An expansion for self-interacting random walks

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Abstract

We derive a perturbation expansion for general self-interacting random walks, where steps are made on the basis of the history of the path. Examples of models where this expansion applies are reinforced random walk, excited random walk, the true (weakly) self-avoiding walk, loop-erased random walk, and annealed random walk in random environment. In this paper we show that the expansion gives rise to useful formulae for the speed and variance of the random walk, when these quantities are known to exist. The results and formulae of this paper have been used elsewhere by the authors to prove monotonicity properties for the speed (in high dimensions) of excited random walk and related models, and certain models of random walk in random environment. We also derive a law of large numbers and central limit theorem (with explicit error terms) directly from this expansion, under strong assumptions on the expansion coefficients. The assumptions are shown to be satisfied by excited random walk in high dimensions with small excitation parameter, a model of reinforced random walk with underlying drift and small reinforcement parameter, and certain models of random walk in random environment under strong ellipticity conditions.

1 Introduction

Recently, many models of random walks with a certain self-interaction have been introduced. A few examples are self-reinforced random walks [11, 32, 34], excited random walks [3, 28, 29, 38, 39], true-self avoiding walks and loop-erased random walks. Proofs in these models often rely on martingale methods, or explicit comparisons to random walk properties. In some of the examples, laws of large numbers are derived. The difficulty is that the limiting parameters are rather implicit, so that it is hard to derive analytical properties of them. For example, it is quite reasonable to assume that the drift for excited random walk is monotone increasing in the excitement parameter for each $d \geq 2$, but a proof of this fact is currently missing. Similarly, it has not been proved that the speed for once-reinforced random walk on the tree is monotone decreasing in the reinforcement parameter (see [11]). See [33] for a survey of self-interacting random walks with reinforcement.

In the past decades, the lace expansion has proved to be an extremely useful technique to investigate a variety of models above their upper-critical dimension, where Gaussian limits are expected. Examples are self-avoiding walks above 4 dimensions [6, 14, 35, 36, 37], lattice trees above 8 dimensions [9, 10, 13, 24], the contact process above 4 dimensions [20, 21], oriented percolation above 4 dimensions [23, 30, 31], and percolation above 6 dimensions [12, 15, 16]. An essential ingredient in the proofs is the fact that the above models are *self-repellent*. There are many more models where a Gaussian limit is expected above a certain upper critical dimension, but using the lace expansion for these models is hard as they

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are not strictly self-repellent. In this paper, we perform a first step for a successful application of the lace expansion methodology, namely, we derive the expansion for general self-interacting random walks. The goal is to use this expansion for some of the simpler self-interacting stochastic processes available.

We will study a particular version of once-reinforced random walk, where the initial weights are such that the corresponding random walk has a *non-zero drift*. A similar situation was investigated in [11], where once-reinforced random walk was investigated on the tree. We expect that our method can be adapted to the tree setting to reprove some results in [11], when the reinforcement parameter is sufficiently small. We also study *excited random walk*, where the random walker has a drift in the direction of the first component each time when the walker visits a new site. It was shown that this process has ballistic behaviour when $d \geq 2$ in [3, 28, 29], while there is no ballistic behaviour in one dimension. For a third application of our method we study random walk in (partially) random environments, similar to those considered in [5].

In this paper we give a (self-contained) proof of a law of large numbers ($d \geq 6$) and a central limit theorem $d \geq 9$) when the excitation parameter is sufficiently small. We also derive a law of large numbers and central limit theorem for the once-reinforced random walk with drift when the reinforcement is sufficiently small compared to the drift. These results were completed in 2006. Since then, substantial progress has been made on these two models. Using renewal techniques, a strong law of large numbers and invariance principle has been proved for the excited random walk in dimensions $d \geq 2$ [4], while laws of large numbers and local central limit theorems are obtained for a large class of ballistic self-interacting random walks (including the reinforced random walk with drift in all dimensions) in [27]. We prove similar results for annealed random walks in partially random environment similar to those in [5] in the perturbative regime, with the difference being that the probability of taking a step in each coordinate may be random. We believe that our results for random walk in random environment are new.

The renewal techniques often give strong results (as described above), but currently do not provide much insight into how the results depend on the underlying parameters. As is done in this paper, our expansion can be used independently to prove (sometimes weaker) results in the perturbative regime. In doing so we obtain formulae and estimates of error terms for some of the relevant quantities of interest. This is one of the main advantages of our method, but as illustrated by recent applications (see Section 2.5) we see a *combination* of our expansion with renewal and ergodic methods to be highly informative.

2 The main results

We start by introducing some notation. A path ω is a sequence $\{\omega_i\}_{i=0}^{\infty}$ for which $\omega_i \in \mathbb{Z}^d$ for all $i \geq 0$. We obtain *random walk* when the random vector $\{\omega_{i+1} - \omega_i\}_{i=0}^{\infty}$ is an i.i.d. sequence. We let $\mathbb{P}$ be the law of a random walk law starting at the origin. We write $\vec{\omega}_n$ for the vector

$$\vec{\omega}_n = (\omega_0, \ldots, \omega_n), \tag{2.1}$$

that is, for the first n positions of the walk and its starting point. Let

$$D(x) = \mathbb{P}(\omega_1 = x) \tag{2.2}$$

be the random walk transition probability, so that

$$\mathbb{P}(\vec{\omega}_n = (x_0, x_1, \ldots, x_n)) = \prod_{i=0}^{n-1} D(x_{i+1} - x_i). \tag{2.3}$$

We restrict our attention to D with finite range L so that $\sum_{x:|x|>L} D(x) = 0$ and all moments of D exist. For self-interacting random walks, a similar expression to (2.3) is valid, but the term appearing in the product may depend on the history of the path. Let $\mathbb{Q}_{(x_0)}$ denote the law of a self-interacting random walk $\vec{\omega} = (\omega_0, \omega_1, \dots)$ started at $\omega_0 = x_0$, i.e.,

$$\mathbb{Q}_{(x_0)}(\vec{\omega}_n = (x_0, x_1, \dots, x_n)) = \prod_{i=0}^{n-1} p^{\vec{x}_i}(x_i, x_{i+1}), \tag{2.4}$$

where

$$p^{\vec{x}_i}(x_i, x_{i+1}) = \mathbb{Q}_{(x_0)}(\omega_{i+1} = x_{i+1} | \vec{\omega}_i = \vec{x}_i).$$

In other words, for a general path $\vec{x}_i$, we write $p^{\vec{x}_i}(x_i, x_{i+1})$ for the conditional probability that the walk steps from x_i to x_{i+1}, given the history of the entire path $\vec{x}_i = (x_0, \dots, x_i)$. It is crucial to our analysis that our self-interacting random walk law is translation invariant, i.e. for all $\vec{x}_n$,

$$\mathbb{Q}_{(x_0)}(\vec{\omega}_n = (x_0, x_1, \dots, x_n)) = \mathbb{Q}_{(o)}(\vec{\omega}_n = (o, x_1 - x_0, \dots, x_n - x_0)).$$

We henceforth write $\mathbb{Q} = \mathbb{Q}_{(o)}$ and drop the dependence on the starting point x_0 from the notation when the history $\vec{x}_n$ of the path is given, e.g. $\mathbb{Q}(\cdot | \vec{\omega}_n = \vec{x}_n) = \mathbb{Q}_{(x_0)}(\cdot | \vec{\omega}_n = \vec{x}_n)$.

The goal of this paper is to investigate the *two-point function*

$$c_n(x) = \mathbb{Q}(\omega_n = x). \tag{2.5}$$

In this paper, we will derive an expansion for the two-point function in full generality. However, for the analytical results we will focus on directed once-edge-reinforced random walks, excited random walks, and random walks in partially random environments. In Sections 2.1, 2.2, and 2.3 below, we will define the models and state the results.

2.1 Once edge-reinforced random walk with drift

In this section, we introduce an example of a once edge-reinforced random walk with drift. For a directed edge b, denote the number of times the edge b is traversed up to time t by

$$\ell_t(b) = \sum_{i=1}^{t} I_{\{(\omega_{i-1}, \omega_i) = b\}}, \tag{2.6}$$

where I_A denotes the indicator of the event A, and let $t \mapsto \beta_t$ be a sequence of $\mathbb{R}$-valued reinforcement parameters. We use $w_s(b)$ to denote the *weight of the edge* b at time s. The main assumption for our reinforced random walk is that $w_0(b)$ is translation invariant, and that

$$\sum_x x w_0(0, x) \neq 0. \tag{2.7}$$

Define $w_s(b)$ recursively by

$$w_t(b) = w_{t-1}(b) + I_{\{(\omega_{t-1}, \omega_t) = b\}} \beta_{\ell_t(b)}. \tag{2.8}$$

We define a directed version of edge-reinforced random walk (ERRW) by setting

$$p^{\vec{\omega}_i}(x_i, x_{i+1}) = \frac{w_i(x_i, x_{i+1})}{\sum_y w_i(x_i, y)}. \tag{2.9}$$

We will deal with directed *once-reinforced random walks*, where $\beta_t = \beta \delta_{t,1}$ is taken sufficiently small, however our results extend to directed *boundedly-reinforced random walks*, where we assume that

$$\beta = \sum_{t=0}^{\infty} |\beta_t| < \infty, \quad \text{is sufficiently small.} \tag{2.10}$$

The parameters β_s are allowed to be negative (provided $w_0(b)+\sum_{t=1}^{m}\beta_t$ remains bounded away from 0). Note that (2.7) implies that the random walk distribution arising for $\beta = 0$ has *non-zero drift.*

We denote by $\mathbb{Q}_\beta$ the distribution of the above once-reinforced random walk with drift, and we let $\mathbb{E}_\beta$ denote expectation with respect to $\mathbb{Q}_\beta$. We denote by $\mathrm{Var}_\beta(\omega_n)$ the covariance matrix of the random vector ω_n under the measure $\mathbb{Q}_\beta$. We also denote convergence in distribution by $\stackrel{d}{\Longrightarrow}$, convergence in probability under the law $\mathbb{P}$ by $\stackrel{\mathbb{P}}{\longrightarrow}$ and write $\mathcal{N}(0,\Sigma)$ for the multivariate normal distribution with mean the zero vector and covariance matrix Σ.

Theorem 2.1 (A CLT for finitely reinforced random walk with drift)**.** *Fix $d \geq 1$ and assume* (2.7)*. There exist $\beta_0 = \beta_0(d, w_0) > 0$, $\theta = \theta(\beta, w_0, d) \in [-1,1]^d$ and finite $\Sigma = \Sigma(\beta, w_0, d)$ such that, for all $\beta \leq \beta_0$,*
(a)

$$\mathbb{E}_\beta[\omega_n] = \theta n[1 + O(\frac{1}{n})]. \tag{2.11}$$

(b)

$$\mathrm{Var}_\beta(\omega_n) = \Sigma n[1 + O(\frac{1}{n})]. \tag{2.12}$$

(c) ω_n satisfies a central limit theorem under $\mathbb{Q}_\beta$, that is,

$$\frac{\omega_n - \theta n}{\sqrt{n}} \stackrel{d}{\Longrightarrow} \mathcal{N}(0,\Sigma). \tag{2.13}$$

As noted in the introduction, this result has since been strengthened in [27], although without error estimates. Parts of our methods apply to once-reinforced random walk where the initial weights induce no drift. However, we are currently unable to prove the bounds on the expansion coefficients, one of the crucial steps in the analysis. We shall comment on this issue in more detail in Section 6.5 below. In Section 3.3-3.4, we shall further give formulas for the speed and variance appearing in Theorem 2.1.

2.2 Excited random walk

In this section, we introduce excited random walk (ERW), which is the second model to which we shall apply our expansion method. It is defined for $\beta \in [0,1]$ by taking

$$p^{\vec{\omega}_i}(x_i, x_{i+1}) = p_0(x_{i+1} - x_i) I_{\{x_i \in \vec{\omega}_{i-1}\}} + p_\beta(x_{i+1} - x_i)\big[1 - I_{\{x_i \in \vec{\omega}_{i-1}\}}\big], \tag{2.14}$$

where $\{x_i \in \vec{\omega}_{i-1}\}$ denotes the event that $x_i = \omega_j$ for some $0 \leq j \leq i-1$, and where

$$p_0(x) = \frac{1}{2d} I_{\{|x|=1\}} \tag{2.15}$$

is the nearest-neighbour step distribution and

$$p_\beta(x) = \frac{1 + \beta e_1 \cdot x}{2d} I_{\{|x|=1\}}. \tag{2.16}$$

Here $e_1 = (1, 0, \ldots, 0)$ and $x \cdot y$ is the inner-product between x and y. In words, the random walker gets excited and has a positive drift in the direction of the first coordinate each time he/she visits a new site.

That ERW has a positive drift (in the sense of a lower bound) was established for ERW in $d \geq 4$ in [3], for $d = 3$ in [28], and for $d = 2$ in [29]. For $d = 1$, it is known that ERW is recurrent and diffusive (except the trivial case $\beta = 1$) [7]. Many generalisations of this model, described in terms of cookies, have also been studied (see for example [38], [1], [2]).

We denote by $\mathbb{Q}_\beta$ the distribution of the above excited random walk started at the origin, and we let $\mathbb{E}_\beta$ denote expectation with respect to $\mathbb{Q}_\beta$. We denote by $\mathrm{Var}_\beta(\omega_n)$ the covariance matrix of the random vector ω_n under the measure $\mathbb{Q}_\beta$.

Our main result for excited random walk is the following theorem:

Theorem 2.2 (A CLT for ERW above 8 dimensions)**.** *Fix $d > 8$. Then, there exists $\beta_0 = \beta_0(d) > 0$, $\theta = (\theta_1(\beta, d), 0, \dots, 0)$ and finite $\Sigma = \Sigma(\beta, d)$ such that, for all $\beta \leq \beta_0$,*
(a)

$$\mathbb{E}_\beta[\omega_n] = \theta n[1 + O(\frac{1}{n})]. \tag{2.17}$$

(b)

$$\mathrm{Var}_\beta(\omega_n) = \Sigma n[1 + O(\frac{\log n}{n^{1\wedge\frac{d-7}{2}}})]. \tag{2.18}$$

(c) ω_n satisfies a central limit theorem under $\mathbb{Q}_\beta$, that is,

$$\frac{\omega_n - \theta n}{\sqrt{n}} \overset{d}{\Longrightarrow} \mathcal{N}(0, \Sigma). \tag{2.19}$$

Unfortunately, our methods do not apply to general $d \geq 2$. However, when $d > 5$, we *can* prove a weak law of large numbers:

Theorem 2.3 (A LLN for ERW above 5 dimensions)**.** *Fix $d > 5$. Then, there exists $\beta_0 = \beta_0(d) > 0$ and $\theta = (\theta_1(\beta, d), 0, \dots, 0)$ such that, for all $\beta \leq \beta_0$,*
(a)

$$\mathbb{E}_\beta[\omega_n] = \theta n[1 + O(\frac{\log n}{n^{1\wedge\frac{d-5}{2}}})]. \tag{2.20}$$

(b) ω_n satisfies a law of large numbers under $\mathbb{Q}_\beta$, that is,

$$\frac{\omega_n}{n} \overset{\mathbb{Q}_\beta}{\longrightarrow} \theta. \tag{2.21}$$

As remarked in the introduction, these results have since been strengthened considerably in [4], and indeed a strong law of large numbers was already implicit in [38]. One of the key purposes of this paper is to obtain analytically tractable formulae for the coefficients $\theta(\beta, d)$ and $\Sigma(\beta, d)$ in the central limit theorem (see Section 3.2), allowing for a proof that $\beta \mapsto \theta(\beta, d)$ is monotonically increasing. This result has since been proved for $d \geq 9$ in [19], making crucial use of the methodology in this paper:

Theorem 2.1 ([19])**.** *For excited random walk in dimensions $d \geq 9$, the velocity θ_1 is a strictly increasing function of $\beta \in [0, 1]$.*

2.3 Random walk in a partially random environment

In this section, we introduce a model of random walk in (partially) random environment (RWpRE). The model we consider is a (nearest-neighbour, for simplicity) random walk in $\mathbb{Z}^d$, where $d = d_0 + d_1$ with $d_1 \geq 5$. The random environment has the property that the random walker observed only when stepping in the coordinates $d_0 + 1, \dots, d$, behaves as a simple random walk in d_1 dimensions. This is similar to the model studied in [5], but with two important differences. Firstly, our results will only apply in the perturbative regime, where the transition probabilities are sufficiently close to their expected values. Secondly we allow the probability of stepping in the $d_0 + 1, \dots, d$ coordinates to depend on the environment, provided that this probability is bounded away from 0, uniformly in the environment. This situation is not allowed in [5], so our results for this model can be seen as a non-trivial extension to [5] in the perturbative regime.

