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An existence theorem for set differential inclusions in a semilinear metric space

by

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Abstract: Using the notion of continuous approximate selections, we establish an existence theorem for set differential inclusions in a semi-linear metric space.

Keywords: approximate continuous selections, Cauchy problem, set differential inclusions, Hukuhara derivative.

1. Introduction

In recent years the development of the calculus in metric spaces has attracted some attention (Ambrosio, Tilli, 2004; Lakshmikantham et al., 2006). Earlier, De Blasi, Iervolino (1969) (see also Brandào, De Blasi, Iervolino, 1970) started the investigation of set differential equations (SDEs) in semilinear metric spaces. This has now evolved into the theory of SDEs as an independent discipline, (see Lakshmikantham, 2004; Lakshmikantham et al., 2006; Plotnikov, Tumbrukaki, 2000; Plotnikov, Plotnikova, 1997; Plotnikov, Rashkov, 1999). On the other hand, SDEs are useful in other areas of mathematics. For example, SDEs are used, as an auxiliary tool, to prove existence results for differential inclusions (Tolstonogov, 2000). Also, one can employ SDEs in the investigation of fuzzy differential equations (see Lakshmikantham, Mohapatra, 2003). For alternative approaches to fuzzy differential equations and additional bibliography see Rzeżuchowski, Wąsowski (2001). Moreover, SDEs are a natural generalization of the usual ordinary differential equations in finite (or infinite) dimensional Banach spaces.

In this paper, we consider set differential inclusions (SDIs) in semilinear metric spaces, and prove an existence result under the assumption that the right hand side is upper semicontinuous. Our approach has its origin in the classical Severini method (Severini, 1898) for solving ordinary differential equations based on the construction of a sequence of "regular" approximating differential equations and on a compactness argument for the corresponding solutions. For differential inclusions in $\mathbb{R}^k$ the Severini approach was adopted by Cellina (1970), in combination with an approximate selection theorem (Cellina, 1969). In our setting, by using an analogous approximate selection result, valid in α-convex metric spaces, we construct, as in Cellina (1970), a sequence of "regular" approximating SDE's and show the convergence of the corresponding solutions. However, unlike the $\mathbb{R}^k$ case, the proof that the limit is actually a solution of the given SDI is more delicate, and it is carried out by a technical argument, in which an important characterization theorem of integrals, due to Hermes (1968), plays a crucial role.

Our presentation intends to be elementary and self contained. For this reason direct proofs are supplied also for a few auxiliary results which are essentially contained in the monograph by Hu and Papageorgiou (1997).

The rest of the paper is organized as follows. In Section 2, we introduce the terminology and formulate the Cauchy problem under study. Section 3 contains a few auxiliary results that we use in the proof of our existence theorem. In Section 4 we establish the existence of solutions to the Cauchy problem for SDIs.

2. Notations and preliminaries

Let Z be a nonempty metric space with a metric ρ, and

$$P(Z) = \{A \subset Z \mid A \text{ is bounded and nonempty}\}.$$

For $z \in Z$ and $\phi \neq A \subset Z$, set

$$d(z, A) = \inf_{a\in A} \rho(z, a).$$

Moreover, if $A, B \in P(Z)$ put

$$e(A, B) = \sup_{a\in A} d(a, B) \text{ and } e(B, A) = \sup_{b\in B} d(b, A).$$

These notions will occasionally be used also when A, B are nonempty subsets of Z.

By $B_Z(a, r)$ and $B_Z[a, r]$ we mean an open and a closed ball in Z, with center a and radius r.

Let T be a metric space. A multifunction $\Phi : T \to P(Z)$ is called upper semicontinuous (u.s.c) (resp. lower semicontinuous (l.s.c)) at $x_0 \in T$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$e(\Phi(x), \Phi(x_0)) < \epsilon, \quad (\text{resp. } e(\Phi(x_0), \Phi(x)) < \epsilon), \text{ for every } x \in B_T(x_0, \delta).$$

Φ is said to be continuous at x_0 if it is u.s.c. and l.s.c.x_0.

