# AN EXISTENCE THEOREM FOR A VOLTERRA INTEGRAL EQUATION WITH DEVIATING ARGUMENTS*

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## ABSTRACT

An existence theorem is proved for a nonlinear Volterra integral equation with deviating arguments.

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## 1. INTRODUCTION

The theory of nonlinear Volterra integral equations with deviating arguments and functional integral equations have been studied by many authors [1,2,5,6]. Banas [4] has proved an existence theorem for functional integral equations and Balachandran [1] has proved an existence theorem for a nonlinear Volterra integral equation with deviating argument. In this paper we shall derive a set of sufficient conditions for the existence of a solution of nonlinear Volterra integral equations with deviating arguments. This result is a generalization of the results in [1,4].

## 2. BASIC ASSUMPTIONS

Let $p(t)$ be a given continuous function defined on the interval $[0,\infty)$ and taking real positive values. Denote $C([0,\infty), p(t):\boldsymbol{R}^n)$ by $C_p$, the set of all continuous functions from $[0,\infty)$ into $\boldsymbol{R}^n$ such that

$$\sup \{|x(t)|p(t):\ t \geq 0\} < \infty.$$

It has been proved [7] that $C_p$ forms a real Banach space with regard to the norm

$$\|x\| = \sup \{ |x(t)|p(t):\ t \geq 0\}.$$

If $x \in C_p$ then we will denote by $\omega^T(x,\varepsilon)$ the usual modulus of continuity of $x$ on the interval $[0,T]$, that is,

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$$\omega^T(x,\varepsilon) = \sup\{|x(t) - x(s)|: |t - s| \le \varepsilon;\ t,s \in [0,T]\}.$$

Our existence theorem is based on the following lemma.

Lemma (See [3].): *Let E be a bounded set in the space $C_p$. If all functions belonging to E are equicontinuous on each interval* [0,T] *and*

$$\lim_{T\to\infty}\{|x(t)|p(t):\ t \ge T\} = 0$$

*uniformly with respect to E, then E is relatively compact in $C_p$.*

Consider the nonlinear Volterra integral equation with deviating arguments

$$(1)\quad x(t) = G(x)g(t,x(h_1(t)),\ x(h_2(t)),\dots,x(h_n(t))) + \int_0^t K(t,s,x(H_1(s)),\dots,x(H_m(s)))ds$$

where $x$, $H$ and $K$ are $n$-vectors and $G$ is a real-valued function. Assume the following conditions.

(*i*) Let $\Delta = \{(t,s):\ 0 \le s \le t < \infty\}$. The kernel $K$: $\Delta \times R^{nm} \to R^n$ is continuous and there exist continuous functions $m$: $\Delta \to [0,\infty)$, $a$: $[0,\infty) \to (0,\infty)$, and $b$: $[0,\infty) \to [0,\infty)$ such that

$$|K(t,s,x_1,x_2,\dots,x_m)| \le m(t,s) + a(t)b(s)\sum_{i=1}^{m}|x_i|$$

for all $(t,s) \in \Delta$ and $(x_1,x_2,\dots,x_m) \in R^{nm}$.

In order to formulate other assumptions let us define $L(t) = \int_0^t a(s)b(s)ds, \quad t \ge 0;$ furthermore, let us take an arbitrary number $M > 0$ and consider the space $C_p$ with $p(t) = [a(t)e^{ML(t)+t}]^{-1}$.

(*ii*) There exists a constant $A > 0$ such that for any $t \in [0,\infty)$ the following inequality holds.

$$\int_0^t m(t,s)ds \le Aa(t)e^{ML(t)}$$

(*iii*) For $i = 1, 2,\dots,n$ the functions $h_i$: $[0,\infty) \to [0,\infty)$ are continuous, $h_i(0) = 0$, $h_i(t) \le t$ for $t \ge 0$ and there exists a positive real number $B_i$ such that $a(h_i(t)) \le B_i a(t)$.

(*iv*) G: $C_p \to [0,\infty)$ is continuous and bounded. Assume $|G(x)| \le k_1$ where $k_1$ is a positive constant.