To be more precise about the model that we study, we require some additional notation. Let $d = d_0 + d_1 \geq 6$ with $d_1 \geq 5$. Let U_d be the set of unit vectors in $\mathbb{Z}^d$, and $\mathcal{P}(U_d)$ be the set of probability measures on U_d. Let $W. = \{W.(u)\}_{u\in U_d}$ denote an element of $\mathcal{P}(U_d)$ and let μ be a probability measure on $\mathcal{P}(U_d)$, satisfying the following:

(1) The weight assigned to U_{d_0} is not too large, i.e. there exists some $\delta > 0$ such that

$$\mu\Big(\sum_{u\in U_{d_0}} W.(u) \leq 1-\delta\Big) = 1.$$

(2) The weights assigned to $U_d \setminus U_{d_0}$ are "fair", i.e. for each $v \in U_d \setminus U_{d_0}$,

$$\mu\left(W.(v) = \frac{1-\sum_{u\in U_{d_0}} W.(u)}{2d_1}\right) = 1.$$

(3) The weights cannot vary too much, i.e. there exists some $\beta < 1$ such that for each $u \in U_d$,

$$\mu\left(|W.(u) - E_\mu[W.(u)]| < \beta\right) = 1, \tag{2.22}$$

where E_μ denotes expectation with respect to μ.

Let ν be the product measure on $\mathcal{P}(U_d)^{\mathbb{Z}^d}$ obtained from μ, i.e. under ν, $\{W_x\}_{x\in\mathbb{Z}^d}$ are independent with distribution μ. The RWpRE in environment $W = \{W_x\}_{x\in\mathbb{Z}^d}$ is the Markov chain $\{X_n\}_{n\geq 0}$ such that $P_W(X_0 = 0) = 1$, and $P_W(X_{n+1} = X_n + u|X_0,\dots,X_n) = W_{X_n}(u)$. The annealed RWpRE is the (non-Markovian) random walk with law $\mathbb{Q}$ obtained by averaging over all environments, i.e.

$$\mathbb{Q}(\vec{\omega}_n = \vec{x}_n) = \int P_W(\vec{X}_n = \vec{x}_n)d\nu.$$

The annealed transition probabilities are given by

$$p^{\vec{x}_i}(x_i, x_{i+1}) = \mathbb{Q}(\omega_{i+1} = x_{i+1}|\vec{\omega}_i = \vec{x}_i) = \mathbb{E}[W_{x_i}(x_{i+1}-x_i)|\vec{\omega}_i = \vec{x}_i]. \tag{2.23}$$

Our main result for RWpRE is the following theorem, in which $\mathbb{Q}_\beta$ denotes the above annealed law for a fixed (sufficiently small) choice of β.

Theorem 2.4 (A CLT for RWpRE for $d_1 > 7$)**.** *Fix $d_1 > 7$ and $d_0 \geq 1$. Then, for every $\delta > 0$ there exists $\beta_0 = \beta_0(d_1, d_0, \delta) > 0$, $\theta(\beta,\delta,d_1,d_0)$ and finite $\Sigma = \Sigma(\beta,\delta,d_1,d_0)$ such that, for all $\beta \leq \beta_0$,*
(a)

$$\mathbb{E}_\beta[\omega_n] = \theta n\big[1 + O(\frac{1}{n})\big]. \tag{2.24}$$

(b)

$$\mathrm{Var}_\beta(\omega_n) = \Sigma n\big[1 + O(\frac{\log n}{n^{1\wedge\frac{d_1-6}{2}}})\big]. \tag{2.25}$$

(c) ω_n satisfies a central limit theorem under $\mathbb{Q}_\beta$, that is,

$$\frac{\omega_n - \theta n}{\sqrt{n}} \stackrel{d}{\Longrightarrow} \mathcal{N}(0,\Sigma). \tag{2.26}$$

Theorem 2.5 (A LLN for RWpRE for $d_1 > 4$)**.** *Fix $d_1 > 4$ and $d_0 \geq 1$. Then for each δ, there exists $\beta_0 = \beta_0(d_1, d_0, \delta) > 0$ and $\theta = \theta(d_1, d_0, \delta)$ such that, for all $\beta \leq \beta_0$,*
(a)

$$\mathbb{E}_\beta[\omega_n] = \theta n[1 + O(\frac{\log n}{n^{1 \wedge \frac{d_1-4}{2}}})]. \tag{2.27}$$

(b) ω_n satisfies a law of large numbers under $\mathbb{Q}_\beta$, that is,

$$\frac{\omega_n}{n} \stackrel{\mathbb{Q}_\beta}{\longrightarrow} \theta. \tag{2.28}$$

In the above theorems, the β_0 arising from our analysis can be taken larger as δ increases.

Although results of a similar nature appear in [5] and some of the references therein, we believe that this is a new result. In particular, we do not assume that the random components of the environment are isotropic, nor that they have mean zero, nor that the random walker is transient in any particular direction. However, as in [5], our analysis relies heavily on the fact that simple random walk in d_1 dimensions is, loosely speaking, *very transient*.

2.4 Overview of the method

The main tool used is a *perturbation expansion for the two-point function.* Such an expansion is often called a *lace expansion*, and takes the form of a recurrence relation

$$c_{n+1}(x) = \sum_y D(y) c_n(x-y) + \sum_y \sum_{m=2}^{n+1} \pi_m(y) c_{n+1-m}(x-y) \tag{2.29}$$

for certain expansion coefficients $\{\pi_m\}_{m=2}^{\infty}$, and where

$$D(x) = p^o(o, x) \tag{2.30}$$

is the transition probability function for the first step. A recurrence relation such as (2.29) is derived for the oriented percolation and self-avoiding walk two-point functions, and plays an essential part in the proofs that these models are Gaussian above the upper-critical dimension. For self-avoiding walk, $c_n(x)$ equals the number of n-step self-avoiding walks starting at 0 and ending at x, and $\sum_x c_n(x)$ equals the total number of self-avoiding walks, which grows exponentially at a certain rate that needs to be determined in the course of the proof. For self-interacting random walks, $\sum_x c_n(x) = 1$. This essential difference gives rise to a difference in the strategy for proofs.

In any lace expansion analysis, there are three main steps. The first is the expansion in (2.29), which, for general self-interacting random walks, will be derived in Section 3. The second step is to derive bounds on the lace expansion coefficients. These bounds will be derived in Section 6. The final step is the analysis of the recurrence relation, using the bounds on the lace expansion coefficients. For this analysis, we will make use of induction. The inductive analysis in this paper is intended for the perturbative regime (sufficiently small β), and is similar to the one in [17], where a lace expansion was used to prove ballistic behaviour and a central limit theorem for general one-dimensional weakly self-avoiding walk models. In turn, this induction was inspired by the analyses in [18, 22].

In the induction argument, we shall make use of the characteristic function of the end-point of the n-step self-interacting random walk, which is the Fourier transform

$$\hat{c}_n(k) = \sum_{x \in \mathbb{Z}^d} e^{ik \cdot x} c_n(x). \tag{2.31}$$

Taking the Fourier transform of (2.29) yields

$$\hat{c}_{n+1}(k) = \hat{D}(k)\hat{c}_n(k) + \sum_{m=2}^{n+1} \hat{\pi}_m(k)\hat{c}_{n+1-m}(k) \tag{2.32}$$

We shall present two separate induction arguments. The first proves a law of large numbers as in Theorem 2.3 under relatively weak assumptions on the expansion coefficients, the second is a more involved induction argument proving the central limit theorem as in Theorems 2.1 and 2.2 under stronger assumptions on the expansion coefficients.

The remainder of the paper is organised as follows. In Section 3, we present the expansion for self-interacting random walks, which applies in the general context described in Section 2. We also establish the formulae for the limiting speed and variance of the endpoint of the walk, assuming that these quantities exist. In Sections 4 and 5, we describe the induction arguments for the law of large numbers and central limit theorem respectively. In Section 6, we prove the bounds on the lace expansion coefficients for the two models under consideration. In Section 3.4, we prove the formula for the variance stated in Theorem 3.2 in Section 3.

2.5 Recent applications of this method

In this paper we have concentrated on deriving the expansion and on obtaining laws of large numbers and central limit theorems under strong conditions on the expansion coefficients. However in Sections 3.3 and 3.4 we also obtain formulae for the speed and variance, when these quantities are known to exist, under much weaker conditions on the expansion coefficients.

The speed of excited random walk is known to exist in all dimensions, e.g. see [4] and the results of this paper give a formula for that speed. This formula is shown in [19] to be monotone increasing in the excitation parameter in dimensions $d \geq 9$. Multi-excited random walks in high dimensions, where there can be infinitely many cookies at each site, with positive or negative drifts are studied in [25]. A result of [5] using cut-times and ergodicity shows that the speed of this model exists in high dimensions. A formula for the annealed speed is then given by the results of this paper and it is shown in [25] that in high dimensions the velocity is positive if the first cookie drift is sufficiently positive and is continuous and monotone in certain parameters. It is then possible to give examples of excited random walks in non-trivial cookie environments that have zero speed. In [26], certain models of random walk in i.i.d. random environment (where there is a transient random walk component and where at each site either the left or right step is not available) are studied. In these models the existence of the speed is given by an extension of the LLN obtained in [5], a formula is provided by this paper, and it is possible to prove monotonicity of the speed as a function of the probability p that the right step is available at the origin.

3 The expansion for self-interacting random walks

In this section, we perform and discuss the expansion for interacting random walks. In Section 3.1, we derive the expansion in (2.29), in Section 3.2, we discuss the consequences of our expansion, and in Sections 3.3 and 3.4, respectively, we identify the speed and variance from our expansion formula, assuming that they exist and that the expansion formulae converge.

3.1 Derivation of the expansion

Before we can start to prove (2.29), we need some more notation. We will make use of the convolution of functions, which is defined for absolutely summable functions f, g on $\mathbb{Z}^d$ by

$$(f * g)(x) = \sum_y f(y)g(x-y), \tag{3.1}$$

so that we can rewrite (2.29) as

$$c_{n+1}(x) = (D * c_n)(x) + \sum_{m=2}^{n+1} (\pi_m * c_{n+1-m})(x). \tag{3.2}$$

If $\vec{\eta}$ and $\vec{x}$ are two paths of length at least j and m respectively and such that $\eta_j = x_0$, then the concatenation $\vec{\eta}_j \circ \vec{x}_m$ is defined by

$$(\vec{\eta}_j \circ \vec{x}_m)_i = \begin{cases} \eta_i & \text{when } 0 \le i \le j, \\ x_{i-j} & \text{when } j \le i \le m+j. \end{cases} \tag{3.3}$$

Given $\vec{\eta}_m$, we define a probability measure $\mathbb{Q}^{\vec{\eta}_m}$ on walk paths starting from η_m, by specifying its value on particular cylinder sets (in a consistent manner) as follows

$$\mathbb{Q}^{\vec{\eta}_m}(\vec{\omega}_n = (x_0, x_1, \ldots, x_n)) \equiv \prod_{i=0}^{n-1} p^{\vec{\eta}_m \circ \vec{x}_i}(x_i, x_{i+1}), \tag{3.4}$$

and extending the measure to all finite-dimensional cylinder sets in the natural (consistent) way. We write $\mathbb{E}^{\vec{\eta}_m}$ for the expected value with respect to $\mathbb{Q}^{\vec{\eta}_m}$, and define

$$c_n^{\vec{\eta}_m}(\eta_m, x) = \mathbb{Q}^{\vec{\eta}_m}(\omega_n = x). \tag{3.5}$$

Any path of length $n+1$ is a path of length 1 concatenated with a path of length n, so that, in terms of the above notation, we can use (2.30) to rewrite

$$c_{n+1}(x) = \sum_{\vec{\omega}_1^{(0)}} D(\omega_1^{(0)}) \sum_{\vec{\omega}_n^{(1)}:\omega_1^{(0)} \to x} \prod_{i=0}^{n-1} p^{\vec{\omega}_1^{(0)} \circ \vec{\omega}_i^{(1)}}(\omega_i^{(1)}, \omega_{i+1}^{(1)}). \tag{3.6}$$

If we had $p^{\vec{\omega}_1^{(0)} \circ \vec{\omega}_i^{(1)}} = p^{\vec{\omega}_i^{(1)}}$ for all $\vec{\omega}_1^{(0)} \circ \vec{\omega}_i^{(1)}$, then we would be back in the random walk case, since we would arrive at

$$c_{n+1}(x) = \sum_{\vec{\omega}_1^{(0)}} D(\omega_1^{(0)}) \sum_{\vec{\omega}_n^{(1)}:\omega_1^{(0)} \to x} \prod_{i=0}^{n-1} p^{\vec{\omega}_i^{(1)}}(\omega_i^{(1)}, \omega_{i+1}^{(1)}) = (D * c_n)(x). \tag{3.7}$$

For interacting random walks, $p^{\vec{\omega}_1^{(0)} \circ \vec{\omega}_i^{(1)}}$ does not equal $p^{\vec{\omega}_i^{(1)}}$ in general, and we are left to deal with the difference between the two. For given $\vec{\eta}_m$ and $\vec{x}_i$ we can write

$$p^{\vec{\eta}_m \circ \vec{x}_i}(x_i, x_{i+1}) = p^{\vec{x}_i}(x_i, x_{i+1}) + (p^{\vec{\eta}_m \circ \vec{x}_i} - p^{\vec{x}_i})(x_i, x_{i+1}). \tag{3.8}$$

With this substitution, we have that

$$\prod_{i=0}^{n-1} p^{\vec{\omega}_1^{(0)} \circ \vec{\omega}_i^{(1)}}(\omega_i^{(1)}, \omega_{i+1}^{(1)}) = \prod_{i=0}^{n-1} \left[p^{\vec{\omega}_i^{(1)}}(\omega_i^{(1)}, \omega_{i+1}^{(1)}) + (p^{\vec{\omega}_1^{(0)} \circ \vec{\omega}_i^{(1)}}(\omega_i^{(1)}, \omega_{i+1}^{(1)}) - p^{\vec{\omega}_i^{(1)}}(\omega_i^{(1)}, \omega_{i+1}^{(1)}))\right]. \tag{3.9}$$

In (3.9), the first term has 'forgotten' the first step, while the second term makes up for this mistake. We would like to expand out the product in (3.9). Note that for all $\{a_i\}_{i=0}^{n-1}$ and $\{b_i\}_{i=0}^{n-1}$,

$$\prod_{i=0}^{n-1}(a_i+b_i)=\prod_{i=0}^{n-1}a_i+\sum_{j=0}^{n-1}\Big(\prod_{i=0}^{j-1}(a_i+b_i)\Big)b_j\Big(\prod_{i=j+1}^{n-1}a_i\Big), \tag{3.10}$$

where the empty products arising in $\prod_{i=0}^{j-1}(a_i+b_i)$ when $j=0$ and $\prod_{i=j+1}^{n-1}a_i$ when $j=n-1$, are defined to be equal to 1. Applying this to (3.6) with

$$a_i=p^{\vec{\omega}_i^{(1)}}(\omega_i^{(1)},\omega_{i+1}^{(1)}),\qquad b_i=(p^{\vec{\omega}_1^{(0)}\circ\vec{\omega}_i^{(1)}}-p^{\vec{\omega}_i^{(1)}})(\omega_i^{(1)},\omega_{i+1}^{(1)}),$$

we arrive at

$$\begin{aligned}c_{n+1}(x)=&\sum_{\vec{\omega}_1^{(0)}}D(\omega_1^{(0)})\sum_{\vec{\omega}_n^{(1)}:\omega_1^{(0)}\to x}\prod_{i=0}^{n-1}p^{\vec{\omega}_i^{(1)}}(\omega_i^{(1)},\omega_{i+1}^{(1)})\\&+\sum_{j=0}^{n-1}\sum_{\vec{\omega}_1^{(0)}}D(\omega_1^{(0)})\sum_{\vec{\omega}_n^{(1)}:\omega_1^{(0)}\to x}\Big[\prod_{i=0}^{j-1}p^{\vec{\omega}_1^{(0)}\circ\vec{\omega}_i^{(1)}}(\omega_i^{(1)},\omega_{i+1}^{(1)})\Big](p^{\vec{\omega}_1^{(0)}\circ\vec{\omega}_j^{(1)}}-p^{\vec{\omega}_j^{(1)}})(\omega_j^{(1)},\omega_{j+1}^{(1)})\\&\times\Big[\prod_{i=j+1}^{n-1}p^{\vec{\omega}_i^{(1)}}(\omega_i^{(1)},\omega_{i+1}^{(1)})\Big].\end{aligned}\tag{3.11}$$