Now set

$$\mathfrak{X} = \{A \subset I\!R^k \,|\, A \text{ is compact, convex and nonempty}\},$$

and equip $\mathfrak{X}$ with the Pompeiu-Hausdorff metric

$$h(A,B) = \max\{e(A,B), e(B,A)\}, \quad A, B \in \mathfrak{X}.$$

Under this metric $\mathfrak{X}$ is a complete metric space.

Set $I = [0,1]$. We denote by $C(I,\mathfrak{X})$ the space of all continuous maps $X : I \to \mathfrak{X}$ equipped with the metric of uniform convergence. Clearly, $C(I,\mathfrak{X})$ is a complete metric space.
Furthermore, set

$$\mathcal{C}(\mathfrak{X}) = \{\mathcal{A} \subset \mathfrak{X} \,|\, \mathcal{A} \text{ is compact, convex and nonempty}\}.$$

$\mathcal{C}(\mathfrak{X})$ is endowed with the corresponding Pompeiu-Hausdorff metric

$$H(\mathcal{A},\mathcal{B}) = \max\{e(\mathcal{A},\mathcal{B}), e(\mathcal{B},\mathcal{A})\}, \quad \mathcal{A},\mathcal{B} \subset \mathcal{C}(\mathfrak{X}),$$

under which it becomes a complete metric space.
Occasionally, we consider the space

$$\mathfrak{Y} = \{A \subset I\!R^k \,|\, A \text{ is compact and nonempty}\},$$

and equip it with the Pompeiu-Hausdorff metric h.

A map $U : I \to \mathfrak{Y}$ is said to be *measurable* if the set $\{t \in I \,|\, U(t) \cap C \neq \phi\}$ is (Lebesgue) measurable for every closed set $C \subset I\!R^k$. Observe that if $U : I \to \mathfrak{Y}$ is measurable, then so is the map $t \to \overline{co}\, U(t)$, $t \in I$, where $\overline{co}\, U(t)$ stands for the closed convex hull of $U(t)$. By $m(J)$ we mean the (Lebesgue) measure of $J \subset I\!R$.
The space $I \times \mathfrak{X}$ is endowed with the metric

$$\max\{|t - t'|, h(X,X')\}, \quad (t,X),(t',X') \in I \times \mathfrak{X}.$$

Now, consider the Cauchy problem

$$\text{(CP)} \quad DX(t) \in \Phi(t, X(t)), \quad X(0) = X_0.$$

Here $DX(t)$ denotes the *Hukuhara derivative* of X at t, (see Hukuhara, 1967), $X_0 \in \mathfrak{X}$ and Φ is supposed to satisfy the following assumptions:

(h_1) $\Phi : I \times \mathfrak{X} \to \mathcal{C}(\mathfrak{X})$ is u.s.c.

(h_2) $H(\Phi(t,X), \{0\}) \leq M$ for every $(t,X) \in I \times \mathfrak{X}$, $M \geq 1$.

A map $X : I \to \mathfrak{X}$ is called *a solution of the Cauchy problem (CP)* if there exists a measurable map $U : I \to \mathfrak{X}$ such that:

$$X(t) = X_0 + \int_0^t U(s)\, ds \quad t \in I,$$

$$U(t) \in \Phi(t, X(t)), \quad t \in I, a.e.$$

Here the integral is in the sense of Hukuhara (1967). (This integral certainly exists, if U is measurable and the real map $t \to h(U(t), \{0\})$ is integrable on I.)

By virtue of De Blasi, Iervolino (1969), Hukuhara (1967), if X is a solution of the Cauchy problem, then X is continuous on I, has Hukuhara derivative DX a.e. in I, and $DX(t) = U(t)$, for $t \in I$, a.e.

