Алгебра и анализ
Том. 16 (2004), вып. 2

St. Petersburg Math. J.
Vol. 16 (2005), No. 2, Pages 417–422
S 1061-0022(05)00858-7
Article electronically published on March 9, 2005

AN EXAMPLE OF A PERIODIC MAGNETIC SCHRÖDINGER OPERATOR WITH DEGENERATE LOWER EDGE OF THE SPECTRUM

R. G. SHTERENBERG

Dedicated to my dear teacher M. Sh. Birman

Abstract. The structure of the lower edge of the spectrum of a periodic magnetic Schrödinger operator is investigated. It is known that in the nonmagnetic case the energy depends quadratically on the quasimomentum in a neighborhood of the lower edge of the spectrum of the operator. An example of a magnetic Schrödinger operator is constructed for which energy is partially degenerate with respect to one component of the quasimomentum.

§0. Introduction

0.1. Let Γ be a lattice in $\mathbb{R}^d$, $d \geq 1$, and let Ω be an elementary cell of the lattice Γ. Γ-periodic differential operators (DO's) can be partially diagonalized with the help of the Gelfand transformation (see, e.g., [Sk]). Then the initial DO is represented as a direct integral of a family of DO's that act on the torus associated with Ω. The operators in this family depend on a parameter $\mathbf{k} \in \mathbb{R}^d$ (called the *quasimomentum*).

We consider *lower semibounded selfadjoint* DO's. For most of DO's of mathematical physics, the spectrum of the corresponding operators acting in $L_2(\Omega)$ is discrete. The eigenvalues $E_j(\mathbf{k})$, $j = 1, 2, \dots$, arranged in nondecreasing order, depend on $\mathbf{k}$ continuously. Then the spectrum of the initial DO has a *band structure*, with bands corresponding to the *band functions* $E_j(\cdot)$. For convenience, we assume that the lower edge of the spectrum is $\lambda = 0$. For some problems, it turns out that their solution requires only the knowledge of the approximate behavior of the first band function $E_1(\mathbf{k})$ near its minimum point. In such cases we talk about the *threshold effects* at the point $\lambda = 0$. It is a nontrivial task to judge whether or not a given effect is a threshold one. As an example, we point out the problem of the discrete spectrum that appears to the left of the point $\lambda = 0$ when a periodic DO is perturbed by a decaying negative potential. If this potential decays not too fast, then the crucial contribution is of threshold nature. Another important case of a threshold effect at the point $\lambda = 0$ is the behavior of a periodic DO in the small period limit.

0.2. The investigation of threshold effects becomes much easier if the periodic DO in question admits an appropriate ("regular") factorization. In [BSu], a wide class of elliptic periodic second order DO's was distinguished, and a fairly deep analysis of the lower edge

2000 *Mathematics Subject Classification.* Primary 35J10, 35P15.
Key words and phrases. Periodic operator, magnetic Schrödinger operator, lower edge of the spectrum, threshold effects, degeneration.
Supported by RFBR (grant no. 02-01-00798).

of the spectrum was developed for that class. In the scalar case, this class of DO's is described by the following condition: *the operator M admits the representation*

$$M = \overline{f(\mathbf{x})}\mathbf{b}(\mathbf{D})^* G(\mathbf{x})\mathbf{b}(\mathbf{D})f(\mathbf{x}), \quad \mathbf{b}(\mathbf{D}) := \sum_{j=1}^{d} D_j \mathbf{b}_j, \tag{0.1}$$

where $\mathbf{D} = \operatorname{col}\{D_1, \dots, D_d\} = -i\nabla = -i\operatorname{col}\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_d}\}$, $f(\mathbf{x})$ *is a* Γ*-periodic function,* $G(\mathbf{x})$ *is a* Γ*-periodic* $(m \times m)$*-matrix-valued function* $(m \geq d)$, *and the* $\mathbf{b}_j$, $j = 1, 2, \dots, d$, *form a set of linearly independent vectors in* $\mathbb{C}^m$. *Moreover,*

$$f, f^{-1} \in L_\infty(\mathbb{R}^d); \quad c_0\mathbf{1} \leq G(\mathbf{x}) \leq c_1\mathbf{1}, \ \ \mathbf{x} \in \mathbb{R}^d, \ \ 0 < c_0 \leq c_1 < \infty. \tag{0.2}$$

Under conditions (0.2), the factorization (0.1) will be called a *proper* factorization. The following statements are direct consequences of the existence of a proper factorization.