The first term equals $(D*c_n)(x)$ by (3.7). To rewrite the second term, we need some more notation. We abbreviate

$$\Delta_{j+1}^{(1)}=(p^{\vec{\omega}_1^{(0)}\circ\vec{\omega}_j^{(1)}}-p^{\vec{\omega}_j^{(1)}})(\omega_j^{(1)},\omega_{j+1}^{(1)}), \tag{3.12}$$

so that (3.11) becomes

$$\begin{aligned}c_{n+1}(x)&=(D*c_n)(x)+\sum_{j=0}^{n-1}\sum_{\vec{\omega}_1^{(0)}}D(\omega_1^{(0)})\sum_{\vec{\omega}_n^{(1)}:\omega_1^{(0)}\to x}\Big[\prod_{i=0}^{j-1}p^{\vec{\omega}_1^{(0)}\circ\vec{\omega}_i^{(1)}}(\omega_i^{(1)},\omega_{i+1}^{(1)})\Big]\Delta_{j+1}^{(1)}\Big[\prod_{i=j+1}^{n-1}p^{\vec{\omega}_i^{(1)}}(\omega_i^{(1)},\omega_{i+1}^{(1)})\Big]\\&=(D*c_n)(x)+\sum_{j=0}^{n-1}\sum_{\vec{\omega}_1^{(0)}}D(\omega_1^{(0)})\sum_{\vec{\omega}_{j+1}^{(1)}:\omega_0^{(1)}=\omega_1^{(0)}}\Big[\prod_{i=0}^{j-1}p^{\vec{\omega}_1^{(0)}\circ\vec{\omega}_i^{(1)}}(\omega_i^{(1)},\omega_{i+1}^{(1)})\Big]\Delta_{j+1}^{(1)}\\&\qquad\times\sum_{(\omega_{j+2}^{(1)},\dots,\omega_n^{(1)}):\omega_n^{(1)}=x}\Big[\prod_{i=j+1}^{n-1}p^{\vec{\omega}_i^{(1)}}(\omega_i^{(1)},\omega_{i+1}^{(1)})\Big].\end{aligned}\tag{3.13}$$

From (3.5), we have that

$$\sum_{(\omega_{j+2}^{(1)},\dots,\omega_n^{(1)}):\omega_n^{(1)}=x}\Big[\prod_{i=j+1}^{n-1}p^{\vec{\omega}_i^{(1)}}(\omega_i^{(1)},\omega_{i+1}^{(1)})\Big]=c_{n-j-1}^{\vec{\omega}_{j+1}^{(1)}}(\omega_{j+1}^{(1)},x). \tag{3.14}$$

Therefore, (3.13) is equal to

$$c_{n+1}(x)=(D*c_n)(x)+\sum_{j=0}^{n-1}\sum_{\vec{\omega}_1^{(0)}}D(\omega_1^{(0)})\sum_{\vec{\omega}_{j+1}^{(1)}}\mathbb{Q}^{\vec{\omega}_1^{(0)}}(\vec{\omega}_j=\vec{\omega}_j^{(1)})\Delta_{j+1}^{(1)}\,c_{n-j-1}^{\vec{\omega}_{j+1}^{(1)}}(\omega_{j+1}^{(1)},x) \tag{3.15}$$

For the second step of the expansion, we note that a type of two-point function $c^{\vec{\omega}^{(1)}_{j+1}}_{n-j-1}(\omega^{(1)}_{j+1}, x)$ appears on the right side of (3.15). The second step of the expansion involves expanding out the dependence of this two-point function on the history $\vec{\omega}^{(1)}_{j+1}$. Given $\vec{\omega}^{(1)}_{j+1}$ we write

$$c^{\vec{\omega}^{(1)}_{j+1}}_{n-j-1}(\omega^{(1)}_{j+1}, x) = c_{n-j-1}(\omega^{(1)}_{j+1}, x) + \left(c^{\vec{\omega}^{(1)}_{j+1}}_{n-j-1}(\omega^{(1)}_{j+1}, x) - c_{n-j-1}(\omega^{(1)}_{j+1}, x) \right). \tag{3.16}$$

The contribution to (3.15) from the first term on the right of (3.16) is

$$\sum_{j=0}^{n-1} \sum_{y} \left[\sum_{\vec{\omega}^{(0)}_1} D(\omega^{(0)}_1) \sum_{\vec{\omega}^{(1)}_{j+1}} \mathbb{Q}^{\vec{\omega}^{(0)}_1}(\vec{\omega}_j = \vec{\omega}^{(1)}_j) \Delta^{(1)}_{j+1} I_{\{\omega^{(1)}_{j+1} = y\}} \right] c_{n-j-1}(x-y) \equiv \sum_{m=2}^{n+1} \sum_{y} \pi^{(1)}_m(y) c_{n+1-m}(x-y), \tag{3.17}$$

where, for $m \geq 2$,

$$\pi^{(1)}_m(y) = \sum_{\vec{\omega}^{(0)}_1} D(\omega^{(0)}_1) \sum_{\vec{\omega}^{(1)}_{m-1}} \mathbb{Q}^{\vec{\omega}^{(0)}_1}(\vec{\omega}_{m-2} = \vec{\omega}^{(1)}_{m-2}) \Delta^{(1)}_{m-1} I_{\{\omega^{(1)}_{m-1} = y\}} \tag{3.18}$$

To investigate the contribution to (3.15) from the term in brackets on the right of (3.16), we consider the difference between $c^{\vec{\eta}_m}_n(\eta_m, x)$ and $c_n(\eta_m, x)$ for general $\vec{\eta}_m$, n and x. We first write

$$c^{\vec{\eta}_m}_n(\eta_m, x) = \sum_{\vec{\omega}^*_n : \eta_m \to x} \prod_{i=0}^{n-1} p^{\vec{\eta}_m \circ \vec{\omega}^*_i}(\omega^*_i, \omega^*_{i+1}), \tag{3.19}$$

and then use (3.8) and (3.10) to end up with

$$c^{\vec{\eta}_m}_n(\eta_m, x) = c_n(\eta_m, x) + \sum_{j=0}^{n-1} \sum_{\vec{\omega}^*_n : \eta_m \to x} \Big[\prod_{i=0}^{j-1} p^{\vec{\eta}_m \circ \vec{\omega}^*_i}(\omega^*_i, \omega^*_{i+1}) \Big] (p^{\vec{\eta}_m \circ \vec{\omega}^*_j} - p^{\vec{\omega}^*_j})(\omega^*_j, \omega^*_{j+1}) \prod_{i=j+1}^{n-1} p^{\vec{\omega}^*_i}(\omega^*_i, \omega^*_{i+1}) \Big]. \tag{3.20}$$

Therefore, similarly to (3.13)–(3.15), we obtain

$$c^{\vec{\eta}_m}_n(\eta_m, x) = c_n(\eta_m, x) + \sum_{j=0}^{n-1} \sum_{\vec{\omega}^*_{j+1}} \mathbb{Q}^{\vec{\eta}_m}(\vec{\omega}_j = \vec{\omega}^*_j) \Delta^*_{j+1} c^{\vec{\omega}^*_{j+1}}_{n-j-1}(\omega^*_{j+1}, x). \tag{3.21}$$

In (3.21), the first term is a regular two-point function, i.e., it does not depend on the history $\vec{\eta}_m$. In the correction term a history-dependent two-point function $c^{\vec{\omega}^*_{j+1}}_{n-j-1}$ appears to which we can iteratively use (3.21). Thus, with $m = j+2$,

$$\begin{aligned} c_{n+1}(x) =& (D * c_n)(x) + \sum_{m=2}^{n+1} (\pi^{(1)}_m * c_{n-m+1})(x) \qquad (3.22) \\ &+ \sum_{j_1, j_2} \sum_{\vec{\omega}^{(0)}_1} D(\omega^{(0)}_1) \sum_{\vec{\omega}^{(1)}_{j_1+1}} \sum_{\vec{\omega}^{(2)}_{j_2+1}} \mathbb{Q}^{\vec{\omega}^{(0)}_1}(\vec{\omega}_{j_1} = \vec{\omega}^{(1)}_{j_1}) \Delta^{(1)}_{j_1+1} \mathbb{Q}^{\vec{\omega}^{(1)}_{j_1+1}}(\vec{\omega}_{j_2} = \vec{\omega}^{(2)}_{j_2}) \Delta^{(2)}_{j_2+1} c^{\vec{\omega}^{(2)}_{j_2+1}}_{n-j_1-j_2-2}(\omega^{(2)}_{j_2+1}, x), \end{aligned}$$

where we write, for $N \geq 1$,

$$\Delta^{(N)}_{j_N+1} = (p^{\vec{\omega}^{(N-1)}_{j_{N-1}+1} \circ \vec{\omega}^{(N)}_{j_N}} - p^{\vec{\omega}^{(N)}_{j_N}})(\omega^{(N)}_{j_N}, \omega^{(N)}_{j_N+1}), \tag{3.23}$$

with $j_0 \equiv 0$.

For $N \geq 1$, we let $\mathcal{A}_{m,N} = \{\vec{j} \in \mathbb{Z}_+^N : j_1 + \cdots + j_N = m - N - 1\}$ and further define

$$\pi_m^{(N)}(y) = \sum_{\vec{j}\in\mathcal{A}_{m,N}} \sum_{\vec{\omega}_1^{(0)}} \sum_{\vec{\omega}_{j_1+1}^{(1)}} \cdots \sum_{\vec{\omega}_{j_N+1}^{(N)}} I_{\{\omega_{j_N+1}^{(N)}=y\}} D(\omega_1^{(0)}) \prod_{n=1}^{N} \Delta_{j_n+1}^{(n)} \prod_{i_n=0}^{j_n-1} p^{\vec{\omega}_{j_{n-1}+1}^{(n-1)}\circ\vec{\omega}_{i_n}^{(n)}} \left(\omega_{i_n}^{(n)}, \omega_{i_n+1}^{(n)}\right) \tag{3.24}$$

$$= \sum_{\vec{j}\in\mathcal{A}_{m,N}} \sum_{\vec{\omega}_1^{(0)}} \sum_{\vec{\omega}_{j_1+1}^{(1)}} \cdots \sum_{\vec{\omega}_{j_N+1}^{(N)}} I_{\{\omega_{j_N+1}^{(N)}=y\}} D(\omega_1^{(0)}) \prod_{n=1}^{N} \Delta_{j_n+1}^{(n)} \mathbb{Q}^{\vec{\omega}_{j_{n-1}+1}^{(n-1)}}(\vec{\omega}_{j_n} = \vec{\omega}_{j_n}^{(n)}).$$

which is zero when $N + 1 > m$. Note that (3.24) reduces to (3.18) in the case $N = 1$. Then define

$$\pi_m(y) = \sum_{N=1}^{\infty} \pi_m^{(N)}(y). \tag{3.25}$$

We emphasize that, conditionally on $\vec{\omega}_{j_M+1}^{(M)}$, the probability measure $\mathbb{Q}_{M+1}^{\vec{\omega}_{j_M+1}^{(M)}}$ is the law of $\vec{\omega}_{j_{M+1}+1}^{(M+1)}$, i.e., that $\vec{\omega}_{j_M+1}^{(M)}$ acts as the history for $\vec{\omega}_{j_{M+1}+1}^{(M+1)}$.

Equation (2.29) follows by iteratively replacing the two-point function in (3.21) by using the equality (3.21), until the second term on the right of (3.21) vanishes. This must happen when $N = n + 1$. This completes the derivation of the expansion.

3.2 Discussion of the expansion

In this section, we discuss the consequences of the expansion in (2.29).

The lace expansion coefficients. The lace expansion coefficients involve the factors

$$\Delta_{j_N+1}^{(N)} = (p^{\vec{\omega}_{j_{N-1}+1}^{(N-1)}\circ\vec{\omega}_{j_N}^{(N)}} - p^{\vec{\omega}_{j_N}^{(N)}})(\omega_{j_N}^{(N)}, \omega_{j_N+1}^{(N)}) \tag{3.26}$$

in (3.23). This difference is identically zero when the histories $\vec{\omega}_{j_{N-1}+1}^{(N-1)} \circ \vec{\omega}_{j_N}^{(N)}$ and $\vec{\omega}_{j_N}^{(N)}$ give the same transition probabilities to go from $\omega_{j_N}^{(N)}$ to $\omega_{j_N+1}^{(N)}$. For excited random walk, $\Delta_{j_N+1}^{(N)}$ is non-zero precisely when $\omega_{j_N}^{(N)}$ has already been visited by $\vec{\omega}_{j_{N-1}+1}^{(N-1)}$, but not by $\vec{\omega}_{j_N-1}^{(N)}$, so that

$$\begin{aligned} |\Delta_{j_N+1}^{(N)}| \leq& |\Delta_{j_N+1}^{(N)}| I_{\{\omega_{j_N}^{(N)} \in \vec{\omega}_{j_{N-1}+1}^{(N-1)}\}} I_{\{\omega_{j_N}^{(N)} \notin \vec{\omega}_{j_N-1}^{(N)}\}} \\ \leq& C\beta I_{\{\omega_{j_N}^{(N)} \in \vec{\omega}_{j_{N-1}+1}^{(N-1)}\}} I_{\{\omega_{j_N}^{(N)} \notin \vec{\omega}_{j_N-1}^{(N)}\}} \leq C\beta I_{\{\omega_{j_N}^{(N)} \in \vec{\omega}_{j_{N-1}}^{(N-1)}\}}. \end{aligned} \tag{3.27}$$

For once-edge-reinforced random walk, the difference (3.23) is nonzero exactly when the vertex $\omega_{j_N}^{(N)}$ has already been visited by $\vec{\omega}_{j_{N-1}+1}^{(N-1)}$ via an edge that was not traversed by $\vec{\omega}_{j_N}^{(N)}$. Therefore, we also have for once-edge-reinforced random walk that

$$|\Delta_{j_N+1}^{(N)}| \leq C\beta I_{\{\omega_{j_N}^{(N)} \in \vec{\omega}_{j_{N-1}}^{(N-1)}\}}. \tag{3.28}$$

For RWpRE, a similar bound holds as follows. From (2.23),

$$\Delta_{j_N+1}^{(N)} = \mathbb{E}[W_{\omega_{j_N}^{(N)}}(\omega_{j_N+1}^{(N)} - \omega_{j_N}^{(N)}) | \vec{\omega}_{j_{N-1}+1+j_N} = \vec{\omega}_{j_{N-1}+1}^{(N-1)} \circ \vec{\omega}_{j_N}^{(N)}] - \mathbb{E}[W_{\omega_{j_N}^{(N)}}(\omega_{j_N+1}^{(N)} - \omega_{j_N}^{(N)}) | \vec{\omega}_{j_N} = \vec{\omega}_{j_N}^{(N)}].$$

Figure 1: The diagrams for $\pi_m^{(N)}$, $N = 1, \ldots, 5$, arising from the expansion and the bound (3.28) for both models. The subwalks (indicated by different shades) in the diagrams have the previous subwalk as their history. An intersection of two subwalks and a small factor β appears at each vertex.

By definition the random environment is site-wise independent, so the only information about $W_{\omega_{j_N}^{(N)}}$ contained in the history of the path is in the departures from the site $\omega_{j_N}^{(N)}$. Trivially every departure from $\omega_{j_N}^{(N)}$ by $\vec{\omega}_{j_N}^{(N)}$ is also a departure from $\omega_{j_N}^{(N)}$ by $\vec{\omega}_{j_{N-1}+1}^{(N-1)} \circ \vec{\omega}_{j_N}^{(N)}$, and any additional departures from this site by $\vec{\omega}_{j_{N-1}+1}^{(N-1)} \circ \vec{\omega}_{j_N}^{(N)}$ are actually departures from $\omega_{j_N}^{(N)}$ by $\vec{\omega}_{j_{N-1}}^{(N-1)}$. Thus $\Delta_{j_N+1}^{(N)}$ is non-zero only if $\omega_{j_N}^{(N)} \in \vec{\omega}_{j_{N-1}}^{(N-1)}$. It then follows immediately from (2.22) that

$$|\Delta_{j_N+1}^{(N)}| \leq 2\beta I_{\{\omega_{j_N}^{(N)} \in \vec{\omega}_{j_{N-1}}^{(N-1)}\}}. \tag{3.29}$$

We conclude that for all models under consideration, each factor $|\Delta_{j_N+1}^{(N)}|$:

1. enforces an intersection between the path and its previous history;
2. gives rise to a factor β, making $\pi_m^{(N)}(y)$ small when β is sufficiently small and N is large.

The quantities $\pi_m^{(N)}(y)$ combined with the bound (3.28) for both models, can be represented by diagrams of the form displayed in Figure 1 for $N = 1, \ldots, 5$. The first step is special, as it has no history. Thereafter, each subwalk $\vec{\omega}_{j_i+1}^{(i)}$ (indicated by shading in Figure 1) has the previous subwalk $\vec{\omega}_{j_{i-1}+1}^{(i-1)}$ as its history. The apparent similarity with the self-avoiding walk diagrams (see for example [14]) is natural due to the intersections enforced by the factors $\Delta_{j_i+1}^{(i)}$ as described above. A small factor β arises from each intersection (represented by vertices in Figure 1), and the number of intersections increases with the complexity of the diagram.