3. Auxiliary results

Given $\varphi : I \times \mathfrak{X} \to \mathfrak{X}$ and $\Phi : I \times \mathfrak{X} \to \mathcal{C}(\mathfrak{X})$, the $\mathfrak{X}$- graph of φ, denoted by $Gr\,\varphi$ and the $\mathfrak{X}$ - graph of Φ denoted by $Gr\,\Phi$, are defined by

$$Gr\,\varphi = \{(t, X, \varphi(t, X)) \in I \times \mathfrak{X} \times \mathfrak{X} |\, (t, X) \in I \times \mathfrak{X}\},$$
$$Gr\,\Phi = \{(t, X, Y) \in I \times \mathfrak{X} \times \mathfrak{X} |\, (t, X) \in I \times \mathfrak{X} \text{ and } Y \in \Phi(t, X)\}.$$

The $\mathbb{R}^k$ - graph of a map $\varphi : I \to \mathfrak{X}$, denoted by graph φ, is defined by

$$\text{graph}\,\varphi = \{(t, y) \in I \times \mathbb{R}^k | t \in I, y \in \varphi(t)\}.$$

Let $\Phi : I \times \mathfrak{X} \to \mathcal{C}(\mathfrak{X})$ and $\epsilon > 0$. Any continuous function $\varphi_\epsilon : I \times \mathfrak{X} \to \mathfrak{X}$ satisfying $e(Gr\,\varphi_\epsilon, Gr\,\Phi) < \epsilon$ is called a continuous approximate selection of Φ.

Remark 1 Observe that $e(Gr\,\varphi, Gr\,\Phi) \le \epsilon$ if and only if for every $(t, X) \in I \times \mathfrak{X}$, there exist $(t', X') \in I \times \mathfrak{X}$ and $Y' \in \Phi(t', X')$ such that

$$|t' - t| < \epsilon, \quad h(X', X) < \epsilon, \quad h(Y', \varphi(t, X)) < \epsilon.$$

The reverse implication is valid with $\le$ in place of $<$.

The following is a metric version of theorem of Cellina (1969).

Theorem 1 (De Blasi, Pianigiani, 2004a) *Let $\Phi : I \times \mathfrak{X} \to \mathcal{C}(\mathfrak{X})$ satisfy the assumptions (h_1) and (h_2). Then, there exists a sequence $\{\varphi_n\}$ continuous approximate selections $\varphi_n : I \times \mathfrak{X} \to \mathfrak{X}$ of Φ such that:*

$$(a_1) \quad h(\varphi_n(t, X), \{0\}) \le M, \quad \textit{for every } (t, X) \in I \times \mathfrak{X}, \textit{ and } n \in \mathbb{N}.$$

$$(a_2) \quad e(Gr\,\varphi_n, Gr\,\Phi) \le \frac{1}{n}, \quad \textit{for every } n \in \mathbb{N}.$$

Proof. The space $\mathfrak{X}$, equipped with the convexity map $\alpha : \mathfrak{X} \times \mathfrak{X} \times [0, 1] \to \mathfrak{X}$ given by $\alpha(X, Y, t) = (1 - t)X + tY$, $(X, Y) \in \mathfrak{X}^2$, $t \in [0, 1]$ is an α- convex metric space. Moreover, $B_{\mathfrak{X}}[0, M]$ is an α- convex subset of $\mathfrak{X}$. As Φ is u.s.c and its values are contained in $B_{\mathfrak{X}}[0, M]$, the statement follows from Proposition 5.1 of De Blasi, Pianigiani (2004a). ■

In the sequel we shall employ a result of Hermes (1968), where the following definition of measurability is used.

A map $U : I \to \mathfrak{Y}$ is called *Borel measurable* if graph U is a Borel subset of $I \times \mathbb{R}^k$. Observe that any Borel measurable map $U : I \to \mathfrak{Y}$ is also measurable. The following is a special case of Theorem 2.1 in Hermes (1968).