(*v*) The function g: $[0,\infty) \times R^{n^2} \to R^n$ is continuous and satisfies the conditions

$$|g(t,x_1,x_2,\dots,x_n) - g(t,y_1,y_2,\dots,y_n)| \le \sum_{i=1}^{n}\alpha_i(t)|x_i - y_i|$$

where $\alpha_i(t)$ is continuous such that

$\alpha_i(t) \le e^{M[L(t) - L(h_i(t))]}$ for $t \ge 0$, for $i = 1,2,\dots,n$ and $|g(t,0,\dots,0)| \le a(t)e^{ML(t)}$.

(*vi*) For $i = 1,2,\ldots,m$, the functions $H_i\colon [0,\infty) \to [0,\infty)$ are continuous and satisfy the following conditions:

$$L(H_i(t)) - L(t) \le N_i$$

where $N_i$ is a positive constant, and

$$a(H_i(t))/a(t) \le (M/m)[1 - k_1(1+B) - A]e^{-MN_i}$$

where $B = \sum_{i=1}^{n} B_i$ and we assume $k_1(1+B) + A < 1$.

## 3. EXISTENCE THEOREM

Theorem: *Assume that the hypotheses* (*i*) *through* (*vi*) *hold; then equation* (1) *has at least one solution* $x$ *in the space* $C_p$ *such that* $|x(t)| \le a(t)e^{ML(t)}$ *for any* $t \ge 0$.

*Proof:* Define a transformation $F$ in the space $C_p$ by

$$(2) \qquad (Fx)(t) = G(x)g(t,x(h_1(t)), x(h_2(t)),\ldots,x(h_n(t))) + \int_0^t K(t,s,x(H_1(s)),\ldots,x(H_m(s)))ds.$$

From our assumptions we observe that $(Fx)(t)$ is continuous on the interval $[0,\infty)$. Define the set $E$ in $C_p$ by

$$E = \{x \in C_p\colon |x(t)| \le a(t)e^{ML(t)}\}.$$

Clearly $E$ is nonempty, bounded, convex, and closed in $C_p$. Now we prove that $F$ maps the set $E$ into itself. Take $x \in E$. Then from our assumptions we have

$$|(Fx)(t)| \le |G(x)|\,|g(t,x(h_1(t)),\ldots,x(h_n(t)))| + \int_0^t |K(t,s,x(H_1(s)),\ldots,x(H_m(s)))|ds.$$

$$\le k_1|g(t,x(h_1(t)),\ldots,x(h_n(t))) - g(t,0,\ldots,0)| + k_1|g(t,0,\ldots,0)| + \int_0^t m(t,s)ds$$

$$+ a(t)\int_0^t b(s)\sum_{i=1}^{m} |x(H_i(s))|ds$$

$$\le k_1\sum_{i=1}^{n} \alpha_i(t)|x(h_i(t))| + k_1 a(t)e^{ML(t)} + Aa(t)e^{ML(t)}$$

$$+ a(t)\int_0^t b(s)\sum_{i=1}^{m} a(H_i(s))e^{ML(H_i(s))}ds$$

$$\le k_1\sum_{i=1}^{n} e^{M[L(t) - L(h_i(t))]}a(h_i(t))e^{ML(H_i(t))} + k_1 a(t)e^{ML(t)} + Aa(t)e^{ML(t)}$$

$$+ a(t)([1 - k_1(1+B) - A]/m) \sum_{i=1}^{m} e^{-MN_i} \int_0^t Mb(s)a(s)e^{MN_i} e^{ML(s)}ds$$

$$\leq k_1 Ba(t)e^{ML(t)} + k_1 a(t)e^{ML(t)} + Aa(t)e^{ML(t)} + a(t)[1 - k_1(1+B) - A]e^{ML(t)}$$

$$= a(t)e^{ML(t)},$$

which proves that $FE$ is a subset of $E$.