The speed and variance. By convention our vectors are considered to be column vectors. Thus if $\theta \in \mathbb{R}^d$, then $\theta\theta^t$ is a $d \times d$ matrix with real entries.

The limiting speed $\theta = \theta(\beta, d)$ and covariance matrix $\Sigma = \Sigma(\beta, d)$ appearing in Theorems 2.1–2.3 are given by

$$\theta(\beta, d) = \theta_\varnothing - i \sum_{m=2}^{\infty} \nabla \hat{\pi}_m(0), \tag{3.30}$$

$$\Sigma(\beta, d) = \Sigma_\varnothing - \theta\theta^t - \sum_{m=2}^{\infty} \nabla^2 \Big[e^{-i\theta \cdot k(m-1)} \hat{\pi}_m(k) \Big]_{k=0}, \tag{3.31}$$

where $\theta_\varnothing$ is the expected drift of the transition probability $D = p^o$, i.e.,

$$\theta_\varnothing = \sum_{x\in\mathbb{Z}^d} xD(x), \tag{3.32}$$

while $\Sigma_\varnothing$ is the covariance matrix of $D = p^o$ given by

$$(\Sigma_\varnothing)_{i,j} = \sum_{x\in\mathbb{Z}^d} x_i x_j D(x), \tag{3.33}$$

and $\nabla f(k)$ is the vector of derivatives of $k \mapsto f(k)$, while $\nabla^2 f(k)$ is the matrix consisting of the double derivatives of $k \mapsto f(k)$.

These formulas can be heuristically derived from the recurrence relation (2.29). Indeed, take the Fourier transform to obtain

$$\hat{c}_{n+1}(k) = \hat{D}(k)\hat{c}_n(k) + \sum_{m=2}^{n+1} \hat{\pi}_m(k)\hat{c}_{n+1-m}(k). \tag{3.34}$$

Now replace $\hat{c}_l(k)$ throughout the recurrence relation by $e^{i\theta\cdot kl-\frac{1}{2}k^t\Sigma kl}$, in accordance with Theorem 2.1(c)–2.2(c). Then, dividing by $e^{i\theta\cdot kn-\frac{1}{2}k^t\Sigma kn}$, we obtain

$$e^{i\theta\cdot k-\frac{1}{2}k^t\Sigma k} \approx \hat{D}(k) + \sum_{m=2}^{n+1} \hat{\pi}_m(k)e^{-i\theta\cdot k(m-1)+\frac{1}{2}k^t\Sigma k(m-1)}. \tag{3.35}$$

Expanding to linear order in k yields (3.30) and expanding to second order in k yields (3.31), when we note that Σ (as defined in (2.12) and (2.18)) must be symmetric, and

$$\hat{\pi}_m(0) = 0 \quad \text{and} \quad \hat{D}(k) = 1 + ik\cdot\theta_\varnothing - \frac{1}{2}k^t\Sigma_\varnothing k + O(|k|^3). \tag{3.36}$$

The results in this paper, as well as the proofs, follow part of the ideas in [17], where it was shown that certain weakly self-avoiding walk models in $d = 1$ behave ballistically.

3.3 The formula for the speed

In this section, we show that, when the speed is proved elsewhere to exist, *and* our formula for the speed in (3.30) converges, then in fact (3.30) identifies the speed. For example, for ERW in dimensions $d = 2, \dots, 5$, where Theorem 2.3 does not apply, it is known (e.g. [4]) that the speed exists almost surely.

Theorem 3.1 (The speed formula)**.** *If* $\lim_{n\to\infty}\sum_{m=2}^n\sum_x x\pi_m(x)$ *exists and* $n^{-1}\omega_n \xrightarrow{\mathbb{Q}} \theta$*, then*

$$\theta = \sum_x xp^o(o,x) + \sum_{m=2}^{\infty}\sum_x x\pi_m(x). \tag{3.37}$$

Proof. Multiplying (2.29) by $x = y+(x-y)$, summing, and using the facts that $\sum_x c_n(x) = \sum_x p^o(o,x) = 1$ and $\sum_x \pi_m(x) = 0$, we obtain

$$\sum_x xc_{n+1}(x) = \sum_y yp^o(o,y) + \sum_x xc_n(x) + \sum_{m=2}^{n+1}\sum_y y\pi_m(y). \tag{3.38}$$

Now $\sum_x x c_n(x) = \mathbb{E}[\omega_n]$, so rearranging (3.38) we obtain

$$\mathbb{E}[\omega_{n+1} - \omega_n] = \theta_\varnothing + \sum_{m=2}^{n+1} \sum_y y \pi_m(y). \tag{3.39}$$

The right hand side converges if and only if the left hand side does. Thus, under the assumption that $\lim_{n\to\infty} \sum_{m=2}^{n} \sum_x x\pi_m(x) \equiv \tilde{\theta} - \theta_\varnothing$ exists, we obtain that

$$\lim_{n\to\infty} \mathbb{E}[\omega_{n+1} - \omega_n] = \tilde{\theta}. \tag{3.40}$$

In turn, (3.40) implies that

$$\lim_{n\to\infty} \mathbb{E}[n^{-1}\omega_n] = \tilde{\theta}. \tag{3.41}$$

When $n^{-1}\omega_n \xrightarrow{\mathbb{Q}} \theta$, by bounded convergence and the fact that $|\omega_n| \leq nL$ since the maximal step size of our self-interacting random walks is L, we have that

$$\lim_{n\to\infty} \mathbb{E}[n^{-1}\omega_n] = \theta, \tag{3.42}$$

so that, as required, $\theta = \tilde{\theta}$. □

3.4 The formula for the variance

In this section we prove a result about the variance of the endpoint of the walk, similar to that obtained above for the speed. Define $a_m^{[i]} := \sum_y y^{[i]}\pi_m(y)$. Then, we have the following formula for the variance of self-interacting random walks in terms of the lace expansion coefficients:

Theorem 3.2 (The variance formula)**.** *Suppose that for each $i, j \in \{1, 2, \ldots, d\}$,*

$$\lim_{n\to\infty} \frac{\mathbb{E}[\omega_n^{[i]}\omega_n^{[j]}] - \mathbb{E}[\omega_n^{[i]}]\mathbb{E}[\omega_n^{[j]}]}{n} = \Sigma_{ij}, \qquad \text{and} \qquad \sum_{m=2}^{\infty} \sum_y y^{[i]} y^{[j]} \pi_m(y) < \infty, \tag{3.43}$$

and that either

(i) $\mathbb{E}[\omega_n] = 0$ *for each n, or*

(ii) $n^{-1}\omega_n \xrightarrow{\mathbb{Q}} \theta$*, and* $\sum_{m=2}^{\infty}(m-1)|a_m^{[i]}| < \infty$.

Then

$$\Sigma_{ij} = (\Sigma_\varnothing)_{ij} - \theta^{[i]}\theta^{[j]} - \sum_{m=2}^{\infty} \left[\theta^{[i]}(m-1)a_m^{[j]} + \theta^{[j]}(m-1)a_m^{[i]} - \sum_y y^{[i]}y^{[j]}\pi_m(y) \right]. \tag{3.44}$$

The proof of Theorem 3.2 is an adaptation of that of the speed formula in Theorem 3.1 above, and is deferred to Section 7.

4 Induction for the weak law of large numbers

In this section we prove a law of large numbers from the recurrence relation (2.29), or, more precisely, its Fourier transform (3.34), assuming certain bounds on the coefficients $\hat{\pi}_m(k)$. The bounds roughly correspond to upper bounds on the accuracy of the Taylor approximation of $\hat{\pi}_m(k)$ up to *first* order.

We start by formulating a general assumption (which must be verified for a specific model), and prove the main result, Theorem 4.1, under this assumption.

Assumption (LLN). *There exists a sequence $\{b_m\}_{m\geq 1}$, independent of β and with $b_1 \geq 1$, and a constant $\varepsilon_\beta = \varepsilon_\beta(d)$ satisfying $\lim_{\beta\to 0}\varepsilon_\beta(d) = 0$ such that*

$$\hat{\pi}_m(0) = 0, \qquad |\nabla\hat{\pi}_m(0)| \leq \varepsilon_\beta b_m, \qquad |\nabla^2\hat{\pi}_m(0)| \leq \varepsilon_\beta m b_m, \tag{4.1}$$

and uniformly in $k \in [-\pi,\pi]^d$,

$$|\hat{\pi}_m(k)| \leq \varepsilon_\beta |k| b_m, \qquad |\hat{\pi}_m(k) - k\cdot\nabla\hat{\pi}_m(0)| \leq \varepsilon_\beta |k|^2 m b_m, \tag{4.2}$$

where

$$B \equiv \sum_{m=1}^{\infty} b_m < \infty \quad \text{and} \quad B' \equiv \sup_n \frac{(\log(n\vee 3))^2}{n}\sum_{m=1}^{n} m b_m < \infty. \tag{4.3}$$

Theorem 4.1 (Weak law of large numbers)**.** *When Assumption (LLN) holds, there exist $\beta_0 = \beta_0(d) > 0$ and $\theta = \theta(\beta)$ such that for all $\beta \leq \beta_0$,*

$$\mathbb{E}_\beta[\omega_n] = \theta n\Big[1 + O\Big(\frac{1}{n}\sum_{m=1}^{\infty}(n\wedge m)b_m\Big)\Big]. \tag{4.4}$$

Furthermore, there exists $C>0$ such that for every $k \in \mathbb{R}^d$,

$$\log\left(\mathbb{E}_\beta[e^{ik\cdot\omega_n/n}]\right) = ik\cdot\theta + O\left(\frac{|k|}{n}e^{C|k|}\sum_{m=1}^{\infty}(n\wedge m)b_m\right) + O\left(\frac{|k|^2}{n}\sum_{m=1}^{n} m b_m\right), \tag{4.5}$$

where the constant θ given by (3.30) *is model dependent.*

Remark 4.1. *Observe that $n^{-1}\sum_{m=1}^{\infty}(n\wedge m)b_m = o(1)$ and $n^{-1}\sum_{m=1}^{n} m b_m = o(1)$ when* (4.3) *holds. Thus* (4.5) *implies that $\lim_{n\to\infty}\mathbb{E}_\beta[e^{ik\cdot\omega_n/n}] = e^{ik\cdot\theta}$, which is equivalent to the statement of convergence in probability, $\omega_n/n \overset{\mathbb{Q}_\beta}{\longrightarrow} \theta$.*

Note that since D has finite range, there exists a constant $C_1 \geq 1$ independent of β such that

$$|\hat{D}(k) - 1 - ik\cdot\theta_\varnothing| \leq C_1|k|^2, \tag{4.6}$$

and let $K_1 = 2C_1$, which is independent of β.

We will frequently use the following lemma, whose proof follows easily by applying Taylor's Theorem at $t=0$ to the map from $\mathbb{R}\to\mathbb{C}$ given by $t\mapsto e^{tx}$:

Lemma 4.2. *For all $x\in\mathbb{C}, j\in\mathbb{N}$,*

$$\left|e^x - \sum_{l=0}^{j}\frac{x^l}{l!}\right| \leq \frac{|x|^{j+1}}{(j+1)!}e^{|\mathrm{Re}(x)|},$$

where $\mathrm{Re}(x)$ *is the real part of x.*

Set $\theta_1 = \theta_\varnothing$, and, for $n \geq 2$, we define the following approximation to θ:

$$\theta_n = \theta_\varnothing - i \sum_{m=2}^{n} \nabla \hat{\pi}_m(0). \tag{4.7}$$

Our induction hypothesis for the law of large numbers in Theorem 4.1 is that the following bound holds for all $\beta \leq \beta_0$, some $\delta < 1$ independent of β and all $0 \leq j \leq n$:

For $|k| \leq \delta \log(n \vee 3)/n$ and some $K \geq 1$ independent of β we can write,

$$\hat{c}_j(k) = \exp\Big[\sum_{l=1}^{j} \big(ik \cdot \theta_l + e_l(k)\big)\Big] \quad \text{where} \quad |e_j(k)| \leq K|k|^2 \sum_{l=1}^{j} l b_l, \tag{4.8}$$

where the empty sum, arising when $j = 0$, is defined to be 0, and where, for $n = 0$, the equation is valid for all $k \in [-\pi, \pi]^d$.

The initialisation of the induction (the $n = 0$ case) holds trivially since $1 = e^0$. In Section 4.1 we will advance the induction hypothesis. In Section 4.2 we will use it to prove Theorem 4.1.

4.1 The LLN induction advanced

We fix $n \geq 0$. The induction step will be achieved as soon as we are able to write

$$\frac{\hat{c}_{n+1}(k)}{\hat{c}_n(k)} = \exp\left[ik \cdot \theta_{n+1} + e_{n+1}(k)\right], \tag{4.9}$$

for $e_{n+1}(k)$ satisfying the required bound. For this, we write

$$\frac{\hat{c}_{n+1}(k)}{\hat{c}_n(k)} = 1 + ik \cdot \theta_{n+1} + e'_{n+1}(k) \tag{4.10}$$

and then set

$$e_{n+1}(k) = \log\left[1 + ik \cdot \theta_{n+1} + e'_{n+1}(k)\right] - ik\theta_{n+1}. \tag{4.11}$$

The following lemma is a trivial consequence of (3.32) and (4.7):

Lemma 4.3. *We have $|\theta_\varnothing| \leq L$ and when Assumption (LLN) holds we have $|\theta_n| \leq L + \varepsilon_\beta B$ for every n.*

Let

$$B_n = \sum_{m=1}^{n} m b_m. \tag{4.12}$$

We note that by the second bound in (4.3), and uniformly in k such that $|k| \leq \delta n^{-1} \log(n \vee 3)$, we have

$$B_{n+1}|k| \leq \delta B', \qquad n B_n |k|^2 \leq \delta^2 B'. \tag{4.13}$$

These bounds will be frequently used in what follows.

Choose $\beta_0 > 0$ so that $\varepsilon_\beta \leq 1$ for all $\beta \leq \beta_0$, and suppose that the required bound (4.8) holds for $e'_{n+1}(k)$ with constant K_1. By Lemma 4.3, $|k||\theta_{n+1}| + |e'_{n+1}(k)| \leq 1/2$ for $|k| \leq \delta \log(n \vee 3)/n$ when $\delta \leq (2(L + B + K_1 B') \log 3)^{-1}$. Therefore we may apply Taylor's Theorem $|\log(1+x) - x| \leq 4|x|^2$ for $|x| \leq 1/2$, to (4.11). This implies that when the required bound holds for $e'_{n+1}(k)$ with constant K_1, it also holds for $e_{n+1}(k)$ for some K independent of β, since the terms of order k in (4.11) cancel. Specifically, if $|e'_{n+1}(k)| \leq K_1 |k|^2 B_{n+1}$, then, using also (4.13) and $(x+y)^2 \leq 2x^2 + 2y^2$,

$$\begin{aligned} |e_{n+1}(k)| &\leq 4(|k||\theta_{n+1}| + |e'_{n+1}(k)|)^2 + |e'_{n+1}(k)| \\ &\leq 8|k|^2(L+B)^2 + 8K_1^2 B_{n+1}^2 |k|^4 + K_1 B_{n+1}|k|^2 \leq |k|^2(8(L+B)^2 + 8B_{n+1}K_1^2 \delta B' + B_{n+1}K_1) \\ &\leq K B_{n+1}|k|^2, \end{aligned} \tag{4.14}$$

for $K \geq 8(L+B)^2 + 8\delta B' K_1^2 + K_1$, which is independent of β.