THEOREM 2 (Hermes, 1968) *Let $\{\Omega_n\}$ be a sequence of Borel measurable maps $\Omega_n : I \to \mathfrak{Y}$ satisfying $\Omega_n(t) \subset B_{\mathbb{R}^k}[0, M]$, $t \in I$, and suppose that the corresponding sequence $\{Y_n\}$, where*

$$Y_n(t) = (A)\int_0^t \Omega_n(s)\,ds, \quad t \in I,$$

converges uniformly to a map $Y : I \to \mathfrak{Y}$. Then, there exists a Borel measurable map $\Omega : I \to \mathfrak{Y}$ with $\Omega(t) \subset B_{\mathbb{R}^k}[0, M], t \in I$, such that

$$Y(t) = (A)\int_0^t \Omega(s)\,ds, \quad t \in I.$$

The symbol (A) before the above integrals is to point out that they are understood in the sense of Aumann (1965).

PROPOSITION 1 *Let $\{U_n\}$ be a sequence of measurable maps $U_n : I \to \mathfrak{X}$ satisfying $U_n(t) \subset B_{\mathbb{R}^k}[0, M]$, $t \in I$, and suppose that the corresponding sequence $\{X_n\}$, where*

$$X_n(t) = \int_0^t U_n(s)\,ds, \quad t \in I,$$

converges uniformly to a map $X : I \to \mathfrak{X}$. Then, there exists a measurable map $U : I \to \mathfrak{X}$, with $U(t) \subset B_{\mathbb{R}^k}[0, M]$, $t \in I$ such that

$$X(t) = \int_0^t U(s)\,ds, \quad t \in I.$$

Proof. By Lusin-Pliś Theorem, for every $n \in \mathbb{N}$ there exists a closed set $I_n \in I$ with $m(I \backslash I_n) < \frac{1}{n}$ such that U_n restricted to I_n is continuous. Define $\Phi : I \to \mathcal{C}(\mathfrak{X})$ by

$$\phi(t) = \begin{cases} \{U_n(t)\}, & t \in I_n \\ \mathcal{B}, & t \in I \backslash I_n, \end{cases}$$

where $\mathcal{B} = \{X \in \mathfrak{X} | \, X \subset B_{\mathbb{R}^k}[0, M]\}$. Φ is lower semicontinuous on I and thus by De Blasi, Pianigiani (2004b), it admits a continuous selection $\Omega_n : I \to \mathfrak{X}$ satisfying $\Omega_n(t) = U_n(t)$, $t \in I_n$ and $\Omega_n(t) \subset B_{\mathbb{R}^k}[0, M]$, $t \in I$. Moreover, Ω_n is Borel measurable, for graph Ω_n is a compact subset of $I \times \mathbb{R}^k$.

We claim that the sequence $\{Y_n\}$, given by

$$Y_n(t) = \int_0^t \Omega_n(s)\,ds, \quad t \in I, \tag{1}$$

converges uniformly to $X : I \to \mathfrak{X}$. In fact, denoting by χ_J the characteristic function of $J \subset I$, for each $t \in I$ we have

$$\begin{aligned} h(Y_n(t), X(t)) &\leq h\left(\int_0^t \Omega_n(s)\,ds, \int_0^t U_n(s)\,ds\right) + h(X_n(t), X(t)) \\ &= h\Big(\int_0^t \Omega_n(s)\chi_{I_n}(s)\,ds + \int_0^t \Omega_n(s)\chi_{I\setminus I_n}(s)\,ds, \\ &\quad \int_0^t U_n(s)\chi_{I_n}(s)\,ds + \int_0^t U_n(s)\chi_{I\setminus I_n}(s)\,ds\Big) + h(X_n(t), X(t)) \\ &\leq h\Big(\int_0^t \Omega_n(s)\chi_{I_n}(s)\,ds, \int_0^t U_n(s)\chi_{I_n}(s)\,ds\Big) \\ &\quad + h\Big(\int_0^t \Omega_n(s)\chi_{I\setminus I_n}(s)\,ds, \int_0^t U_n(s)\chi_{I\setminus I_n}(s)\,ds\Big) \\ &\quad + \max_{t\in I} h(X_n(t), X(t)) \\ &\leq 2Mm(I\setminus I_n) + \max_{t\in I} h(X_n(t), X(t)). \end{aligned}$$

Since the latter quantity vanishes as $n \to \infty$, the sequence $\{Y_n\}$ converges uniformly to X on I.