1) The first band function $E_1(\mathbf{k})$ has a nondegenerate minimum at the point $\mathbf{k} = 0$ (and then $E_1(0) = 0$). The corresponding quadratic form can be estimated explicitly; the constants in estimates are well controlled.

2) We have $E_1(\mathbf{k}) > 0$ for $k \neq 0 \pmod{\Gamma}$.

3) $\min_{\mathbf{k}} E_2(\mathbf{k}) > 0$.

Sometimes, the DO in question is given not in the form (0.1), but can be rewritten properly. In [BSu] it was noted that, in particular, the periodic Schrödinger operator $-\operatorname{div} g(\mathbf{x})\nabla + V(\mathbf{x})$ with a positive matrix-valued function (metric) $g(\mathbf{x})$ admits a proper factorization. The same is true for the two-dimensional periodic Pauli operator $(\mathbf{D}-\mathbf{A})^2 \pm (\partial_1 A_2 - \partial_2 A_1)$, which is a particular case of the magnetic Schrödinger operator

$$(\mathbf{D} - \mathbf{A})^* g(\mathbf{x})(\mathbf{D} - \mathbf{A}) + V(\mathbf{x}). \tag{0.3}$$

Here $\mathbf{A}(\mathbf{x})$ is a Γ-periodic $\mathbb{R}^2$-valued function.

0.3. We consider the "pure magnetic" Schrödinger operator

$$H(t) = (\mathbf{D} - t\mathbf{A})^2, \quad t \in \mathbb{R}. \tag{0.4}$$

Here, for convenience, we have introduced the parameter t. The operator (0.4) is already factorized, but this factorization is not of the form (0.1). The lower edge of the spectrum of the operator $H(t)$ is positive for $t \neq 0$ unless the vector-valued function $\mathbf{A}$ is potential. For an appropriate $\theta(t) > 0$, the lower edge of the spectrum of the operator $H(t) - \theta(t)I$ is equal to $\lambda = 0$, while the initial factorization disappears.

For sufficiently small t, the operator $H(t) - \theta(t)I$ can be shown to admit a proper factorization under conditions on the coefficients that are not very restrictive. However, in general, if t is not assumed to be small, a proper factorization fails.

Observe that the situation may be much different if the electric potential in (0.3) is coordinated properly with the magnetic potential. For example, the lower edge of the spectrum of the two-dimensional Pauli operator is always equal to $\lambda = 0$, and a proper factorization exists.

In the present paper we construct an example of a pure magnetic Schrödinger operator such that not only does it fail to admit a proper factorization, but also the lower edge of its spectrum is partially degenerate. Namely, the first band function $E_1(\mathbf{k})$ of this operator near its minimum point is of fourth (but not second) order in one of the components of the quasimomentum. Such degeneration of the lower edge of the spectrum leads to some particular features of the threshold effects that could not be observed in the nonmagnetic situation. A more detailed discussion of the effects arising in this case will be presented in another paper, where we shall also examine the existence of a proper factorization for the operator (0.3) with sufficiently small $\mathbf{A}$.

0.4. Acknowledgements. In essence, my interest in the topics treated here was initiated due to papers by M. Sh. Birman and T. A. Suslina. The author is sincerely grateful to both of them for indicating the problem and for fruitful discussions during the preparation of this text.