The rest of this section will be devoted to the proof of the following lemma:

Lemma 4.4. *There exists β_0 such that for all $\beta \leq \beta_0$, if $e_j(k)$ satisfies the bound in (4.8) for all $j \leq n$ and $|k| \leq \delta(n+1)^{-1}\log((n+1) \vee 3)$ then for such k,*

$$|e'_{n+1}(k)| \;\; \leq \;\; K_1 B_{n+1}|k|^2. \tag{4.15}$$

Proof. Divide the recursion relation (2.32) by $\hat{c}_n(k)$ and use the equality $\hat{\pi}_m(0) = 0$ of (4.1) to obtain

$$\frac{\hat{c}_{n+1}(k)}{\hat{c}_n(k)} \;\; = \;\; \hat{D}(k) + \sum_{m=2}^{n+1} [\hat{\pi}_m(k) - \hat{\pi}_m(0)] \frac{\hat{c}_{n+1-m}(k)}{\hat{c}_n(k)}. \tag{4.16}$$

We can rewrite (4.16) as

$$\frac{\hat{c}_{n+1}(k)}{\hat{c}_n(k)} \;\; = \;\; 1 \; + ik \cdot \theta_{n+1} + e'_{n+1}(k),$$

where

$$e'_{n+1}(k) \;\; = \;\; [\hat{D}(k) - 1 - ik \cdot \theta_{\varnothing}] + \sum_{m=2}^{n+1} \Big[\hat{\pi}_m(k) - k \cdot \nabla \hat{\pi}_m(0)\Big] + \sum_{m=2}^{n+1} \hat{\pi}_m(k) [\frac{\hat{c}_{n+1-m}(k)}{\hat{c}_n(k)} - 1].$$

The first term is taken care of by (4.6). Furthermore, by (4.2), we have that

$$\sum_{m=1}^{n+1} |\hat{\pi}_m(k) - k \cdot \nabla \hat{\pi}_m(0)| \leq \varepsilon_\beta |k|^2 \sum_{m=1}^{n+1} m b_m = \varepsilon_\beta B_{n+1}|k|^2. \tag{4.17}$$

Finally, using Lemma 4.2 and the induction hypothesis (4.8) for $e_l(k)$ with $l \leq n$, which is allowed since $|k| \leq \delta \log((n+1) \vee 3)/(n+1)$ implies that also $|k| \leq \delta \log(n \vee 3)/n$,

$$\begin{aligned} \Big|\frac{\hat{c}_{n+1-m}(k)}{\hat{c}_n(k)} - 1\Big| =& \Big| \exp\Big[- \sum_{l=n+1-m}^{n} (ik \cdot \theta_l + e_l(k))\Big] - 1 \Big| \leq m(|k|(L+B) + K B_{n+1}|k|^2) e^{mKB_{n+1}|k|^2} \\ \leq& m(|k|(L+B) + KB_{n+1}|k|^2) e^{KB'\delta^2}, \end{aligned} \tag{4.18}$$

by the second inequality in (4.13).

Using the first bound in (4.2), it follows that

$$\begin{aligned} \Big| \sum_{m=2}^{n+1} \hat{\pi}_m(k) [\frac{\hat{c}_{n+1-m}(k)}{\hat{c}_n(k)} - 1] \Big| \leq& \sum_{m=2}^{n+1} |k| \varepsilon_\beta b_m e^{KB'\delta^2} m(|k|(L+B) + B_{n+1}K|k|^2) \\ \leq& \varepsilon_\beta |k|^2 B_{n+1} e^{KB'\delta^2} \Big((L+B) + B_{n+1}K|k|\Big) \\ \leq& \varepsilon_\beta |k|^2 B_{n+1} e^{KB'\delta^2} ((L+B) + \delta B' K), \end{aligned} \tag{4.19}$$

where in the last inequality we used the first bound in (4.13). Summarising (4.17)-(4.19) we have

$$|e'_{n+1}(k)| \leq C_1 |k|^2 + \varepsilon_\beta B_{n+1} |k|^2 + \varepsilon_\beta e^{KB'\delta^2} ((L+B) + \delta B' K) B_{n+1} |k|^2. \tag{4.20}$$

Recall that $K_1 = 2C_1$ and $B_{n+1} \geq b_1 \geq 1$. We choose β_0 sufficiently small so that both $\varepsilon_\beta \leq 1$ for all $\beta \leq \beta_0$ and

$$\varepsilon_\beta \Big(1 + e^{KB'\delta^2} ((L+B) + \delta B' K)\Big) \leq \frac{K_1}{2}. \tag{4.21}$$

Then, we conclude that $C_1 + \varepsilon_\beta B_{n+1}(1 + e^{KB'\delta^2}((L+B) + \delta B'K)) \leq K_1 B_{n+1}$ and therefore (4.15) holds as required for all $\beta \leq \beta_0$. This completes the proof of Lemma 4.4. □

4.2 Proof of Theorem 4.1

To prove (4.4), we note that from (4.8), which is now known to be valid for all n,

$$\mathbb{E}_\beta[\omega_n] = -i\nabla\hat{c}_n(0) = -i\sum_{j=1}^{n}[i\theta_j + \nabla e_j(0)]. \tag{4.22}$$

Since $|e_j(k)| = O(|k|^2)$, we have that $\nabla e_j(0) = 0$. Therefore,

$$\mathbb{E}_\beta[\omega_n] = \sum_{j=1}^{n}\theta_j = n\theta + \sum_{j=1}^{n}[\theta_j - \theta]. \tag{4.23}$$

By (3.30), (4.7) and (4.1), we have that

$$\sum_{j=1}^{n}[\theta_j - \theta] = i\sum_{j=1}^{n}\sum_{s=j+1}^{\infty}\nabla\hat{\pi}_s(0) = i\sum_{s=2}^{\infty}(n\wedge(s-1))\nabla\hat{\pi}_s(0) = O\left(\sum_{s=1}^{\infty}(n\wedge s)b_s\right). \tag{4.24}$$

For (4.5), let $k\in\mathbb{R}^d$. Then for $n\geq e^{\delta^{-1}|k|}$ we can apply (4.8) in the form

$$\hat{c}_n(kn^{-1}) = e^{ikn^{-1}\cdot\theta n}\exp\Big[\sum_{l=1}^{n}[ikn^{-1}\cdot(\theta_l-\theta) + e_l(kn^{-1})]\Big], \tag{4.25}$$

with

$$|e_j(kn^{-1})| \leq K\frac{|k|^2}{n^2}\sum_{l=1}^{j} lb_l. \tag{4.26}$$

By (4.24),

$$\sum_{l=1}^{n} ikn^{-1}\cdot(\theta_l-\theta) = O\left(\frac{|k|}{n}\sum_{s=1}^{\infty}(n\wedge s)b_s\right). \tag{4.27}$$

Similarly,

$$\sum_{j=1}^{n}|e_j(kn^{-1})| \leq K\frac{|k|^2}{n^2}\sum_{j=1}^{n}\sum_{l=1}^{j} lb_l \leq K\frac{|k|^2}{n^2}\sum_{l=1}^{n}(n-l+1)lb_l \leq K\frac{|k|^2}{n}\sum_{l=1}^{n} lb_l. \tag{4.28}$$

Together (4.27) and (4.28) prove (4.5) for $n\geq e^{\delta^{-1}|k|}$.

For $n < e^{\delta^{-1}|k|}$ the result is trivial by writing

$$ikn^{-1}\cdot\omega_n = ik\cdot\theta + O(|k|(L+\theta)) = ik\cdot\theta + O(|k|e^{\delta^{-1}|k|}n^{-1}). \tag{4.29}$$

□

5 Induction for the central limit theorem

In this section we prove a central limit theorem from the recurrence relation (2.29), or more precisely its Fourier transform (3.34), assuming certain bounds on the coefficients $\hat{\pi}_m(k)$. The bounds roughly correspond to upper bounds on the accuracy of the Taylor approximation of $\hat{\pi}_m(k)$ up to *second* order, and the argument is an extension of the one in Section 4. In this section, for a $d\times d$ matrix Σ, we define its L^1-norm by

$$|\Sigma| = \sum_{i,j=1}^{d}|(\Sigma)_{ij}|. \tag{5.1}$$

We start by formulating a general assumption, and prove the main result, Theorem 5.1, under this assumption.

Assumption (CLT). *There exists a non-increasing sequence $\{b_m\}_{m\geq 1}$ independent of β with $b_1 \geq 1$, and a constant ε_β with $\lim_{\beta\downarrow 0} \varepsilon_\beta = 0$, such that*
(i)

$$\hat{\pi}_m(0) = 0, \qquad |\nabla \hat{\pi}_m(0)| \leq \varepsilon_\beta b_m, \qquad |\nabla^2 \hat{\pi}_m(0)| \leq \varepsilon_\beta m b_m. \tag{5.2}$$

(ii) for all $k \in [-\pi, \pi]^d$,

$$\left|\hat{\pi}_m(k) - k \cdot \nabla \hat{\pi}_m(0)\right| \leq \varepsilon_\beta |k|^2 m b_m, \qquad \left|\hat{\pi}_m(k) - k \cdot \nabla \hat{\pi}_m(0) - \frac{1}{2} k^t \nabla^2 \hat{\pi}_m(0) k\right| \leq \varepsilon_\beta |k|^3 m^2 b_m. \tag{5.3}$$

Moreover, $B^* \equiv \sum_{m=1}^{\infty} m b_m < \infty$, and there exists $\gamma \in (0, 1/2)$ such that

$$d_n \equiv \sum_{m=2}^{n} m b_m \sum_{l=n+1-m}^{n} b_{l+1} = o(1), \quad \text{as } n \to \infty, \text{ and}$$

$$a_n \equiv \sum_{m=1}^{n} m^{2+\gamma} b_m \leq C_a \sqrt{\frac{n}{\log(n+1)}}, \quad \text{for all } n \text{ and some } C_a \geq 1. \tag{5.4}$$

Similarly to (4.12), we define

$$A_n = \sum_{j=1}^{n} a_j, \quad D_n = \sum_{j=1}^{n} d_j, \quad E_n = \sum_{m=1}^{\infty} (m \wedge n) m b_m. \tag{5.5}$$

We will prove a generalised version of Theorems 2.1 and 2.2, which is formulated below:

Theorem 5.1 (Central limit theorem)**.** *When Assumption (CLT) holds, there exist $\beta_0 = \beta_0(d) > 0$, and $\theta = \theta(\beta)$, and $\Sigma = \Sigma(\beta)$ such that, for all $\beta \leq \beta_0$,*
(a)

$$\mathbb{E}_\beta[\omega_n] = \theta n \left[1 + O\left(\frac{1}{n}\right)\right]. \tag{5.6}$$

(b)

$$\mathrm{Var}_\beta(\omega_n) = \Sigma n + O(D_n) + O(E_n). \tag{5.7}$$

(c) there exists $C > 0$ such that for every $k \in \mathbb{R}^d$

$$\log\left(\mathbb{E}_\beta\big[e^{ik\cdot\frac{(\omega_n - \theta n)}{\sqrt{n}}}\big]\right) = -\frac{1}{2} k^t \Sigma k + O(|k| e^{C|k|^2} n^{-1/2}) + O\big(|k|^3 n^{-3/2} A_n\big) + O\big(|k|^2 e^{C|k|^2} n^{-1}(D_n + E_n)\big). \tag{5.8}$$

The constants θ and Σ (given by (3.30), (3.31)*) are model dependent.*

It is not hard to see that each of the O terms in (5.8) is indeed an error term when we assume that Assumption (CLT) holds. However, in the general set-up in Assumption (CLT), it is not clear to us which term on the right-hand side of (5.8) is typically the largest.

Note that since D has finite range, there exists a constant $C_2 \geq 1$ independent of β such that

$$|\hat{D}(k) - 1 - ik \cdot \theta_\varnothing - \frac{1}{2} k^t \Sigma_\varnothing k| \leq C_2 |k|^3, \tag{5.9}$$

and let $K_2 = 2C_2$, which is independent of β.

Recall (4.7) and define the following approximation to Σ:

$$\Sigma_n = \Sigma_\varnothing - \theta_n \theta_n^t - \sum_{m=2}^{n} \nabla^2 \Big[e^- i\theta_n \cdot k(m-1) \hat{\pi}_m(k)\Big]_{k=0}. \tag{5.10}$$

Let $B \equiv \sum_m b_m$ and $d^* \equiv \sup_n d_n$.

Our induction hypothesis for the central limit theorem is that the following bound holds for all $\beta \le \beta_0$, all $0 \le j \le n$, and some $\delta \in (0,1)$, independent of β:

For k such that $|k|^2 \le \delta \log(n \vee 3)/n$, and some K independent of β we can write,

$$\hat{c}_j(k) = \exp\Big[\sum_{l=1}^{j}\big[ik\cdot\theta_l - \frac{1}{2}k^t\Sigma_l k + r_l(k)\big]\Big] \quad \text{with } |r_j(k)| \le K(|k|^2 d_j + |k|^3 a_j), \tag{5.11}$$

where again the empty sum appearing when $j = 0$ is defined to be zero, and for $n = 0$, (5.11) is assumed to hold for all $k \in [-\pi,\pi]^d$.

The initialisation of the induction ($n = 0$ case) holds trivially as $1 = e^0$.

5.1 The CLT induction advanced

We follow the same strategy as in Section 4.1, now expanding the Fourier transform one order further. We fix $n \ge 0$. The induction step will be achieved as soon as we are able to write

$$\frac{\hat{c}_{n+1}(k)}{\hat{c}_n(k)} = \exp\big[ik\cdot\theta_{n+1} - \frac{1}{2}k^t\Sigma_{n+1}k + r_{n+1}(k)\big], \tag{5.12}$$

for $r_{n+1}(k)$ satisfying the required bound. For this, we write

$$\frac{\hat{c}_{n+1}(k)}{\hat{c}_n(k)} = 1 + ik\cdot\theta_{n+1} - \frac{1}{2}k^t(\Sigma_{n+1} + \theta_{n+1}\theta_{n+1}^t)k + r'_{n+1}(k) \tag{5.13}$$

and then set

$$r_{n+1}(k) = \log\big[1 + ik\cdot\theta_{n+1} - \frac{1}{2}k^t(\Sigma_{n+1} + \theta_{n+1}\theta_{n+1}^t)k + r'_{n+1}(k)\big] - ik\cdot\theta_{n+1} + \frac{1}{2}k^t\Sigma_{n+1}k. \tag{5.14}$$

The following lemma is an easy consequence of (3.33) and (5.10):

Lemma 5.2. *We have $|\theta_\varnothing| \le L$ and $|\Sigma_\varnothing| \le d^2L^2$, and, when Assumption (CLT) holds, for all n, $|\theta_n| \le L + \varepsilon_\beta B$, and*

$$\begin{aligned} |\Sigma_n| &\le d^2L^2 + (L+\varepsilon_\beta B)^2 + 2d^2(L+\varepsilon_\beta B)B^* + \varepsilon_\beta B^*, \quad \textit{and} \\ |\Sigma_n + \theta_n\theta_n^t| &\le d^2L^2 + 2d^2(L+\varepsilon_\beta B)B^* + \varepsilon_\beta B^*. \end{aligned} \tag{5.15}$$

Suppose that the required bound (5.11) holds for $r'_{n+1}(k)$ with constant K_2. Then, by the assumption on a_j in (5.4), we have that, for k satisfying $|k|^2 \le \delta\log(n\vee 3)/n \le 2\delta$, and since $\delta < 1$,

$$|r'_{n+1}(k)| \le K_2\delta(d^* + \sqrt{\delta}C_a) \le K_2\delta(d^* + C_a). \tag{5.16}$$

Choose β_0 so that $\varepsilon_\beta \le 1$ for all $\beta \le \beta_0$, so that, by Lemma 5.2, for k satisfying $|k|^2 \le \delta\log(n\vee 3)/n \le 2\delta$ in (5.11), and using $L, B^* \ge 1$,

$$\begin{aligned} &|k||\theta_{n+1}| + \frac{1}{2}|k|^2|\Sigma_{n+1} + \theta_{n+1}\theta_{n+1}^t| + |r'_{n+1}(k)| \\ &\quad\le \sqrt{2\delta}(L+\varepsilon_\beta B) + \delta\Big(d^2L^2 + (L+\varepsilon_\beta B)^2 + 2d^2(L+\varepsilon_\beta B)B^* + \varepsilon_\beta B^*\Big) + K_2\delta(d^* + C_a) \\ &\quad\le \sqrt{2\delta}(L+B) + \delta\Big(5d^2(L+\varepsilon_\beta B)B^* + K_2(d^*+C_a)\Big) \le 1/2, \end{aligned} \tag{5.17}$$

when we take $\delta \le \delta^*$, which is defined by

$$\delta^* = \min\Big\{(L+B)^{-2}/32, \Big(4\big(5d^2(L+B)B^* + K_2(d^*+C_a)\big)\Big)^{-1}\Big\}. \tag{5.18}$$

Therefore we may apply Taylor's Theorem $|\log(1+x) - x + \frac{x^2}{2}| \le 8|x|^3$ for $|x| \le 1/2$ to (5.14). This implies that when the required bound holds for $r'_{n+1}(k)$ with constant K_2, it also holds for $r_{n+1}(k)$ for some K independent of β, since the terms of order k and $|k|^2$ in (5.14) cancel. Specifically, if $|r'_{n+1}(k)| \le K_2(|k|^2 d_{n+1} + |k|^3 a_{n+1})$ then using Taylor's Theorem, followed by the assumed bound on $r'_{n+1}(k)$, we obtain

$$\begin{aligned}
|r_{n+1}(k)| \le& |r'_{n+1}(k)| + |k||\theta_{n+1}|\left(\frac{1}{2}|k|^2|\Sigma_{n+1} + \theta_{n+1}\theta^t_{n+1}| + |r'_{n+1}(k)|\right) \\
&+ \frac{1}{2}\left(\frac{1}{2}|k|^2|\Sigma_{n+1} + \theta_{n+1}\theta^t_{n+1}| + |r'_{n+1}(k)|\right)^2 \\
&+ 8\left(|k||\theta_{n+1}| + \frac{1}{2}|k|^2|\Sigma_{n+1} + \theta_{n+1}\theta^t_{n+1}| + |r'_{n+1}(k)|\right)^3 \\
\le& CK_2^3(|k|^2 d_{n+1} + |k|^3 a_{n+1}) \le K(|k|^2 d_{n+1} + |k|^3 a_{n+1}),
\end{aligned} \tag{5.19}$$

when $K \ge CK_2^3$. Here $C \ge 1$ is a constant that depends on C_a, B, B^*, d^*, L, d, but is independent of β and δ, and we have used that $|r'_{n+1}(k)|^2 \le CK_2^2(|k|^2 d_{n+1} + |k|^3 a_{n+1})$ since $a_{n+1} \ge 1$ and $|k|^2 \le \delta \log(n \vee 3)/n$, and similarly for $|r'_{n+1}(k)|^3$.