Furthermore, from (1), by the equality of Aumann's and Hukuhara's integrals (see De Blasi, Lasota, 1968), for each $n \in I\!\!N$ we have

$$Y_n(t) = (A)\int_0^t \Omega_n(s)\,ds, \quad t \in I,$$

where Ω_n is Borel measurable.

Then, by Theorem 2, there exists a Borel measurable (hence measurable) map $\Omega : I \to \mathfrak{Y}$ with $\Omega(t) \subset B_{I\!\!R^k}[0, M]$, $t \in I$, satisfying

$$X(t) = (A)\int_0^t \Omega(s)\,ds, \quad t \in I. \tag{2}$$

Define $U : I \to \mathfrak{X}$ by $U(t) = \overline{co}\,\Omega(t)$, $t \in I$. Clearly, U is measurable, $U(t) \subset B_{I\!\!R^k}[0, M]$, $t \in I$ and, moreover, by Aumann (1965),

$$(A)\int_0^t \Omega(s)\,ds = (A)\int_0^t U(s)\,ds, \quad t \in I. \tag{3}$$

As the latter integral is equal to the corresponding Hukuhara integral of U, combining (2) and (3) gives

$$X(t) = \int_0^t U(s)\,ds, \quad t \in I.$$

This completes the proof. ∎

PROPOSITION 2 *Let* $U : I \to \mathfrak{X}$ *and* $\Psi : I \to \mathcal{C}(\mathfrak{X})$ *be measurable maps. Then the function* $t \to d(U(t), \Psi(t))$ *is measurable on* I.

Proof. In fact, $\mathfrak{X}$ is a complete, separable metric space, and thus, by Himmelberg (1975, Theorems 3.5 (iii) and 5.6) there exists a countable family $\{V_n\}$ of measurable selectors $V_n : I \to \mathfrak{X}$ of Ψ such that $\Psi(t) = cl\{V_n(t)\}_{n=1}^{\infty}, \quad t \in I$. Here, the closure is in $\mathfrak{X}$. Since $d(U(t), \Psi(t)) = \inf_n h(U(t), V_n(t))$, the statement follows. ■

For $\mathcal{A} \in \mathcal{C}(\mathfrak{X})$ and $r > 0$ set $N_{\mathfrak{X}}[\mathcal{A}, r] = \{X \in \mathfrak{X} | d(X, \mathcal{A}) \leq r\}$.

PROPOSITION 3 *For* $\mathcal{A} \in \mathcal{C}(\mathfrak{X})$ *and* $r > 0$ *we have* $N_{\mathfrak{X}}[\mathcal{A}, r] \in \mathcal{C}(\mathfrak{X})$.

Proof. Let $X, X' \in N_{\mathfrak{X}}[\mathcal{A}, r]$ and $\lambda, \lambda' \geq 0$ with $\lambda + \lambda' = 1$, be arbitrary. Take $A, A' \in \mathcal{A}$ so that $h(X, A) = d(X, \mathcal{A})$ and $h(X', A') = d(X', \mathcal{A})$, Then, $d(\lambda X + \lambda' X', \mathcal{A}) \leq h(\lambda X + \lambda' X', \lambda A + \lambda' A') \leq \lambda h(X, A) + \lambda' h(X', A') \leq r$, and thus $N_{\mathfrak{X}}[A, r]$ is convex. The compactness of $N_{\mathfrak{X}}[A, r]$ being obvious, the statement follows. ■

4. Existence theorem

THEOREM 3 *Let* $\Phi : I \times \mathfrak{X} \to \mathcal{C}(\mathfrak{X})$ *satisfy the assumptions* (h_1), (h_2), *and let* $X_0 \in \mathfrak{X}$. *Then, the Cauchy problem (CP) has at least one solution* $X : I \to \mathfrak{X}$.

