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AN EXAMPLE OF AN ODD DIMENSIONAL POSITIVELY PINCHED RIEMANNIAN MANIFOLD

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Introduction. In the theory of geodesics the following theorems are well-known.

THEOREM 1. (Klingenberg) *Let M be a complete simply connected Riemannian manifold* (dim$M \geqq 3$). *If the sectional curvature* K_σ *for any 2-plane* σ *of M satisfies*

$$\frac{1}{4}L < K_\sigma \leqq L,$$

then the distance from an arbitrary point p to the cut locus C(p) is not smaller than $\pi/\sqrt{L}$ *i.e.*

$$d(p,C(p)) \geqq \pi/\sqrt{L}.$$

THEOREM 2. (Klingenberg [2]) *Let M be a compact simply connected Riemannian manifold of even-dimension. If the sectional curvature* K_σ *for any 2-plane* σ *of M satisfies*

$$0 < K_\sigma \leqq L,$$

then the distance from an arbitrary point p to the cut locus C(p) is not smaller than $\pi/\sqrt{L}$, *i.e.*

$$d(p,C(p)) \geqq \pi/\sqrt{L}.$$

M. Berger [1] has shown that Theorem 2 dose not hold good for the case of dim $M=3$, by giving a counter example which is a compact reductive homogeneous space with the canonical connection of the first kind. Its construction leads us to give examples for any odd dimension. The purpose of this paper is to consider a Riemannian manifold of any odd dimension for which Theorem 2 dose not hold good. The author would like to express her sincere gratitude to Professor S. Tachibana and Dr. S. Tanno for their many valuable suggestions.

NOTATIONS. Let M^{2n+1} be a $(2n+1)$-dimensional Riemannian manifold with local coordinates (x^i) and metric tensor g. With respect to the natural basis, denoting by g_{ji} the components of g, Christoffel symbol $\left\{ {h \atop ji} \right\}$ and curvature tensor $R_{kji}{}^h$ are given by

$$\left\{ {h \atop ji} \right\} = \frac{1}{2} g^{ht}(\partial_j g_{ti} + \partial_i g_{tj} - \partial_t g_{ji}) \tag{0.1}$$

$$R_{kji}{}^h = \partial_k \left\{ {h \atop ji} \right\} - \partial_j \left\{ {h \atop ki} \right\} + \left\{ {h \atop kt} \right\} \left\{ {t \atop ji} \right\} - \left\{ {h \atop jt} \right\} \left\{ {t \atop ki} \right\} \tag{0.2}$$

where $(g^{kj}) = (g_{ji})^{-1}$.

1. M^{2n+1} is called an almost contact Riemannian manifold when it admits a 1-form $\eta = \eta_j\, dx^j$ and a 2-form $\varphi = 1/2\ \varphi_{ji}\, dx^j \wedge dx^i$ satisfying

$$\xi^j \eta_j = 1, \qquad \xi^k \varphi_{ik} = 0 \tag{1.1}$$

$$\varphi_j{}^k \varphi_k{}^i = -\delta_j^i + \eta_j \xi^i \tag{1.2}$$

where $\varphi_j{}^i = g^{ik}\varphi_{jk}$ and $\xi^i = g^{ik}\eta_k$.

An almost contact structure is called a Sasakian structure, if the set (φ, ξ, η, g) satisfies

$$\nabla_k \varphi_{ji} = \eta_j g_{ki} - \eta_i g_{kj}, \tag{1.3}$$

which implies that

$$\nabla_j \eta_i = \varphi_{ji} \tag{1.4}$$

and

$$R_{kji}{}^h \eta_h = \eta_j g_{ki} - \eta_k g_{ji}. \tag{1.5}$$

2. In the following we shall consider a $(2n+1)$-dimensional Sasakian space

M^{2n+1} with structure (φ, ξ, η, g). We introduce a new metric $^*g$ on M^{2n+1} defined by

$$^*g_{ji} = g_{ji} - b\eta_j\eta_i \quad (2.1)$$ [1)]

where b is a certain constant $(0 < b < 1)$.