Most of this section will be devoted to the proof of the following lemma:

Lemma 5.3. *If* (5.11) *holds for all $j \le n$ and $|k|^2 \le \delta(n+1)^{-1}\log((n+1)\vee 3)$ then for such k*

$$|r'_{n+1}(k)| \quad \le K_2(|k|^2 d_{n+1} + |k|^3 a_{n+1}). \tag{5.20}$$

5.1.1 Proof of Lemma 5.3

The proof involves expressing $r'_{n+1}(k)$ as a sum of three terms and showing that each term is bounded in absolute value by the right hand side of (5.20).

Recall (5.13), then

$$r'_{n+1}(k) \quad = I + II,$$

where

$$\begin{aligned}
I &= [\hat{D}(k) - 1 - ik\cdot\theta_\varnothing + \frac{1}{2}k^t\Sigma_\varnothing k] + \sum_{m=2}^{n+1}\Big[\hat{\pi}_m(k) - k\cdot\nabla\hat{\pi}_m(0) - \frac{1}{2}k^t\nabla^2\hat{\pi}_m(0)k\Big], \\
II &= \sum_{m=2}^{n+1}\Big[\hat{\pi}_m(k)[\frac{\hat{c}_{n+1-m}(k)}{\hat{c}_n(k)} - 1] + k\cdot\nabla\hat{\pi}_m(0)i(m-1)k\cdot\theta_{n+1}\Big].
\end{aligned}$$

We will bound $|I|$ and $|II|$, and then choose β_0 sufficiently small so that $|I|+|II|$ satisfies the bound on the right hand side of (5.20). By (5.9) and (5.3) in Assumption (CLT), and the fact that $a_{n+1} \ge 1$ we have

$$|I| \quad \le C_2|k|^3 + \textstyle\sum_{m=2}^{n+1}\varepsilon_\beta|k|^3 m^2 b_m \le (C_2+\varepsilon_\beta)|k|^3 a_{n+1}. \tag{5.21}$$

To bound II, we first split $II = II_1 + II_2$, with

$$
\begin{aligned}
II_1 &= \sum_{m=2}^{n+1} [\hat{\pi}_m(k) - k \cdot \nabla \hat{\pi}_m(0)] \left[\frac{\hat{c}_{n+1-m}(k)}{\hat{c}_n(k)} - 1 \right], \\
II_2 &= \sum_{m=2}^{n+1} k \cdot \nabla \hat{\pi}_m(0) \left[\frac{\hat{c}_{n+1-m}(k)}{\hat{c}_n(k)} - 1 + i(m-1)k \cdot \theta_{n+1} \right].
\end{aligned}
$$

For II_1, we use the first bound in (5.3) in Assumption (CLT) and Lemma 4.2 for $j = 0$, to get

$$
\begin{aligned}
|II_1| &\leq \sum_{m=2}^{n+1} |\hat{\pi}_m(k) - k \cdot \nabla \hat{\pi}_m(0)| \left| \frac{\hat{c}_{n+1-m}(k)}{\hat{c}_n(k)} - 1 \right| \\
&\leq \sum_{m=2}^{n+1} \varepsilon_\beta |k|^2 m b_m \left| \exp\left[- \sum_{l=n+2-m}^{n} \left[ik \cdot \theta_l - \frac{1}{2} k^t \Sigma_l k + r_l(k) \right] \right] - 1 \right|, \\
&\leq \varepsilon_\beta |k|^2 \sum_{m=2}^{n+1} m b_m e^{\chi_{m,n}(k)} \sum_{l=n+2-m}^{n} \left[|k||\theta_l| + \frac{1}{2} |k|^2 |\Sigma_l| + |r_l(k)| \right],
\end{aligned}
$$

with

$$
\chi_{m,n}(k) = \sum_{l=n+2-m}^{n} \left[\frac{|k|^2}{2} |\Sigma_l| + |r_l(k)| \right]. \tag{5.22}
$$

Since a_n is increasing, for $|k|^2 \leq \delta \log((n+1) \vee 3)/(n+1)$, we have that

$$
\chi_{m,n}(k) \leq m|k|^2 (C + Kd^* + a_n|k|) \leq m|k|^2 (C + Kd^* + \sqrt{\delta} K C_a), \tag{5.23}
$$

where we recall that $d^* = \sup_n d_n$. Also, for $|k|^2 \leq \delta \log((n+1) \vee 3)/(n+1)$,

$$
m|k|^2 \leq \delta \log(m \vee 3) \frac{\log((n+1) \vee 3)}{n+1} \frac{m}{\log(m \vee 3)} \leq \delta \log(m \vee 3), \tag{5.24}
$$

since $x \mapsto \frac{\log(x \vee 3)}{x}$ is decreasing for $x \geq 0$. As a result, we obtain that, with $\nu = \delta(C + Kd^* + \sqrt{\delta} K C_a)$,

$$
e^{\chi_{m,n}(k)} \leq (m \vee 3)^\nu. \tag{5.25}
$$

Note that, by picking $\delta > 0$ sufficiently small, we can make $\nu < \gamma$.

For $|k|^2 \leq \delta \log((n+1) \vee 3)/(n+1)$ it follows from Lemma 5.2 and (5.11), using a similar argument as in (5.23), that

$$
|k||\theta_l| + \frac{|k|^2}{2} |\Sigma_l| + |r_l(k)| \leq \Big((L + \varepsilon_\beta B) + (\sqrt{\delta} C + \sqrt{\delta} K d^* + \delta K C_a) \Big) |k| \equiv C_K^{(1)} |k|. \tag{5.26}
$$

Therefore,

$$
|II_1| \quad \leq \quad C_K^{(1)} \varepsilon_\beta |k|^3 \sum_{m=2}^{n+1} m^{2+\gamma} b_m = C_K^{(1)} \varepsilon_\beta |k|^3 a_{n+1}. \tag{5.27}
$$

For II_2 we use (5.11) and Lemma 4.2 for $j=1$ to obtain

$$
\begin{aligned}
|II_2| &\le |k| \sum_{m=3}^{n+1} \varepsilon_\beta b_m \left| \exp\left[-\sum_{l=n+2-m}^{n} \Big[ik\cdot\theta_l - \frac{1}{2} k^t \Sigma_l k + r_l(k) \Big] \right] - 1 + ik\cdot\theta_{n+1}(m-1) \right| \\
&\le \varepsilon_\beta |k| \sum_{m=2}^{n+1} b_m \sum_{l=n+2-m}^{n} \Big[|k||\theta_{n+1}-\theta_l| + \frac{|k|^2}{2}|\Sigma_l| + |r_l(k)| \Big] \\
&\quad + \varepsilon_\beta |k| \sum_{m=2}^{n+1} b_m \left[\sum_{l=n+2-m}^{n} \Big[|k||\theta_l| + \frac{|k|^2}{2}|\Sigma_l| + |r_l(k)| \Big] \right]^2 e^{\chi_{m,n}(k)}.
\end{aligned}
$$

For $|k|^2 \le \delta \log((n+1)\vee 3)/(n+1)$, the second sum can be bounded, using (5.26) and (5.25), as

$$
(C_K^{(1)})^2 3^\gamma \varepsilon_\beta |k|^3 \sum_{m=2}^{n+1} m^{2+\gamma} b_m \equiv C_K^{(2)} \varepsilon_\beta |k|^3 a_{n+1}. \tag{5.28}
$$

By a similar argument as in (5.23), we have for $|k|^2 \le \delta \log((n+1)\vee 3)/(n+1)$,

$$
\frac{|k|^2}{2}|\Sigma_l| + |r_l(k)| \le C(1 + Kd^* + KC_a\sqrt{\delta})|k|^2 \equiv C_K^{(3)}|k|^2. \tag{5.29}
$$

Therefore,

$$
\varepsilon_\beta |k| \sum_{m=2}^{n+1} b_m \sum_{l=n+2-m}^{n} \Big[\frac{|k|^2}{2}|\Sigma_l| + |r_l(k)| \Big] \le C_K^{(3)} \varepsilon_\beta |k|^3 \sum_{m=2}^{n+1} m b_m \le C_K^{(3)} \varepsilon_\beta |k|^3 a_{n+1}. \tag{5.30}
$$

We continue with the remaining contribution to II_2. Since $\{b_m\}_{m\ge 1}$ is a decreasing sequence,

$$
|\theta_{n+1} - \theta_l| \le \varepsilon_\beta \sum_{s=l+1}^{n+1} b_s \le \varepsilon_\beta (n-l+1) b_{l+1}. \tag{5.31}
$$

Thus,

$$
\sum_{m=2}^{n+1} b_m \sum_{l=n+2-m}^{n} |\theta_{n+1}-\theta_l| \le \varepsilon_\beta \sum_{m=2}^{n+1} m b_m \sum_{l=n+2-m}^{n} b_{l+1} = \varepsilon_\beta d_{n+1}. \tag{5.32}
$$

We conclude that

$$
|II_2| \le \varepsilon_\beta (\varepsilon_\beta |k|^2 d_{n+1} + (C_K^{(2)} + C_K^{(3)})|k|^3 a_{n+1}). \tag{5.33}
$$

We have shown that

$$
|I| + |II| \le \varepsilon_\beta^2 |k|^2 d_{n+1} + (C_2 + \varepsilon_\beta + \varepsilon_\beta(C_K^{(1)} + C_K^{(2)} + C_K^{(3)}))|k|^3 a_{n+1}. \tag{5.34}
$$

Choose β_0 sufficiently small so that for all $\beta \le \beta_0$, $\varepsilon_\beta + \varepsilon_\beta(C_K^{(1)} + C_K^{(2)} + C_K^{(3)}) \le \frac{1}{2}K_2$. Recall that $K_2 = 2C_2 \ge 1$. Then for $\beta \le \beta_0$,

$$
|I| + |II| \le (C_2 + \frac{1}{2}K_2)(|k|^2 d_{n+1} + |k|^3 a_{n+1}) \le K_2(|k|^2 d_{n+1} + |k|^3 a_{n+1}), \tag{5.35}
$$

as required. This completes the proof of Lemma 5.3. □

5.2 Proof of Theorem 5.1

We will make use of the following lemma:

Lemma 5.4. *For all $\beta \leq \beta_0$, and all $j \in \mathbb{N}$,*

(i) $\nabla r_j(0) = 0$, *and*

(ii) $|\nabla^2 r_j(0)| \leq 3Kd^2 d_j$.

Proof. The induction hypothesis (5.11), now verified for all j, states that $|r_j(k)| \leq K(|k|^2 d_j + |k|^3 a_j)$. Therefore, letting $[\nabla r_j(0)]_i$ denote the i^{th} coordinate of the vector $\nabla r_j(0)$, we have

$$|[\nabla r_j(0)]_i| = \lim_{k_i \to 0} \frac{|r_j(0,\ldots,0,k_i,0,\ldots,0)|}{|k_i|} \leq \lim_{k_i \to 0} \frac{K(|k_i|^2 d_j + |k_i|^3 a_j)}{|k_i|} = 0 \tag{5.36}$$

Since all partial derivatives of $\hat{c}_n(k)$ up to second order exist and are continuous, and $\hat{c}_n(0) = 1$, we have from (5.13) and (5.14) that all partial derivatives of $r_j(k)$ up to second order exist in a neighbourhood of 0 and are continuous. Let $(\nabla^2 r_j(0))_{lm}$ denote the $(l,m)^{\text{th}}$ entry of the matrix $\nabla^2 r_j(0)$ and suppose that $|r_j(k)| \leq J_1|k|^2 + J_2|k|^3$. We claim that this implies that $|(\nabla^2 r_j(0))_{lm}| \leq 3J_1$ for each m,l, from which part *(ii)* of the lemma follows immediately. Without loss of generality we suppose that $l,m \in \{1,2\}$.

Let $h(k_1,k_2) = r_j(k_1,k_2,0,\ldots,0)$. By the second order mean value theorem, $f_{u_1,u_2}(t) \equiv h(tu_1,tu_2)$ satisfies

$$f_{u_1,u_2}(t) = f_{u_1,u_2}(0) + f'_{u_1,u_2}(0)t + f''_{u_1,u_2}(t^*)\frac{t^2}{2} \tag{5.37}$$

for some $t^* \equiv t^*(t,u_1,u_2) \in (0,t)$.

Now $f_{u_1,u_2}(0) = h(0,0) = 0$ and

$$|f'_{u_1,u_2}(0)| = \lim_{t\to 0}\left|\frac{h(tu_1,tu_2) - h(0,0)}{t}\right| = \lim_{t\to 0}\left|\frac{h(tu_1,tu_2)}{t}\right| \leq \lim_{t\to 0}\left|\frac{C_{u_1,u_2}(t^2+t^3)}{t}\right| = 0 \tag{5.38}$$

where we have used the bound on $|r_j(k)|$ in the last inequality. Thus (5.37) reduces to

$$f_{u_1,u_2}(t) = f''_{u_1,u_2}(t^*)\frac{t^2}{2}, \tag{5.39}$$

and by hypothesis the left hand side is bounded in absolute value by $J_1 t^2(u_1^2+u_2^2) + J_2 t^3 (u_1^2+u_2^2)^{3/2}$.

We now set $t_n = 1/n$ and let $t_n^* = t^*(t_n,u_1,u_2)$. Then for each n, $|f''_{u_1,u_2}(t_n^*)| \leq 2J_1(u_1^2+u_2^2) + 2n^{-1}J_2(u_1^2+u_2^2)^{3/2}$. By the multivariate chain rule $\frac{d}{dt}h(\vec{g}(t)) = \nabla h \cdot \vec{g}'(t)$ we have

$$f''_{u_1,u_2}(t_n^*) = u_1^2 h_{11}(t_n^*u_1,t_n^*u_2) + u_2^2 h_{22}(t_n^*u_1,t_n^*u_2) + 2u_1u_2h_{12}(t_n^*u_1,t_n^*u_2), \tag{5.40}$$

and thus

$$|u_1^2 h_{11}(t_n^*u_1,t_n^*u_2) + u_2^2 h_{22}(t_n^*u_1,t_n^*u_2) + 2u_1u_2h_{12}(t_n^*u_1,t_n^*u_2)| \leq 2J_1(u_1^2+u_2^2) + \frac{J_2}{n}(u_1^2+u_2^2)^{3/2}. \tag{5.41}$$

Putting $u_1 = 1, u_2 = 0$ in (5.41) gives $|h_{11}(t_n^*,0)| \leq 2J_1 + 2n^{-1}J_2$, and similarly $|h_{22}(t_n^*,0)| \leq 2J_1 + 2n^{-1}J_2$. Letting $n \to \infty$ and using the fact that $t_n^* \in (0,t_n)$ (so that $t_n = 1/n \to 0$ implies that $t_n^* \to 0$ as $n \to \infty$) we have $|h_{11}(0,0)| \leq 2J_1$ by continuity of the partial derivatives. Similarly, by taking $u_1 = 0, u_2 = 1$, we obtain $|h_{22}(0,0)| \leq 2J_1$. Next, set $u_1 = u_2 = 1$ in (5.41) and use $|a+b| \leq d \Rightarrow |a| \leq d + |b|$ to see that

$$2|h_{12}(t_n^*u_1,t_n^*u_2)| \leq |h_{11}(t_n^*u_1,t_n^*u_2) + h_{22}(t_n^*u_1,t_n^*u_2)| + 2J_1(u_1^2+u_2^2) + \frac{2J_2}{n}(u_1^2+u_2^2)^{3/2}. \tag{5.42}$$

Now use the triangle inequality and let $n \to \infty$ to get $|h_{12}(0,0)| \leq 3J_1$. $\square$

We are now ready to prove the statements in Theorem 5.1(a)–(c) one by one.