Proof. By Theorem 1, there is a sequence $\{\varphi_n\}$ of continuous maps $\varphi_n : I \times \mathfrak{X} \to \mathfrak{X}$ satisfying the properties (a_1) and (a_2). For each $n \in I\!N$ consider the following Volterra set integral equation

$$X(t) = X_0 + \int_0^t \varphi_n(s, X(s))\, ds. \tag{4}$$

By De Blasi, Iervolino (1969), this equation has a solution $X_n : I \to \mathfrak{X}$, i.e. X_n is continuous and satisfies (4) for each $t \in I$. Hence X_n is also solution of the Cauchy problem

$$DX(t) = \varphi_n(t, X(t)), \quad X(0) = X_0.$$

Consider now the sequence $\{X_n\}$. It is routine to see that the maps X_n are equicontinuous and take their values in some closed ball $B_{\mathfrak{X}}[\{0\}, R] \subset \mathfrak{X}$. As this ball is a compact subset of $\mathfrak{X}$, by Ascoli-Arzelà's theorem there exist a subsequence, say $\{X_n\}$, and a continuous map $X : I \to \mathfrak{X}$ such that

$$X_n \to X, \quad \text{uniformly on } I. \tag{5}$$

Since the sequence $\{X_n - X_0\}$, where

$$X_n(t) - X_0 = \int_0^t \varphi_n(s, X_n(s))\, ds, \quad t \in I,$$

converges uniformly to $X - X_0$, and $\varphi_n(t, X_n(t)) \subset B_{I\!R^k}[0, M]$, by Proposition 1, there exists a measurable map $U : I \to \mathfrak{X}$, with $U(t) \subset B_{I\!R^k}[0, M]$, $t \in I$, such that

$$X(t) = X_0 + \int_0^t U(s)\, ds, \quad \text{for every } t \in I. \tag{6}$$

Claim: The map $X : I \to \mathfrak{X}$ is a solution of the Cauchy Problem (CP).

For this, it suffices to show that

$$d(U(t), \Phi(t, X(t))) = 0, \quad t \in I, \ a.e. \tag{7}$$

Define $\lambda : I \to I\!R$ by $\lambda(t) = d(U(t), \Phi(t, X(t)))$. The map $t \to \Phi(t, X(t))$ is u.s.c., and thus measurable. Hence, by Proposition 2, λ is measurable. Arguing by contradiction, suppose that (7) is not true. Then, there exists $0 < \epsilon < 1$ such that the set $J' = \{t \in I |\, \lambda(t) \geq \epsilon\}$ has measure $m(J') > 0$.

By Pliš theorem, there exists a closed set $J \subset J'$ with $m(J) > 0$ such that U restricted to J is continuous.

Let $\tau \in J$, $0 < \tau < 1$, be a *density point* of J, that is

$$\lim_{\rho \to 0+} \frac{m(I_{\tau,\rho} \cap J)}{2\rho} = 1, \quad \text{where } I_{\tau,\rho} = [\tau - \rho, \tau + \rho].$$

Let $0 < \theta < \frac{\epsilon}{4}$. Since $\lambda(t) \geq \epsilon$ we have

$$B_{\mathfrak{X}}[U(\tau), \frac{\epsilon}{4} + \theta] \cap N_{\mathfrak{X}}[\Phi(\tau, X(\tau)), \frac{\epsilon}{4}] = \phi. \tag{8}$$

Now fix $\rho_0 > 0$, with $I_{\tau,\rho_0} \subset I$, small enough so that the following properties are satisfied:

$$U(t) \in B_{\mathfrak{X}}[U(\tau), \frac{\epsilon}{4}], \quad \text{for every } t \in I_{\tau,\rho_0} \cap J \tag{9}$$

$$\frac{m(I_{\tau,\rho} \backslash J)}{2\rho} < \frac{\theta}{2M}, \quad \text{for every } 0 < \rho < \rho_0. \tag{10}$$

Such a ρ_0 certainly exists, because U restricted to J is continuous and τ is a density point of J.

