AN EXAMPLE INVOLVING A NON-REGULAR $\mathscr{D}$-CLASS IN A SEMIGROUP

T. E. HALL

(Received 27 October 1969)

Communicated by G. B. Preston

1. Introduction and summary

We answer in the negative (by considering an example) the problem posed in exercise 6 of § 2.3, page 62 [1], namely: If a $\mathscr{D}$-class D of a semigroup S is a subsemigroup of S, then is D necessarily bisimple? For any semigroup T, we let $\mathscr{L}_T$, $\mathscr{R}_T$, $\mathscr{H}_T$, $\mathscr{D}_T$ and $\mathscr{J}_T$ denote Green's relations on T.

2. The example

Denote the set of real numbers by R and put $R^+ = \{x \in R\colon x > 0\}$. Consider the following sets of 2×2 matrices over R:

$$K = \left\{\begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix} : a, b \in R^+\right\};$$

$$G = \left\{\begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix} : b \in R^+\right\}; \text{ and } S = K \cup G.$$

Under matrix multiplication, S is a semigroup, K is an ideal of S, and G is a subgroup of S; we note that K is included among the examples in exercises 8, 9, 10 of § 2.1 and exercise 7 of § 5.4 [1].

Take now any elements a, b, c, d in R^+. Using easy calculations, we may show that

$$\begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix} K = \left\{\begin{pmatrix} 1 & 0 \\ a+x & y \end{pmatrix} : x, y \in R^+\right\},$$

$$K \begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix} = \left\{\begin{pmatrix} 1 & 0 \\ z & w \end{pmatrix} \in K : z/w > a/b\right\},$$

$$\begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix} S = \left\{\begin{pmatrix} 1 & 0 \\ a+x & y \end{pmatrix} : y \in R^+, x \in R \text{ and } x \geqq 0\right\},$$

and

$$S \begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix} = \left\{\begin{pmatrix} 1 & 0 \\ z & w \end{pmatrix} \in K : z/w \geqq a/b\right\}.$$

It follows that $\mathscr{L}_K = \mathscr{R}_K = \mathscr{D}_K = \iota_K$, the identity relation on K, while

$$\begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix} \mathscr{R}_S \begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix} \text{ if and only if } a = c, \text{ and}$$

$$\begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix} \mathscr{L}_S \begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix} \text{ if and only if } a/b = c/d.$$

It follows that

$$\begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix} \mathscr{R}_S \begin{pmatrix} 1 & 0 \\ a & ad/c \end{pmatrix} \mathscr{L}_S \begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix}.$$

Clearly now $\mathscr{D}_S \supseteq (K \times K) \cup (G \times G)$, and $\mathscr{D}_S \neq S \times S$ since K is an ideal of S. It follows that K is a $\mathscr{D}_S$-class, and is a subsemigroup of S, but is 'far from' being bisimple.

Remark 1. For any $\mathscr{L}_S$-class, L say, and $\mathscr{R}_S$-class, R say, both contained in the $\mathscr{D}_S$-class K, we have $LR = K$, c.f. exercise 2 of § 2.3, page 61 [1].

Remark 2. Since

$$\mathscr{D}_S \subseteq \mathscr{J}_S \subseteq (K \times K) \cup (G \times G) = \mathscr{D}_S,$$

we see that K is the kernel of S. No principal left ideal of K is also a left ideal of S, c.f. the example given by Clark [2].

References

[1] A. H. Clifford and G. B. Preston, *The Algebraic Theory of Semigroups* (Math. Surveys, number 7, Amer. Math. Soc., Vol. I 1961).

[2] W. E. Clark, 'Remarks on the kernel of a matrix semigroup', *Czechoslovak Math. J.* 15 (1965) 305—310.

Monash University
Clayton, Victoria

https://doi.org/10.1017/S1446788700013653 Published online by Cambridge University Press