Proof of Theorem 5.1(a): Using (5.11) and Lemma 5.4(i), we have

$$\sum_{x \in \mathbb{Z}^d} x c_n(x) \;=\; -i\nabla \hat{c}_n(0) = -i \sum_{j=1}^{n} [i\theta_j + \nabla r_j(0)] = n\theta + \sum_{j=1}^{n} [\theta_j - \theta], \tag{5.43}$$

so that it suffices to prove that

$$\sum_{j=1}^{n} [\theta_j - \theta] = O(1). \tag{5.44}$$

For this, we use (3.30), (4.7) as well as the second bound in (5.2) and to note that

$$\sum_{j=1}^{n} |\theta_j - \theta| \leq \sum_{j=1}^{n} \sum_{m=j+1}^{\infty} |\nabla \hat{\pi}_m(0)| \leq \varepsilon_\beta \sum_{j=1}^{\infty} \sum_{m=j+1}^{\infty} b_m = \varepsilon_\beta \sum_{m=1}^{\infty} m b_m = O(1), \tag{5.45}$$

by the assumption that $B^* = \sum_{m=1}^{\infty} m b_m < \infty$. $\square$

Proof of Theorem 5.1(b): Recall that $\mathrm{Var}_\beta(\omega_n)$ is the covariance matrix of ω_n. Then

$$(\mathrm{Var}_\beta(\omega_n))_{lm} = \sum_x x_l x_m c_n(x) - \left(\sum_x x_l c_n(x)\right)\left(\sum_x x_m c_n(x)\right). \tag{5.46}$$

By (5.11) and Lemma 5.4(i-ii), and writing $[\theta_p]_l$ for the l^{th} component of θ_p,

$$\begin{aligned}
\sum_{x \in \mathbb{Z}^d} x_l x_m c_n(x) \;&=\; -(\nabla^2 \hat{c}_n(0))_{lm} \\
&=\; \sum_{p=1}^{n} \Big((\Sigma_p)_{lm} - (\nabla^2 r_p(0))_{lm}\Big) - \sum_{p,q=1}^{n} \left(i[\theta_p]_l + [\nabla r_p(0)]_l\right)\left(i[\theta_q]_m + [\nabla r_q(0)]_m\right) \\
&=\; \sum_{p=1}^{n} \left((\Sigma_p)_{lm} + O(d_p)\right) + \sum_{p=1}^{n} [\theta_p]_l \sum_{q=1}^{n} [\theta_q]_m.
\end{aligned} \tag{5.47}$$

It follows from (5.43) that $\sum_x x_l c_n(x) = \sum_{p=1}^{n} [\theta_p]_l$ and from (5.46) and (5.47) that

$$\begin{aligned}
(\mathrm{Var}_\beta(\omega_n))_{lm} \;&=\; \sum_{p=1}^{n} \left((\Sigma_p)_{lm} + O(d_p)\right) \\
&=\; n(\Sigma)_{lm} + O\left(\sum_{p=1}^{n} d_p\right) + \sum_{p=1}^{n} ((\Sigma_p)_{lm} - (\Sigma)_{lm}).
\end{aligned}$$

Therefore to complete the proof, it is sufficient to show that for $p \leq n$,

$$|(\Sigma_p)_{lm} - (\Sigma)_{lm}| = O\left(1 \vee (p \wedge n) p b_p\right). \tag{5.48}$$

By (3.31) and (5.10), the left hand side of (5.48) is bounded by

$$
\begin{aligned}
&|[\theta]_l[\theta]_m - [\theta_p]_l[\theta_p]_m| + \sum_{r=p+1}^{\infty} \Big| \Big[(\nabla^2 e^{-i(r-1)k\cdot\theta} \hat{\pi}_r(k))_{lm} \Big]_{k=0} \Big| \\
&\quad + \sum_{r=2}^{p} \Big| \Big[(\nabla^2 e^{i(r-1)k\cdot\theta_p} \hat{\pi}_r(k))_{lm} \Big]_{k=0} - \Big[(\nabla^2 e^{i(r-1)k\cdot\theta} \hat{\pi}_r(k))_{lm} \Big]_{k=0} \Big| \\
&\quad \le |[\theta]_l||[\theta]_m - [\theta_p]_m| + |[\theta_p]_m||[\theta]_l - [\theta_p]_l| \\
&\qquad + \sum_{r=p+1}^{\infty} \Big((r-1)\Big(|[\theta]_m||[\nabla\hat{\pi}_r(0)]_l| + |[\theta]_l||[\nabla\hat{\pi}_r(0)]_m| \Big) + |(\nabla^2\hat{\pi}_r(0))_{lm}| \Big) \\
&\qquad + |\theta_p - \theta| \sum_{r=2}^{p} |[\nabla\hat{\pi}_r(0)]|, \qquad (5.49)
\end{aligned}
$$

since $\hat{\pi}_r(0) = 0$ by (5.2). The first two terms are $O(1)$ using the fact that $|\theta|$ is finite and the $|\theta_p|$ are uniformly bounded together with (5.45). By (5.2), the third term is bounded by

$$\varepsilon_\beta |\theta| \sum_{p=1}^{n} \sum_{r=p+1}^{\infty} r b_r \le \varepsilon_\beta |\theta| \sum_{r=1}^{\infty} (r \wedge n) r b_r,$$

while, again by (5.2), the last term is bounded. This completes the proof. □

Proof of Theorem 5.1(c): Fix $k \in \mathbb{R}^d$. Then for $n \ge e^{\frac{|k|^2}{\delta}}$, we can apply (5.11) in the form

$$
\begin{aligned}
\hat{c}_n(n^{-\frac{1}{2}}k) &= \exp\Big[\sum_{j=1}^{n} (in^{-\frac{1}{2}} k \cdot \theta_j - \frac{1}{2} n^{-1} k^t \Sigma_j k + r_j(n^{-\frac{1}{2}}k)) \Big] \\
&= \exp\, [ik\cdot\theta\sqrt{n} - \frac{1}{2}k^t \Sigma k] \qquad (5.50) \\
&\qquad \times \exp\, [\sum_{j=1}^{n} in^{-\frac{1}{2}} k \cdot [\theta_j - \theta] - \frac{1}{2} n^{-1} k^t \sum_{j=1}^{n} [\Sigma_j - \Sigma] k] \exp \sum_{j=1}^{n} r_j(n^{-\frac{1}{2}}k).
\end{aligned}
$$

From (5.45) we have

$$\Big| \sum_{j=1}^{n} in^{-\frac{1}{2}} k\cdot[\theta_j - \theta] \Big| \le n^{-\frac{1}{2}} |k| \sum_{j=1}^{n} |\theta_j - \theta| = O(n^{-\frac{1}{2}}|k|), \qquad (5.51)$$

and using (5.48) we obtain

$$\Big| \frac{k^t}{2n} \sum_{j=1}^{n} [\Sigma_j - \Sigma] k \Big| \le \frac{|k|^2}{2n} \sum_{j=1}^{n} |\Sigma_j - \Sigma| = O\Big(\frac{|k|^2}{n} \sum_{m=1}^{\infty} (m \wedge n) m b_m \Big). \qquad (5.52)$$

Finally we use (5.11) to get

$$\sum_{j=1}^{n} |r_j(n^{-\frac{1}{2}}k)| \le O\Big(\frac{|k|^2}{n} \sum_{j=1}^{n} d_j \Big) + O\Big(\frac{|k|^3}{n^{3/2}} \sum_{j=1}^{n} a_j \Big). \qquad (5.53)$$

This proves the bound in Theorem 5.1(c) for $n \ge e^{\frac{|k|^2}{\delta}}$. The bound holds trivially for $n \le e^{\frac{|k|^2}{\delta}}$ by writing

$$
\begin{aligned}
ik \cdot \frac{(\omega_n - n\theta)}{\sqrt{n}} + \frac{1}{2} k^t \Sigma k &= O(|k|^2 + |k| n^{\frac{1}{2}}) = O(|k|^2 n^{-1}(D_n + E_n) n + |k| n^{-\frac{1}{2}} n) \\
&= O(|k|^2 n^{-1} (D_n + E_n) e^{\delta^{-1}|k|^2} + |k| n^{-\frac{1}{2}} e^{\delta^{-1}|k|^2}).
\end{aligned}
$$

□

6 Bounds on the lace expansion

In this section, we give bounds on the lace expansion coefficients, and verify that these bounds imply Theorems 2.1, 2.2 and 2.3. We start in Section 6.1 by formulating some general bounds on $\hat{\pi}_m(0), \nabla\hat{\pi}_m(0)$ and $\nabla^2\hat{\pi}_m(0)$ that will reduce the bounds on the derivatives of $\hat{\pi}_m(k)$ to a single bound, which we will prove separately for each model. In Section 6.2, we prove the bounds on the lace expansion coefficients for once edge-reinforced random walk with drift, and complete the proof of Theorem 2.1. In Section 6.3, we prove the bounds on the lace expansion coefficients for excited random walk, and complete the proof of Theorems 2.2–2.3. In Section 6.4 we give the corresponding results for the random walk in partially random environment.

6.1 Reduction to a single bound

Recall (3.24) and the definition $\mathcal{A}_{m,N} = \{(j_1, \dots, j_N) \in \mathbb{Z}_+^N : \sum_{l=1}^N j_l = m - N - 1\}$, and define

$$\pi_m^{(N)}(x,y) = \sum_{\vec{j}\in\mathcal{A}_{m,N}} \sum_{\vec{\omega}_1^{(0)}} \sum_{\vec{\omega}_{j_1+1}^{(1)}} \cdots \sum_{\vec{\omega}_{j_N+1}^{(N)}} I_{\{\omega_{j_N}^{(N)}=x, \omega_{j_N+1}^{(N)}=y\}} D(\omega_1^{(0)}) \prod_{n=1}^{N} \Delta_{j_n+1}^{(n)} \prod_{i_n=0}^{j_n-1} p^{\vec{\omega}_{j_{n-1}+1}^{(n-1)} \circ \vec{\omega}_{i_n}^{(n)}} \left(\omega_{i_n}^{(n)}, \omega_{i_n+1}^{(n)}\right), \tag{6.1}$$

so that

$$\pi_m^{(N)}(y) = \sum_x \pi_m^{(N)}(x,y). \tag{6.2}$$

We also let

$$\pi_m(x,y) = \sum_{N=1}^{\infty} \pi_m^{(N)}(x,y). \tag{6.3}$$

The starting point for the bounds on the lace expansion coefficients for self-interacting random walks is the following proposition:

Proposition 6.1 (Reduction of the bounds on the expansion coefficients)**.**
For a self-interacting stochastic process with range $L < \infty$, where $\pi_m^{(N)}(y)$ is given by (3.24), the following bounds hold:

$$\hat{\pi}_m(0) = 0, \tag{6.4}$$

$$|\nabla\hat{\pi}_m(0)| \leq \sqrt{d}L \sum_{x,y} |\pi_m(x,y)|, \tag{6.5}$$

$$|\nabla^2\hat{\pi}_m(0)| \leq (dL)^2(2m-1) \sum_{x,y} |\pi_m(x,y)|, \tag{6.6}$$

$$|\hat{\pi}_m(k)| \leq |k| L \sum_{x,y} |\pi_m(x,y)|, \tag{6.7}$$

$$\big|\hat{\pi}_m(k) - k\cdot\nabla\hat{\pi}_m(0)\big| \leq |k|^2 m L^2 \sum_{x,y} |\pi_m(x,y)|, \tag{6.8}$$

$$\big|\hat{\pi}_m(k) - k\cdot\nabla\hat{\pi}_m(0) - \frac{1}{2}k\nabla^2\hat{\pi}_m(0)k^t\big| \leq |k|^3 m^2 L^3 \sum_{x,y} |\pi_m(x,y)|. \tag{6.9}$$

Proof. We note that for every $x \in \mathbb{Z}^d$ and $N \geq 1$,

$$\sum_y \Delta_{j_N+1}^{(N)} I_{\{\omega_{j_N}^{(N)}=x, \omega_{j_N+1}^{(N)}=y\}} = \sum_y \left(p^{\vec{\omega}_{j_{N-1}+1}^{(N-1)} \circ \vec{\omega}_{j_N}^{(N)}}(x,y) - p^{\vec{\omega}_{j_N}^{(N)}}(x,y)\right) = 1 - 1 = 0$$

from which it follows immediately that, for every $x \in \mathbb{Z}^d$,

$$\sum_y \pi_m(x,y) = 0. \tag{6.10}$$

Summing (6.10) over x establishes (6.4). Furthermore, again by (6.10), we have that

$$\begin{aligned}[\nabla\hat{\pi}_m(0)]_l &= i\sum_y y_l\pi_m(y) = i\sum_{x,y} y_l\pi_m(x,y) = i\sum_{x,y} x_l\pi_m(x,y) + i\sum_{x,y}[y_l - x_l]\pi_m(x,y) \\ &= i\sum_{x,y}[y_l - x_l]\pi_m(x,y). \end{aligned}\tag{6.11}$$

For walks with range L, we have that $|y_j - x_j| \leq L$, so that

$$|[\nabla\hat{\pi}_m(0)]_l| \leq L\sum_{x,y}|\pi_m(x,y)|, \tag{6.12}$$

which establishes (6.5) since $\sum_{l=1}^d u_l^2 \leq d\max_l |u_l|^2$. Similarly,

$$\begin{aligned}-[\nabla^2\hat{\pi}_m(0)]_{st} &= \sum_y y_sy_t\pi_m(y) = \sum_{x,y} y_sy_t\pi_m(x,y) \\ &= \sum_{x,y} x_sx_t\pi_m(x,y) + \sum_{x,y}[y_s - x_s]x_t\pi_m(x,y) \\ &\quad + \sum_{x,y}[y_t - x_t]x_s\pi_m(x,y) + \sum_{x,y}[y_s - x_s][y_t - x_t]\pi_m(x,y) \\ &= \sum_{x,y}[y_s - x_s]x_t\pi_m(x,y) + \sum_{x,y}[y_t - x_t]x_s\pi_m(x,y) + \sum_{x,y}[y_s - x_s][y_t - x_t]\pi_m(x,y). \end{aligned}$$

We use that $|y_j - x_j| \leq L$ and $|x_j| \leq L(m-1)$ to obtain

$$|[\nabla^2\hat{\pi}_m(0)]_{st}| \leq (2m-1)L^2\sum_{x,y}|\pi_m(x,y)|.$$

This establishes (6.6) by (5.1).

By (6.10),

$$\hat{\pi}_m(k) = \sum_{x,y} e^{ik\cdot y}\pi_m(x,y) = \sum_{x,y} e^{ik\cdot x}[e^{ik\cdot(y-x)} - 1]\pi_m(x,y). \tag{6.13}$$

Since $|x - y| \leq L$, this immediately yields (6.7). Together with (6.11), (6.13) gives

$$\begin{aligned}\hat{\pi}_m(k) - k\cdot\nabla\hat{\pi}_m(0) &= \sum_{x,y}\Big[[1 + (e^{ik\cdot x} - 1)][e^{ik\cdot(y-x)} - 1] - ik\cdot(y-x)\Big]\pi_m(x,y), \\ &= \sum_{x,y}[e^{ik\cdot x} - 1][e^{ik\cdot(y-x)} - 1]\pi_m(x,y) + \sum_{x,y}[e^{ik\cdot(y-x)} - 1 - ik\cdot(y-x)]\pi_m(x,y). \end{aligned}$$

By Lemma 4.2, $|e^{iu} - 1| \leq |u|$ and $|e^{iu} - 1 - iu| \leq \frac{1}{2}u^2$. Together with the finite range properties of the walk this proves (6.8). The final claim is proved similarly by first showing that

$$\begin{aligned}\hat{\pi}_m(k) - k\cdot\nabla\hat{\pi}_m(0) - \frac{1}{2}k\nabla^2\hat{\pi}_m(0)k^t = \sum_{x,y}\Big[&[e^{ik\cdot(y-x)} - 1 - ik\cdot(y-x) + \frac{1}{2}k(y-x)^t(y-x)k^t] \\ &+ [e^{ik\cdot x} - 1 - ik\cdot x][e^{ik\cdot(y-x)} - 1] \\ &+ ik\cdot x[e^{ik\cdot(y-x)} - 1 - ik\cdot(y-x)]\Big]\pi_m(x,y), \end{aligned}$$

and then using $|e^{iu} - 1 - iu + \frac{1}{2}u^2| \leq \frac{1}{6}|u|^3$ together with the previous estimates. □

We conclude that the bounds in (4.1), (4.2), (5.2) and (5.3) follow if we can show that

$$\sum_{x,y} |\pi_m(x,y)| \leq \varepsilon_\beta b_m, \tag{6.14}$$

for some sequence $\{b_m\}_{m\geq 1}$ satisfying the appropriate conditions formulated in Assumptions (LLN) and (CLT). In the following proposition, we state the precise form of our bounds on the lace expansion coefficients, followed by the proof of our main results subject to these bounds.