By hypothesis Φ is u.s.c. and thus there exists $0 < \sigma < \rho_0$ such that

$$e(\Phi(t, Z), \Phi(\tau, X(\tau))) < \frac{\epsilon}{8} \text{ for every } t \in I_{\tau,\sigma} \text{ and } Z \in B_{\mathfrak{X}}[X(\tau), \sigma]. \tag{11}$$

Since X is continuous, there is $0 < \delta < \frac{\sigma}{2}$ such that

$$h(X(t), X(\tau)) < \frac{\sigma}{4} \quad \text{for every } t \in I_{\tau,\delta}. \tag{12}$$

Moreover, by (5), $X_n \to X$ uniformly, and thus there exists a $m_0 \in I\!\!N$ such that

$$h(X_n(t), X(t)) < \frac{\sigma}{4} \quad \text{for every } t \in I_{\tau,\delta},\ n > m_0. \tag{13}$$

Combining (12) and (13) gives

$$h(X_n(t), X(\tau)) < \frac{\sigma}{2}, \text{ for every } t \in I_{\tau,\delta} \text{ and } n \geq m_0. \tag{14}$$

Fix $n_0 \geq m_0$ such that $\frac{1}{n_0} < \min\{\delta, \frac{\epsilon}{8}\}$. Let $n \geq n_0$ and $t \in I_{\tau,\delta}$ be arbitrary. By construction, $e(Gr\,\varphi_n, Gr\,\Phi) < \frac{1}{n}$ and thus, by Remark 1, corresponding to $(t, X_n(t)) \in I \times \mathfrak{X}$ there exist a $(t', X') \in I \times \mathfrak{X}$ and some $Y' \in \Phi(t', X')$ such that

$$|t' - t| < \frac{1}{n}, \quad h(X', X_n(t)) < \frac{1}{n}, \quad h(Y', \varphi_n(t, X_n(t))) < \frac{1}{n}. \tag{15}$$

We have, $t' \in I_{\tau,\sigma}$, and $X' \in B_{\mathfrak{X}}[X(\tau), \sigma]$. In fact,

$$|t' - \tau| \leq |t' - t| + |t - \tau| < \frac{1}{n} + \delta \leq \frac{1}{n_0} + \delta < 2\delta < \sigma,$$

while by virtue of (15) and (14), we have

$$h(X', X(\tau)) \leq h(X', X_n(t)) + h(X_n(t), X(\tau)) < \frac{1}{n} + \frac{\sigma}{2} < \delta + \frac{\sigma}{2} < \sigma.$$

Hence, by (11),

$$e(\Phi(t', X'), \Phi(\tau, X(\tau))) < \frac{\epsilon}{8}. \tag{16}$$

By virtue of (15) and (16), as $Y' \in \Phi(t', X')$, we have

$$\begin{aligned} d(\varphi_n(t, X_n(t)), \Phi(\tau, X(\tau))) &\leq h(\varphi_n(t, X_n(t)), Y') + d(Y', \Phi(t', X')) + \\ &\qquad e(\Phi(t', X'), \Phi(\tau, X(\tau))) \\ &< \frac{1}{n} + \frac{\varepsilon}{8} < \frac{\varepsilon}{4}, \end{aligned}$$

for $\frac{1}{n} \leq \frac{1}{n_0} < \frac{\varepsilon}{8}$.

Therefore,

$$\varphi_n(t, X_n(t)) \in N_{\mathfrak{X}}[\Phi(\tau, X(\tau)), \frac{\varepsilon}{4}] \text{ for every } t \in I_{\tau,\delta} \text{ and } n \geq n_0.$$

Thus, by Proposition 3, for all $n \geq n_0$, we have

$$\begin{aligned} \frac{X_n(\tau+\delta) - X_n(\tau-\delta)}{2\delta} &= \frac{1}{2\delta} \int_{I_{\tau,\delta}} \varphi_n(s, X_n(s))\, ds \\ &\in N_{\mathfrak{X}}[\Phi(\tau, X(\tau)), \frac{\varepsilon}{4}] \end{aligned}$$

from which, letting $n \to \infty$, it follows,

$$\frac{X(\tau+\delta)-X(\tau-\delta)}{2\delta} \in N_{\mathfrak{X}}[\Phi(\tau, X(\tau)), \frac{\varepsilon}{4}]. \tag{17}$$