§1. The main result

1.1. Notation. We use the notation $\mathbf{x} = \{x_1, x_2\} \in \mathbb{R}^2$. Let $\Omega := [-1/2, 1/2)^2$, and let $\widetilde{\Omega} := [-\pi, \pi)^2$ be the cell dual to Ω. The Sobolev space of functions that are square integrable with all derivatives of the first and second order is denoted by $H^2(\mathbb{R}^2)$. By $\widetilde{H}^2(\Omega)$ we denote the class of functions whose $\mathbb{Z}^2$-periodic extensions belong to $H^2_{\mathrm{loc}}(\mathbb{R}^2)$. The symbol $\mathbf{1}$ stands for the unit (2×2)-matrix.

1.2. Construction of the operator. Preliminary remarks. In $L_2(\mathbb{R}^2)$, we consider the operator

$$M_t := (\mathbf{D} - t\mathbf{A}(\mathbf{x}))^2 - t^2, \quad t \in \mathbb{R}, \ \mathbf{x} \in \mathbb{R}^2.$$

Here the magnetic potential $\mathbf{A}(\mathbf{x})$ is given by $\mathbf{A}(\mathbf{x}) = \mathrm{col}\{0, A_2(x_1)\}$, where

$$\begin{aligned} &A_2(x_1) := \operatorname{sgn} x_1, \quad x_1 \in [-1/2, 1/2), \\ &A_2(x_1 + 1) = A_2(x_1), \quad x_1 \in \mathbb{R}. \end{aligned} \tag{1.1}$$

(As we shall see, the term $-t^2$ ensures that the lower edge of the spectrum of M_t is equal to zero for sufficiently small t.) Obviously, on the domain $H^2(\mathbb{R}^2)$ the operator M_t, $t \in \mathbb{R}$, is selfadjoint and lower semibounded. Using (1.1), we can rewrite the operator M_t in the form

$$M_t = D_1^2 + D_2^2 - 2tA_2(x_1)D_2. \tag{1.2}$$

We put

$$\begin{aligned} &\varphi(x_1, x_2) := 2|x_1|, \quad x_1 \in [-1/2, 1/2), \ x_2 \in \mathbb{R}, \\ &\varphi(x_1 + 1, x_2) = \varphi(x_1, x_2), \quad x_1 \in \mathbb{R}, \ x_2 \in \mathbb{R}. \end{aligned} \tag{1.3}$$

Also, we introduce the following family of Hermitian matrices:

$$G_\gamma(\mathbf{x}; t) := \begin{pmatrix} 1 & -i(t\varphi(\mathbf{x}) + \gamma) \\ i(t\varphi(\mathbf{x}) + \gamma) & 1 \end{pmatrix}, \quad \gamma \in \mathbb{R}. \tag{1.4}$$

The identity

$$M_t = \mathbf{D}^* G_\gamma(\mathbf{x}; t)\mathbf{D}, \quad t \in \mathbb{R}, \ \gamma \in \mathbb{R}, \tag{1.5}$$

can be verified directly. Observe that the matrix (1.4) generates the representation (1.5) for any $\gamma \in \mathbb{R}$, though the operator M_t does not depend on γ.

We say that the matrix $G_\gamma(\mathbf{x}; t)$ is *uniformly positive* (for fixed t and γ) if for some constant $c > 0$ the following estimate is fulfilled:

$$G_\gamma(\mathbf{x}; t) \geq c\mathbf{1}, \quad \mathbf{x} \in \mathbb{R}^2, \ c = c(t, \gamma) > 0. \tag{1.6}$$

Lemma 1.1. *The matrix $G_\gamma(\mathbf{x}; t)$ is uniformly positive if and only if*

$$|t| < 2 \quad \text{and} \quad \begin{cases} \gamma \in (-1, 1 - t) & \text{for } \ 0 \leq t < 2, \\ \gamma \in (|t| - 1, 1) & \text{for } -2 < t < 0. \end{cases}$$