LEMMA 1. *The curvature tensor induced from the metric $^*g$ is expressed in the term of the original manifold as*

$$\begin{aligned} {}^*R_{kjih} = {} & R_{kjih} + b(\varphi_{kh}\varphi_{ji} - \varphi_{jh}\varphi_{ki} - 2\varphi_{ih}\varphi_{kj}) \\ & + (b^2 - 2b)(\eta_i\eta_j g_{kh} - \eta_i\eta_k g_{jh} + \eta_h\eta_k g_{ji} - \eta_h\eta_j g_{ki}). \end{aligned} \quad (2.2)$$

PROOF. Putting $(^*g^{kj}) = (^*g_{ji})^{-1}$

we have

$$^*g^{kj} = g^{kj} + \frac{b}{1-b}\xi^k\xi^j . \quad (2.3)$$

Then Christoffel symbol $*\left\{ {k \atop ji} \right\}$ with respect to $^*g$ is

$$*\left\{ {k \atop ji} \right\} = \left\{ {k \atop ji} \right\} + \frac{b}{1-b}\xi^k\eta_t\left\{ {t \atop ji} \right\}$$

$$- \frac{b}{2(1-b)} \{(\partial_i\eta_j + \partial_j\eta_i)\xi^k + b(g^{kt}\varphi_{it}\eta_j + g^{kt}\varphi_{jt}\eta_i)\}$$

and therefore we get

$$*\left\{ {k \atop ji} \right\} = \left\{ {k \atop ji} \right\} + b(\varphi_i{}^k\eta_j + \varphi_j{}^k\eta_i). \quad (2.4)$$

By virture of (1. 1)~(1. 4) and (2. 4) the curvature tensor $^*R_{kji}{}^h$ is expressed as

$$\begin{aligned} {}^*R_{kji}{}^h = {} & R_{kji}{}^h - b\{\eta_i\eta_j\varphi_k{}^h - \eta_i\eta_k\varphi_j{}^h + \xi^h(\eta_k g_{ji} - \eta_j g_{ki}) \\ & + (2\varphi_i{}^h\varphi_{kj} - \varphi_k{}^h\varphi_{ji} + \varphi_j{}^h\varphi_{ki})\}. \end{aligned}$$

1) This transformation is a special one dealt in [5].

If we put

$$*R_{kjih} = *g_{th}*R_{kji}{}^{t},$$

we finally get (2. 2). (Q. E. D.)

For the 2-plane spanned by orthonormal pair X and Y at a point, its sectional curvature is given by

$$*K(X, Y) = -*g(*R(X, Y)X, Y),$$

where $*g(*R(X, Y)X, Y) = *R_{kjih}X^kY^jX^iY^h$ for $X = X^i\partial/\partial x^i$ and $Y = Y^j\partial/\partial x^j$. Then from (1. 5) and (2. 2) we have

LEMMA 2. *For an orthonormal pair* $(X + \alpha\xi, Y)(\alpha$: *real number) such that* $\eta(X) = \eta(Y) = 0$ *we have*

$$*K(X + \alpha\xi, Y) = (1 - \alpha^2)*K(X, Y) + \alpha^2 . \tag{2.5}$$

From which we obtain

LEMMA 3. *If for a pair* (X, Y) *such that* $\eta(X) = \eta(Y) = 0$

$$\delta L \leqq *K(X, Y) \leqq L,$$

then

$$\text{Min. } \{\delta L, 1 - b\} \leqq *K(X,Y) \leqq \text{Max. } \{1-b, L\},$$

for any pair (X, Y).

PROOF. This is shown by the fact that the sectional curvature of the 2-plane which contains ξ is $1 - b$, and also by (2. 5).

In the following we assume that the original manifold is a unit sphere S^{2n+1}. Then the curvature tensor (2. 2) is reduced to the form

$$\begin{aligned} *R_{kjih} = (g_{kh}g_{ji} - g_{jh}g_{ki}) + b(\varphi_{kh}\varphi_{ji} - \varphi_{jh}\varphi_{ki} - 2\varphi_{ih}\varphi_{kj}) \\ + (b^2 - 2b)(\eta_i\eta_j g_{kh} - \eta_i\eta_k g_{jh} + \eta_h\eta_k g_{ji} - \eta_h\eta_j g_{ki}), \end{aligned}$$

and the sectional curvature of the 2-plane defined by an orthonormal pair (X,Y) is

$$*K(X, Y) = 1 + 3bg(\varphi X, Y)^2 + (b^2 - 2b)\{\eta(X)^2 + \eta(Y)^2\} , \tag{2.6}$$

where $\varphi X = (\varphi_j{}^iX^j)\partial/\partial x^i$.