On the other hand, as $\delta < \rho_0$, (9) implies

$$U(t) \in B_{\mathfrak{X}}[U(\tau), \frac{\varepsilon}{4}], \quad \text{for each } t \in I_{\tau,\delta} \cap J. \tag{18}$$

Now, by (6),

$$\begin{aligned} \frac{X(\tau+\delta)-X(\tau-\delta)}{2\delta} &= \frac{1}{2\delta}\int_{I_{\tau,\delta}} U(s)\,ds \\ &= \frac{1}{2\delta}\int_{I_{\tau,\delta}\cap J} U(s)\,ds + \frac{1}{2\delta}\int_{I_{\tau,\delta}\setminus J} U(s)\,ds. \end{aligned} \tag{19}$$

Denote by $\omega_1(\delta)$ and $\omega_2(\delta)$ the last two quantities. We have,

$$\begin{aligned} \omega_1(\delta) &= \frac{m(I_{\tau,\delta}\cap J)}{2\delta}\frac{1}{m(I_{\tau,\delta}\cap J)}\int_{I_{\tau,\delta}\cap J} U(s)\,ds \\ &\in \frac{m(I_{\tau,\delta}\cap J)}{2\delta} B_{\mathfrak{X}}[U(\tau), \frac{\varepsilon}{4}], \text{ by (18)} \\ &= B_{\mathfrak{X}}[\lambda(\delta)U(\tau), \lambda(\delta)\frac{\varepsilon}{4}], \end{aligned} \tag{20}$$

where $\lambda(\delta) = \frac{m(I_{\tau,\delta}\cap J)}{2\delta}$

Any $X \in \mathfrak{X}$ lying in the above ball satisfies

$$\begin{aligned} h(X, U(\tau)) &\le h(X, \lambda(\delta)U(\tau)) + h(\lambda(\delta)U(\tau), U(\tau)) \\ &\le \lambda(\delta)\frac{\varepsilon}{4} + (1-\lambda(\delta))M \\ &\le \frac{\varepsilon}{4} + (1-\lambda(\delta))M. \end{aligned}$$

Since $m(I_{\tau,\delta}\cap J) + m(I_{\tau,\delta}\setminus J) = 2\delta$, one has $1-\lambda(\delta) = \frac{m(I_{\tau,\delta}\setminus J)}{2\delta}$. From this and (10), as $\delta < \rho_0$, it follows that $(1-\lambda(\delta))M < \frac{\theta}{2}$. Therefore, $h(X, U(\tau)) < \frac{\varepsilon}{4} + \frac{\theta}{2}$, showing that $B_{\mathfrak{X}}[\lambda(\delta)U(\tau), \lambda(\delta)\frac{\varepsilon}{4}] \subset B_{\mathfrak{X}}[U(\tau), \frac{\varepsilon}{4} + \frac{\theta}{2}]$. Hence, by (20),

$$\omega_1(\delta) \in B_{\mathfrak{X}}[U(\tau), \frac{\varepsilon}{4} + \frac{\theta}{2}]. \tag{21}$$

Moreover,

$$\begin{aligned} h(\omega_2(\delta), \{0\}) &= h\left(\frac{1}{2\delta}\int_{I_{\tau,\delta}\setminus J} U(s)\,ds, \{0\}\right) \\ &\le \frac{1}{2\delta}\int_{I_{\tau,\delta}\setminus J} h(U(s), \{0\})\,ds \\ &\le M\frac{m(I_{\tau,\delta}\setminus J)}{2\delta} < \frac{\theta}{2}, \text{ by (10)}. \end{aligned} \tag{22}$$

By (19), in view of (21) and (22), it follows

$$\frac{X(\tau+\delta)-X(\tau-\delta)}{2\delta} \in B_{\mathfrak{X}}[U(\tau), \frac{\varepsilon}{4}+\theta]. \tag{23}$$

Since (17) and (23) contradict (8), the claim is true. Hence, $X : I \to \mathfrak{X}$ is a solution of the Cauchy problem (CP). This completes the proof. ■

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