Now we want to prove that F is continuous on the set E. In order to do this we take $F = F_1 + F_2$, where

$$(F_1x)(t) = G(x)g(t,x(h_1(t)),\ldots,x(h_n(t)))$$

and

$$(F_2x)(t) = \int_0^t K(t,s,x(H_1(s)),\ldots,x(H_m(s)))ds.$$

We shall prove the continuity of $F_1$ and $F_2$ separately. Let us fix $\varepsilon > 0$ and take $x,y \in E$ such that $\|x - y\| \leq \varepsilon$. We have

$$\|F_1x - F_1y\| \leq k_1 \sup\{|g(t,x(h_1(t)),\ldots,x(h_n(t)))$$
$$- g(t,y(h_1(t)),\ldots,y(h_n(t)))|\cdot[a(t)e^{ML(t)+t}]^{-1}: t \geq 0\}$$
$$+ |G(x) - G(y)|\sup\{|g(t,y(h_1(t)),\ldots,y(h_n(t)))|\cdot[a(t)e^{ML(t)+t}]^{-1}: t \geq 0\}$$
$$\leq k_1 \sum_{i=1}^{n} \sup\{\alpha_i(t)|x(h_i(t)) - y(h_i(t))|[a(t)e^{ML(t)+t}]^{-1}: t \geq 0\}$$
$$+ |G(x) - G(y)| \sup\{|g(t,y(h_1(t)),\ldots,y(h_n(t))) - g(t,0,\ldots,0)|\cdot[a(t)e^{ML(t)+t}]^{-1}: t \geq 0\}$$
$$+ |G(x) - G(y)| \sup\{|g(t,0,\ldots,0)\cdot[a(t)e^{ML(t)+t}]^{-1}: t \geq 0\}$$
$$\leq k_1 \sum_{i=1}^{n} B_i \sup\{|x(h_i(t)) - y(h_i(t))|[a(h_i(t))]^{-1} e^{M[L(t)-L(h_i(t))]} e^{-ML(t)-t} : t \geq 0\}$$
$$+ |G(x) - G(y)| \sum_{i=1}^{n} \sup\{\alpha_i(t)\, |y(h_i(t))|\, [a(t)\, e^{ML(t)+t}]^{-1} : t \geq 0\}$$
$$+ |G(x) - G(y)| \sup\{e^{-t} : t \geq 0\}$$
$$\leq k_1 \sum_{i=1}^{n} B_i \sup\{|x(h_i(t)) - y(h_i(t))|[a(h_i(t))\, e^{ML(h_i(t)) + h_i(t)}]^{-1} : t \geq 0\}$$
$$+ |G(x) - G(y)| \sum_{i=1}^{n} \sup\{a(h_i(t))e^{ML(t)}[a(t)\, e^{ML(t)+t}]^{-1}: t \geq 0\} + |G(x)-G(y)|$$
$$\leq k_1 B \|x-y\| + B|G(x) - G(y)| + |G(x) - G(y)|.$$

This implies that $F_1$ is continuous in view of *(iv)*.

Now we prove that $F_2$ is continuous on the set $E$. For this let us fix $\varepsilon > 0$ and $x, y \in E$ such that $\| x - y \| \leq \varepsilon$. Further, let us take an arbitrary fixed $T > 0$. In view of *(i)* and *(vi)* the function $K(t, s, x_1, \ldots, x_m)$ is uniformly continuous on

$$[0, T] \times [0, T] \times [-r(H_1(T)), r(H_2(T))] \times \ldots \times [-r(H_m(T)), r(H_m(T))]$$

*where* $r(T) = max\{a(s)e^{ML(s)} : s \in [0, T]$. *Thus, we have for* $t \in [0, T]$

(3) $$/(F_2x)(t) - (F_2y)(t)/ \leq \int_0^t / K(t, s, x(H_1(s)), \ldots, x(H_m(s))) - K(t, s, y(H_1(s)), \ldots, y(H_m(s))) / ds \leq \beta(\varepsilon)$$

where $\beta(\varepsilon)$ is some continuous function such that $\lim_{\varepsilon \to 0} \beta(\varepsilon) = 0$. Further, let us take $t \geq T$. Then we have

$$/(F_2x)(t) - (F_2y)(t)/ \leq /(F_2x)(t)/ + /(F_2y)(t)/ \leq 2a(t)e^{ML(t)}$$

$$\text{and} \quad /(F_2x)(t) - (F_2y)(t)/ p(t) \leq 2e^{-t}.$$

Hence for sufficiently large $T$ we have

(4) $$/(F_2x)(t) - (F_2y)(t)/ p(t) \leq \epsilon .$$

By (3) and (4) we get that $F_2$ is continuous on the set $E$. Hence $F = F_1 + F_2$ is continuous on $E$.