Proof. Clearly, estimate (1.6) is equivalent to the condition

$$\det G_\gamma(\mathbf{x}; t) > 0, \quad \mathbf{x} \in \mathbb{R}^2. \tag{1.7}$$

By (1.3) and (1.4), inequality (1.7) implies that

$$1-(2t|x_1|+\gamma)^2>0, \quad x_1\in[-1/2,1/2).$$

Now, the lemma easily follows. □

Remark 1.2. Lemma 1.1 and identity (1.5) imply that for $|t|<2$ the operator M_t admits a proper factorization of the form (0.1). (However, we cannot take one and the same γ ensuring a proper factorization for all $|t|<2$ (cf. Lemma 1.1).) As a consequence, the point $\lambda=0$ is the lower edge of the spectrum of the operator M_t. In what follows we shall see that the lower edge of the spectrum still coincides with $\lambda=0$ up to $|t|\le 2\sqrt{3}$.

1.3. Investigation of the spectrum of the operator M_t. In $L_2(\Omega)$, we consider the family of operators depending on the parameter $\mathbf{k}\in\mathbb{R}^2$ ($\mathbf{k}$ is the quasimomentum):

$$M_t(\mathbf{k}):=(D_1+k_1)^2+(D_2+k_2)^2-2t\operatorname{sgn}x_1(D_2+k_2), \quad \mathbf{k}=\{k_1,k_2\}\in\mathbb{R}^2. \tag{1.8}$$

Defined on the domain $\widetilde{H}^2(\Omega)$, the operators $M_t(\mathbf{k})$ are selfadjoint and lower semibounded. Each of them has discrete spectrum. Let $\{E_j(\mathbf{k};t)\}$ denote the eigenvalues of the operator $M_t(\mathbf{k})$; we enumerate them in nondecreasing order:

$$E_1(\mathbf{k};t)\le E_2(\mathbf{k};t)\le\cdots\le E_n(\mathbf{k};t)\le\cdots.$$

Let $\{\psi_j(\mathbf{x},\mathbf{k};t)\}$ be the corresponding (periodic) orthonormal eigenfunctions. The functions $E_j(\mathbf{k};t)$ are continuous in $\mathbf{k}$ and t and $(2\pi\mathbb{Z})^2$-periodic in $\mathbf{k}$. In what follows, we consider only the quasimomenta $\mathbf{k}$ belonging to the dual cell $\widetilde{\Omega}$.

The general Floquet–Bloch theory implies that the spectrum of M_t has a band structure, and

$$\sigma(M_t)=\bigcup_{j=1}^{\infty}\mathcal{R}(E_j(\cdot;t)). \tag{1.9}$$

Here the symbol $\mathcal{R}(\cdot)$ stands for the range of a function.

In the present subsection we discuss the structure of the lower edge of the spectrum of the operator M_t in detail. Combined with not too difficult variational arguments, the existence of a proper factorization (1.5) with uniformly positive matrix G_γ leads to some relations for the eigenvalues $E_j(\mathbf{k};t)$ of the operator $M_t(\mathbf{k})$ (cf. [BSu]). In particular, for $|t|<2$ we have

$$\begin{gathered}E_1(\mathbf{k};t)\ge 0,\ \ E_2(\mathbf{k};t)>0, \quad \mathbf{k}\in\widetilde{\Omega};\\ E_1(\mathbf{k};t)=0 \iff \mathbf{k}=0; \quad \psi_1(\mathbf{x},0;t)=1.\end{gathered} \tag{1.10}$$

Furthermore, the minimum of the function $E_1(\cdot,t)$ at the point $\mathbf{k}=0$ is nondegenerate.

