^2 \varphi. \tag{14}$$

IV. Conclusion

In this paper, the equations of motion describing perhaps the most basic electromagnetic wave–charged particle interaction have been shown to have an exact solution. The procedure used here may perhaps be useful in determining exact solutions in more complicated interactions, e.g., a single charged particle in the presence of two or more waves, which include the cases of circular and elliptic polarization. That such solutions exist for the case of circular polarization has been indicated by Roberts and Buchsbaum [1], who found an analytic expression for the energy of the charged particle.

The results presented here are perhaps most applicable to the case in which a free electron finds itself in the beam of a powerful laser, in that region where the amplitude of the associated plane wave is essentially constant transverse to the propagation direction. The results may also be applied to electron motion in a coaxial or parallel line waveguide operating in a TEM mode; since the amplitude of the associated plane wave has transverse variation, the solutions given here can possibly serve as a basis for a perturbative analysis.

Finally, the author wishes to thank the referees and to mention some further references that they suggested. First, Boyd and Sanderson [4] present an approximate treatment of this problem and give references to earlier work. Second, Davis [5] presents general methods for solving first-order nonlinear differential equations such as (4), as well as other types of nonlinear equations.

References

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