Now we prove that $FE$ is relatively compact. For every $x \in E$ we have $Fx \in E$ which gives $|(Fx)(t)|p(t) \leq e^{-t}$. Hence $\lim_{T\to\infty} \sup\{|(Fx)(t)|p(t) : t \geq T\} = 0$ uniformly with respect to $x \in E$.

Furthermore, let us fix $\varepsilon > 0$ and $T > 0$; *and let* $t, s \in [0, T]$ such that $|t-s| \leq \varepsilon$. Then for $x \in E$, we have

$$/(Fx)(t) - (Fx)(s)/ \leq /G(x)/ / g(t, x(h_1(t)), \ldots, x(h_n(t))) - g(s, x(h_1(s)), \ldots, x(h_n(s))) / + / \int_0^t K(t, u, x(H_1(u)), \ldots, x(H_m(u)))du - \int_0^s K(s, u, y(H_1(u)), \ldots, y(H_m(u)))du /$$

$$\leq k_1\ \omega^T (g, \varepsilon) + / \int_0^t K(t, u, x(H_1(u)),\ldots, x(H_m(u))\ du$$

$$- \int_0^s K(t, u, x(H_1(u)),\ldots, x(H_m(u)))\ du/$$

$$+ / \int_0^s K(t, u, x(H_1(\ )),\ldots, x(H_m(u)))\ du$$

$$- \int_0^s K(s, u, x(H_l(u)),\ldots, x(H_m(u)))\ du\ /$$

$$\leq k_l\ \omega^T(g, \varepsilon) + \int_s^t / K(t, u, x(H_l(u)),\ldots, x(H_m(u)))\ /\ du$$

$$+ \int_0^s / K(t, u, x(H_l(u)),\ldots, x(H_m(u)))$$

$$- K(s, \ , x(H_l\ (u)),\ldots, x(H_m(u)))\ /\ du$$

$$\leq k_l\ \omega^T (g, \varepsilon + \varepsilon\, max\{\, m(t, u) + a(t)b(u) \sum_{i=l}^{m} / x(H_i(u) / : o \leq u \leq t \leq T\}$$

$$+ T\omega^T(K, \varepsilon).$$

This tends to zero as $\varepsilon \rightarrow 0$. Thus $FE$ is equicontinuous on $[0, T]$.

Therefore from the lemma $FE$ is relatively compact. Thus the Schauder fixed point theorem guarantees that $F$ has a fixed point $x \in E$ such that $(Fx)\ (t) = x(t)$. Hence the theorem holds.

EXAMPLE: Consider the following nonlinear integral equation:

$$(5) \qquad x(t) = (1/8)\ t\sin x\ (t/2) + \int_0^t [ts + (t^3 + (1/4))\, x(s/3)\ ]\ ds.$$

This is clearly of the form (1) and satisfies all the conditions (i) to (vi) with

$$m = n = 1,\ G(x) = 1/8,\ g(t, x) = t \sin x,$$
$$K(t, s, x) = ts + (t^3 + (1/4))\,x,$$
$$m(t, s) = ts,\ a(t) = t^3 + (1/4),\ b(s) = 1,$$
$$\alpha_1(t) = t,\ h_1(t) = t/2,\ H_1(t) = t/3,$$
$$M = 16,\ A = 1/2,\ B = 1,\ k_1 = 1/8,\ N_1 = 1/16.$$

Therefore from our existence theorem the equation (5) has at least one solution $x$ in the space $C_p$ such that

$$| x(t) | \le (t^3 + 1/4)e^{4(t^4 + t)} \text{ for any } t \ge 0.$$

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