We are interested in the values of t for which relations (1.10) are true. Using the discrete Fourier transformation in x_2, we see that the operator (1.8) can be decomposed into the orthogonal sum of operators

$$(D_1+k_1)^2+(2\pi n_2+k_2)^2-2t(2\pi n_2+k_2)\operatorname{sgn}x_1, \quad n_2\in\mathbb{Z},$$

with periodic boundary conditions, acting on $L_2([-1/2,1/2))$. Thus, the problem concerning the spectrum of the operator $M_t(\mathbf{k})$ reduces to investigation of the spectrum of the operator family

$$\widetilde{M}_t(k_1,\xi):=(D_1+k_1)^2+\xi^2-2t\xi\operatorname{sgn}x_1, \quad k_1\in[-\pi,\pi),\ \ \xi\in\mathbb{R},$$

on the interval $x_1\in[-1/2,1/2)$, with periodic boundary conditions. Let $\widetilde{E}_1(k_1,\xi;t)$, $k_1\in[-\pi,\pi)$, be the smallest eigenvalue of the operator $\widetilde{M}_t(k_1,\xi)$. The operator $\widetilde{M}_t(k_1,\xi)$ has the form $(D_1+k_1)^2+V_\xi(x_1)$. Then, as is well known (see, e.g., [RSi]),

$$\widetilde{E}_1(k_1,\xi;t)>\widetilde{E}_1(0,\xi;t), \quad k_1\in[-\pi,\pi)\backslash\{0\},\ \ \xi\in\mathbb{R}. \tag{1.11}$$

Now we proceed to the study of the spectrum of the operator

$$\widetilde{M}_t(0,\xi) = -\frac{d^2}{dx_1^2} + \xi^2 - 2t\xi \operatorname{sgn} x_1, \quad \xi \in \mathbb{R},$$

on the interval $x_1 \in [-1/2, 1/2)$, with periodic boundary conditions. We distinguish the case where $\xi = 0$. Obviously, this case corresponds to the relations $E_1(0;t) = 0$, $\psi_1(\mathbf{x}, 0; t) = 1$. We consider the problem

$$\widetilde{M}_t(0,\xi)u(x_1) = 0, \quad u(-1/2) = u(1/2), \;\; u'(-1/2) = u'(1/2), \;\; \xi \in \mathbb{R}\backslash\{0\}. \tag{1.12}$$

Obviously, the following condition is necessary for the existence of a nontrivial solution of problem (1.12):

$$\xi^2 < 2|t\xi|, \quad \xi \in \mathbb{R}\backslash\{0\}. \tag{1.13}$$

We put

$$\alpha_\pm := \sqrt{2|t\xi| \pm \xi^2}, \quad \xi^2 < 2|t\xi|, \;\; \xi \in \mathbb{R}\backslash\{0\}.$$

Simple but cumbersome calculations show that problem (1.12) has a nontrivial solution if and only if relation (1.13) and one of the relations

$$\alpha_+ \tanh^{-1}\frac{\alpha_+}{4} = -\alpha_- \tan^{-1}\frac{\alpha_-}{4}, \quad \alpha_+ \tanh\frac{\alpha_+}{4} = \alpha_- \tan\frac{\alpha_-}{4} \tag{1.14}$$

are fulfilled. From the further analysis it is clear that for $|t| \le 2\sqrt{3}$ relations (1.14) fail. Combined with (1.9) and (1.11), this implies that for $|t| \le 2\sqrt{3}$ the point $\lambda = 0$ is the lower edge of the spectrum of the operator M_t and that relations (1.10) are fulfilled.

To complete the study of the lower edge of the spectrum of M_t, we consider again the problem for the first eigenvalue of the operator $M_t(\mathbf{k})$:

$$M_t(\mathbf{k})\psi_1(\mathbf{x}, \mathbf{k}; t) = E_1(\mathbf{k}; t)\psi_1(\mathbf{x}, \mathbf{k}; t).$$