LEMMA 4. *For a pair (X, Y) such that $\eta(X)=\eta(Y)=0$, $^*K(X, Y)$ is pinched as*

$$1 \leqq {}^*K(X, Y) \leqq 1+3b \tag{2.7}$$

and therefore for any pair (X, Y), $^*K(X, Y)$ is pinched as

$$1-b \leqq {}^*K(X, Y) \leqq 1+3b\,. \tag{2.8}$$

PROOF. For a pair (X, Y) such that $\eta(X)=\eta(Y)=0$, we have from (2.6)

$$^*K(X, Y) = 1+3bg(\varphi X, Y)^2$$

which leads us to (2.7), and by virture of Lemma 3, we also have (2.8).

Thus we get

THEOREM. *Let S^{2n+1} be a unit sphere with natural metric g. Then S^{2n+1} has a Sasakian structure (φ, ξ, η, g). If such a change of metric as* (2.1) *is made in S^{2n+1}, S^{2n+1} will be $\left(\frac{1-b}{1+3b}\right)$-pinched; that is, the sectional curvature is pinched as*

$$1-b \leqq {}^*K(X, Y) \leqq 1+3b$$

for any orthonormal pair (X, Y).

3. In the following we assume that our M^{2n+1} is compact and the contact structure η is regular. That is each point of M^{2n+1} has a cubical coordinate neighborhood whose intersection with any trajectory of ξ is a single segment;

$$x^2 = \text{constant}, \cdots, x^{2n+1} = \text{constant}.$$

It is known [4] that in a compact regular Sasakian manifold, all trajectories of ξ are closed geodesic and of the same period. M^{2n+1} being of constant curvature 1 by assumption, the period is 2π. We see from (2.3) that a trajectory of ξ through a point p is also a geodesic with respect to g^*. If we denote by t and s the arc-length with respect to g and $^*g$ respectively, we have the relation

$$s = \sqrt{1-b}\;\; t,$$

since

$$^{*}g(\xi,\xi)=g(\xi,\xi)-b\eta(\xi)^2=1-b\,.$$

Therefore the period of any trajectory of ξ is $2\pi\sqrt{1-b}$ with respect to $^{*}g$.

Generally speaking, if l is a period of a closed geodesic through p, then we have

$$l/2\geqq d(p,C(p))\,.$$

Thus we get the following inequality

$$\pi\sqrt{1-b}\geqq d(p,C(p))\,.$$

In these circumstance, if we choose b satisfying

$$\sqrt{1-b}<\frac{1}{\sqrt{1+3b}}\quad\left(=\frac{\pi}{\sqrt{L}}\right), \tag{3.1}$$

we have

$$\pi/\sqrt{L}>d(p,C(p))\,.$$

That is, Theorem 2 stated in Introduction does not hold good for the case $b>\frac{2}{3}$. Consequently, as we obtain an example what is required, Theorem does not hold good generally for any odd dimension.

REMARK. If we choose $b>\frac{2}{3}$ then the pinching is $\delta>\frac{1}{9}$.

4. The metric (2.1) was introduced under the following geometric consideration which is nothing but Berger's idea.

Let M be a compact regular Sasakian manifold of constant curvature 1 and R be a real line. In the product manifold $M\times R$, we consider a unit vector field

$$H=\xi\cos\alpha-D\sin\alpha$$

where α is a constant $(0<\alpha<\pi/2)$ and D is a unit vector field in R. Let q be a point of $M\times R$ on a trajectory of H through $(p,0)$. If we correspond q to p, we can define a mapping $\pi: M\times R\rightarrow M$. Then a fibred space with invariant metric $(M\times R, M, \pi, H, \tilde{g})$ can be constructed, where $\tilde{g}$ is the product metric of $M\times R$.[2)] The induced metric $^{*}g$ on M by π is nothing but the

2) The notations and definitions of fibred space follow [6].

one given in (2. 1)($b=\cos^2\alpha$). Moreover the projection of any trajectory of a unit vector field U orthogonal to H (i. e. horizontal) is a geodesic in M of the metric $*g$(Prop. 4. 1 [6]). As we have seen, the trajectories are of the same period $2\pi \sin \alpha$. Therefore if we choose α satisfying

$$0 < \sin \alpha < \frac{1}{\sqrt{3}},$$

the base space M is an example. Berger [1] has taken as M a compact simply connected Lie group $SU(2)$.

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