From the above considerations it follows that $E_1(0;t) = 0$ and $\psi_1(\mathbf{x}, 0; t) = 1$ for $|t| \le 2\sqrt{3}$. Moreover, the point $\lambda = 0$ is a simple eigenvalue of $M_t(0)$. It is easy to write down several initial terms of the analytic expansion of the function $E_1(\cdot; t)$ near the point $\mathbf{k} = 0$:

$$E_1(\mathbf{k}; t) = k_1^2 + (1 - \frac{t^2}{12})k_2^2 + \frac{1}{42}\cdot\frac{t^4}{144}k_2^4 - \frac{1}{10}\cdot\frac{t^2}{12}k_1^2k_2^2 + O(|\mathbf{k}|^6).$$

In particular, for $|t| = 2\sqrt{3}$ we have

$$E_1(\mathbf{k}; \pm 2\sqrt{3}) = k_1^2 + \frac{1}{42}k_2^4 - \frac{1}{10}k_1^2k_2^2 + O(|\mathbf{k}|^6).$$

We summarize the results obtained.

Theorem 1.3. 1) *For $|t| < 2\sqrt{3}$ the point $\lambda = 0$ is the lower edge of the spectrum of the operator M_t; relations* (1.10) *are fulfilled; and the function $E_1(\cdot; t)$ is analytic in a neighborhood of $\mathbf{k} = 0$ and has a nondegenerate minimum at the point $\mathbf{k} = 0$. Moreover, for $|t| < 2$ the operator M_t admits a proper factorization.*

2) *For $|t| = 2\sqrt{3}$ the point $\lambda = 0$ is the lower edge of the spectrum of the operator M_t; relations* (1.10) *are fulfilled; and the function $E_1(\cdot; \pm 2\sqrt{3})$ is analytic in a neighborhood of $\mathbf{k} = 0$ and has a minimum of second order in k_1 and of fourth order in k_2 at the point $\mathbf{k} = 0$.*

Remark 1.4. 1) It can be proved that, for $2 \le |t| \le 2\sqrt{3}$, there is no proper factorization for the operator M_t. 2) A more detailed analysis shows that, for $|t| > 2\sqrt{3}$, the operator M_t is not positive anymore. Herewith, at least if $|t| \in (2\sqrt{3}, 2\sqrt{3} + \delta)$, where $\delta > 0$ is sufficiently small, the (negative) minimum of the spectrum of the operator M_t is reached at two different $(\operatorname{mod}(2\pi\mathbb{Z})^2)$ points $\mathbf{k}^{(1)} = \{0, k_2\}$, $\mathbf{k}^{(2)} = \{0, -k_2\}$, $k_2 > 0$.

Remark 1.5. Of course, an example similar to that described in the present subsection can be constructed also for $d \geq 3$. We can simply consider the operator

$$M_t^{(d)} := M_t \otimes I' + I \otimes (-\Delta'), \quad \text{where } \Delta' := \left(\frac{\partial^2}{\partial x_3^2} + \cdots + \frac{\partial^2}{\partial x_d^2} \right),$$

and I, I' are the identity operators relative to x_1, x_2 and $x_3, \ldots, x_d$. Then an analog of Theorem 1.3 is valid.

We have constructed a periodic magnetic Schrödinger operator for which the lower edge of the spectrum is partially degenerate (for one of the components of the quasi-momentum). The author believes that there exist examples of complete degeneration of the lower edge of the spectrum. Such situation seems to come true for a family of two-dimensional operators $\mathcal{M}_t := (\mathbf{D} - t\mathbf{A})^2 - t^2$, where the $\mathbb{Z}^2$-periodic magnetic potential $\mathbf{A}$ is given on Ω by the formula $\mathbf{A}(\mathbf{x}) = \operatorname{col}\{\operatorname{sgn} x_2, \operatorname{sgn} x_1\}$, $\{x_1, x_2\} \in \Omega$, while the parameter $t \in \mathbb{R}$ is varying.

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St. Petersburg State University, Department of Physics, Petrodvorets, St. Petersburg 198504, Russia

E-mail address: roman@RS3759.spb.edu

Received 1/SEP/2003

Translated by THE AUTHOR