Proposition 6.2 (Bounds on the expansion coefficients for each of our models)**.**
(a) For OERRWD, there exist $\beta_0 > 0$ and $\mathcal{J} > 0$ such that for all $|\beta| \leq \beta_0$,

$$\sum_{x,y} |\pi_m(x,y)| \leq C\beta e^{-\mathcal{J}m}, \tag{6.15}$$

where $\mathcal{J}$ depends on β_0, w_0, d but is independent of β.
(b) For ERW with $d-1 > 4$, there exist $\beta_0 > 0$ and $C > 0$ such that for all $\beta \leq \beta_0$

$$\sum_{x,y} |\pi_m(x,y)| \leq \frac{C\beta}{(m+1)^{\frac{d-3}{2}}}. \tag{6.16}$$

(c) For RWpRE with $d_1 > 4$, there exist $\beta_0 > 0$ and $C > 0$ such that for all $\beta \leq \beta_0$

$$\sum_{x,y} |\pi_m(x,y)| \leq \frac{C\beta}{(m+1)^{\frac{d_1-2}{2}}}. \tag{6.17}$$

We now complete the proofs of our main results subject to Proposition 6.2:

Proof of Theorem 2.1 subject to Proposition 6.2(a). We use Proposition 6.1 and 6.2 as well as Theorem 5.1 to complete the proof of Theorem 2.1. When $b_m = e^{-\mathcal{J}m}$, Assumption (CLT) is satisfied. Also, (2.11) is directly implied by (5.6). Furthermore, the error terms in (5.7) can all be seen to be $O(1)$, which proves (2.12). Finally, for each k, as $n \to \infty$ (5.8) implies that

$$\mathbb{E}_\beta[e^{ik\cdot(\omega_n - \theta n)/\sqrt{n}}] \to e^{-\frac{1}{2}k^t\Sigma k}. \tag{6.18}$$

Clearly, this implies (2.13). □

Proof of Theorems 2.2 and 2.3 subject to Proposition 6.2(b). By Propositions 6.1 and 6.2(b), Assumption (LLN) holds with $\varepsilon_\beta = C\beta$ and $b_m = (m+1)^{-(d-3)/2}$ when $d > 5$ (i.e. $d-1 > 4$). Thus, Theorem 4.1 applies, and it is an easy exercise to see that when $b_m = (m+1)^{-(d-3)/2}$ and $d > 5$, the error terms given in (4.4) and (4.5) are sufficient to prove Theorem 2.3.

Similarly, Propositions 6.1 and 6.2(b) show that Assumption (CLT) holds with $\varepsilon_\beta = C\beta$ and $b_m = (m+1)^{-(d-3)/2}$ when $d > 7$ (i.e. $d-1 > 6$). Thus, Theorem 5.1 applies and we now show that when $b_m = (m+1)^{-(d-3)/2}$ and $d > 8$, the error terms given in (5.7) and (5.8) are sufficient to prove Theorem 2.2.

Indeed, note that $E_n = \sum_{m=1}^\infty (m \wedge n) m b_m = O(n^{-(d-9)/2}\log n) = o(n)$, when $d > 7$. Furthermore, by [22, Lemma 3.2] and the fact that $(d-3)/2 > 1$ when $d > 5$, we obtain that $d_n = O(n^{-(d-5)/2})$, so that $D_n = \sum_{m=1}^n d_m = O(1)$ for $d > 7$. Finally, $a_n = \sum_{m=1}^n m^{2+\gamma} b_m = O(n^{\gamma-(d-9)/2} \vee 1)$, which is $O(n^c)$ for some $c < 1/2$ when $d > 8$ (i.e. $d-1 > 7$) and γ is sufficiently small. In this case, also $A_n = \sum_{m=1}^n a_m = o(n^{3/2})$. This identifies all error terms in (5.7) and (5.8). □

Proof of Theorems 2.4 and 2.5 subject to Proposition 6.2(c). Theorems 2.4 and 2.5 follow exactly as in the proofs of Theorems 2.2 and 2.3, when $d_1 > 7$ and $d_1 > 4$, respectively.

□

We will prove (6.14) for once-edge-reinforced random walk in Section 6.2 and for excited random walk in Section 6.3 below.

6.2 Bounds for once-edge-reinforced random walk

In this section we prove Proposition 6.2(a). The bounds in this section are based on the following large deviations estimates.

Lemma 6.1 (Large deviations). *Whenever $\theta_\varnothing \neq 0$, there exist $\beta_0 = \beta_0(D(\cdot), w_0(\cdot)) > 0$ and $\mathcal{I} = \mathcal{I}(D(\cdot), w_0(\cdot)) > 0$ such that for all $|\beta| \leq \beta_0$,*

$$\sup_{\vec{\eta}} \mathbb{Q}_\beta^{\vec{\eta}}(\omega_n = \omega_0) \leq e^{-\mathcal{I}n}, \quad \textit{and} \tag{6.19}$$

$$\sup_{z, \vec{\omega}^{(i-2)}_{j_{i-2}+1}} \sum_{\vec{\omega}^{(i-1)}_{j_{i-1}+1}} \mathbb{Q}_\beta^{\vec{\omega}^{(i-2)}_{j_{i-2}+1}}(\vec{\omega}_{j_{i-1}} = \vec{\omega}^{(i-1)}_{j_{i-1}}) I_{\{\omega^{(i-1)}_{j_{i-1}} = z\}} \mathbb{Q}_\beta^{\vec{\omega}^{(i-1)}_{j_{i-1}+1}}(\omega_{j_i} = \omega^{(i-1)}_l) \leq K e^{-\mathcal{I}(j_{i-1} - l + j_i)}, \tag{6.20}$$

where the supremum is over all $(j_{i-2}+1)$-step random walk paths $\vec{\omega}^{(i-2)}_{j_{i-2}+1}$, and K is a constant that depends only on L, d. The law of the i^{th} walk $\vec{\omega}^{(i)}_{j_i+1}$ depends on the $(i-1)^{\text{st}}$ walk $\vec{\omega}^{(i-1)}_{j_{i-1}+1}$ but not on the $(i-2)^{\text{nd}}$ walk $\vec{\omega}^{(i-2)}_{j_{i-2}+1}$.

Proof. Under $\mathbb{Q}_0$, $\vec{\omega}$ is a simple random walk with bounded increments and non-zero drift (without loss of generality assume the drift is in the positive coordinate direction(s)). It follows that for z sufficiently small (and negative)

$$\mathbb{E}_{\mathbb{Q}_0}[\exp\{z \cdot (\omega_1 - \omega_0)\}] = 1 + z \cdot \mathbb{E}_{\mathbb{Q}_0}[\omega_1 - \omega_0] + O(z^2 L^2) < 1. \tag{6.21}$$

Thus, by Cramér's Theorem (e.g. see [8, Theorem 2.2.30]) there exists $J = J(D(\cdot), w_0(\cdot)) > 0$ such that $\mathbb{Q}_0(\omega_n = \omega_0) \leq e^{-Jn}$ for all n. Let Ω denote the support of $D(x)$. It is easy to show that for every $\beta \in [0, \beta_0]$ and $\vec{\nu}$,

$$p^{\vec{\nu}}(x, y) \leq (1 + C\beta_0)\, D(y - x) \tag{6.22}$$

when $C \geq 1/w_0(x, y)$ (similarly for $\beta \in [-\beta_0, 0]$ when $C \geq (|\Omega| - 1)/(\sum_{u \sim x} w_0(x, u) - \beta_0(|\Omega| - 1))$.

By translation invariance, $w_0(\cdot) \geq W$ is uniformly bounded from below as a function on Ω. We fix

$$\begin{aligned} C &\geq \max\left\{ \frac{|\Omega| - 1}{\frac{1}{2}\sum_{u \sim 0} w_0(0, u)}, \sup_{y \sim 0} \frac{1}{w_0(0, y)} \right\}, \quad \text{and} \\ \beta_0 &\leq \min\left\{ \frac{\sum_{u \sim 0} w_0(0, u)}{2(|\Omega| - 1)}, J/(2C) \right\}, \end{aligned} \tag{6.23}$$

where the constant $C > 0$ shall be determined in the course of the proof, and recall that

$$\mathbb{Q}_\beta^{\vec{\eta}}(\omega_n = x) = \sum_{\vec{\omega}_n : \omega_n = x} \prod_{i=0}^{n-1} p^{\vec{\omega}_i \circ \vec{\eta}}(\vec{\omega}_{i+1} - \vec{\omega}_i). \tag{6.24}$$

The bound (6.19) with $\mathcal{I} = J/2$ follows immediately from this by (6.22) by choosing β_0 sufficiently small so that $\log(1 + C\beta_0) \leq J/2$.

The second bound is obtained similarly, using (6.22) after the l^{th} step of $\vec{\omega}^{(i-1)}$, with the constant arising from the "missing" transition probability corresponding to the sum over $\omega^{(i-1)}_{j_{i-1}+1}$. This proves (6.20) with $\mathcal{I} = J/2$. □

Proof of Proposition 6.2(a). We bound $\sum_{x,y} |\pi^{(N)}_m(x,y)|$ and sum the resulting bound over N. For $N = 1$, $m \geq 2$, (6.1) and (3.28) give

$$\begin{aligned}\sum_{x,y} |\pi^{(1)}_m(x,y)| &\leq \sum_{x,y} \sum_{\omega^{(0)}_1} D(\omega^{(0)}_1) \sum_{\vec{\omega}^{(1)}_{m-1}} \mathbb{Q}^{\vec{\omega}^{(0)}_1}_\beta (\vec{\omega}_{m-2} = \vec{\omega}^{(1)}_{m-2}) |\Delta^{(1)}_{m-1}| I_{\{\omega^{(1)}_{m-2}=x\}} I_{\{\omega^{(1)}_{m-1}=y\}} \\ &\leq C\beta \mathbb{Q}_\beta(\omega_{m-2} = \omega_0) \leq C\beta e^{-\mathcal{I}(m-2)} \leq C\beta e^{-\mathcal{I}m}, \end{aligned} \tag{6.25}$$

where we have applied the first bound of Lemma 6.1 in the last line, and the value of C changes from place to place.

For general N, we have that

$$\sum_{x,y} |\pi^{(N)}_m(x,y)| \leq \sum_{\vec{j} \in \mathcal{A}_{m,N}} \sum_{\vec{\omega}^{(0)}_1} \sum_{\vec{\omega}^{(1)}_{j_1+1}} \cdots \sum_{\vec{\omega}^{(N)}_{j_N+1}} D(\omega^{(0)}_1) \prod_{n=1}^{N} |\Delta^{(n)}_{j_n+1}| \mathbb{Q}^{\vec{\omega}^{(n-1)}_{j_{n-1}+1}}_\beta (\vec{\omega}_{j_n} = \vec{\omega}^{(n)}_{j_n}), \tag{6.26}$$

where, by (3.28),

$$\sum_{\omega^{(i)}_{j_i+1}} |\Delta^{(i)}_{j_i+1}| \leq C\beta \sum_{l_{i-1}=0}^{j_{i-1}} I_{\{\omega^{(i)}_{j_i} = \omega^{(i-1)}_{l_{i-1}}\}}. \tag{6.27}$$

Let $N \geq 2$, and for $q \in \{0,1\}$ let $A_q = \{i \leq N : (N-i) \mod 2 = q\}$ and B_q be the set of $\vec{j} \in \mathcal{A}_{m,N}$ such that $\sum_{i \in A_q} (j_i + 1) \geq m/2$. For $r = 0, \ldots, N-1$, denote by $l_r \leq j_r$ the number of steps in the r^{th} walk $\vec{\omega}^{(r)}_{j_r+1}$ up to the intersection point as in (6.27) (in particular, $l_0 = 0$). Then, combining (6.26) and (6.27),

$$\sum_{x,y} |\pi^{(N)}_m(x,y)| \leq (C\beta)^N \sum_{\vec{j} \in \mathcal{A}_{m,N}} \sum_{\vec{l}} \sum_{\vec{\omega}^{(0)}_1} \sum_{\vec{\omega}^{(1)}_{j_1+1}} \cdots \sum_{\vec{\omega}^{(N)}_{j_N+1}} D(\omega^{(0)}_1) \prod_{n=1}^{N} I_{\{\omega^{(n)}_{j_n} = \omega^{(n-1)}_{l_{n-1}}\}} \mathbb{Q}^{\vec{\omega}^{(n-1)}_{j_{n-1}+1}}_\beta (\vec{\omega}_{j_n} = \vec{\omega}^{(n)}_{j_n}). \tag{6.28}$$

The bound is now split into four cases, depending on whether N is even or odd, and on whether $\vec{j} \in B_0$ or $\vec{j} \in B_1 \setminus B_0$. See Figure 2.

(a) The bound for N even and $\vec{j} \in B_0$. When N is even we bound the contribution to (6.26) from $\vec{j} \in B_0$ by using the following two bounds, the first of which follows immediately from the second bound of Lemma 6.1, while the last holds (with equality) trivially.

The first fact is that for each even $i \in [2, N]$, uniformly in $\vec{\omega}^{(i-2)}_{j_{i-2}+1}$,

$$\sum_{\vec{\omega}^{(i-1)}_{j_{i-1}+1}} I_{\{\omega^{(i-1)}_{j_{i-1}} = \omega^{(i-2)}_{l_{i-2}}\}} \mathbb{Q}^{\vec{\omega}^{(i-2)}_{j_{i-2}+1}}_\beta (\vec{\omega}_{j_{i-1}} = \vec{\omega}^{(i-1)}_{j_{i-1}}) \sum_{\vec{\omega}^{(i)}_{j_i+1}} I_{\{\omega^{(i)}_{j_i} = \omega^{(i-1)}_{l_{i-1}}\}} \mathbb{Q}^{\vec{\omega}^{(i-1)}_{j_{i-1}+1}}_\beta (\vec{\omega}_{j_i} = \vec{\omega}^{(i)}_{j_i}) \leq C e^{-\mathcal{I}(j_i + (j_{i-1} - l_{i-1}))}. \tag{6.29}$$

The second fact is that,

$$\sum_{\vec{\omega}^{(0)}_1} D(\omega^{(0)}_1) \leq 1. \tag{6.30}$$

Figure 2: Illustration of the four cases of the diagrammatic bounds for OERRWD. On the left N is even ($N = 4$) and on the right N is odd ($N = 5$). In the first row $\vec{j} \in B_0$, while in the second, $\vec{j} \in B_1 \setminus B_0$. In each case the thick lines indicate the loops (whose total length is of order half of the total length m of the diagram) that give an exponentially small bound.

By successive applications of (6.29) and lastly (6.30), when N is even we obtain a bound on the contribution to (6.26) from $\vec{j} \in B_0$ (whence $\sum_{i \le N, \text{even}} j_i \ge (m - N)/2$), of

$$(C\beta)^N \sum_{\vec{j} \in B_0} \sum_{\vec{l}} \prod_{2 \le i \le N, \text{ even}} e^{-\mathcal{I}(j_i + (j_{i-1} - l_{i-1}))} \le (C\beta)^N e^{-\mathcal{I}m/2} \sum_{\vec{j} \in B_0} \sum_{\vec{l}} \prod_{2 \le i \le N, \text{ even}} e^{-\mathcal{I}(j_{i-1} - l_{i-1})}, \tag{6.31}$$

where the constant has changed (to accommodate a factor $e^{\mathcal{I}N/2}$). Using the fact that there are at most $j_i + 1$ possible values $\{0, 1, \ldots, j_i\}$ for l_i, this is bounded above by

$$(C\beta)^N e^{-\mathcal{I}m/2} \sum_{\vec{j}} \prod_{i=1}^{N} (j_i + 1), \tag{6.32}$$

which in turn can be bounded by the integral

$$e^{-\mathcal{I}m/2} (C\beta)^N \int_0^{m+3} x_1 \int_0^{m+3-x_1} x_2 \cdots \int_0^{m+3-(x_1+\cdots+x_{N-1})} x_N dx_N \cdots dx_1. \tag{6.33}$$

It is an easy exercise in integration by parts that

$$\int_0^{a - \sum_{i=1}^{j-1} x_i} \frac{x_j}{(2(N-j))!} \left(a - \sum_{i=1}^{j} x_i \right)^{2(N-j)} dx_j = \frac{1}{(2(N-(j-1)))!} \left(a - \sum_{i=1}^{j-1} x_i \right)^{2(N-(j-1))}. \tag{6.34}$$

Applying (6.34) N times, we bound (6.33) by

$$e^{-\mathcal{I}m/2} (C\beta)^N \frac{(m+3)^{2N}}{(2N)!} \le e^{-\mathcal{I}m/2} C\beta (C\beta)^{N/2} \frac{(m+3)^{2N}}{(2N)!}. \tag{6.35}$$

(b) The bound for N even and $\vec{j} \in B_1 \setminus B_0$. When N is even we bound the contribution to (6.26) from $\vec{j} \in B_1 \setminus B_0$ by using the following three facts, the first of which is obtained by simply evaluating the sum, while the second and third follow immediately from Lemma 6.1.

The first fact is that uniformly in $\vec{\omega}^{(N-1)}_{j_{N-1}+1}$,

$$\sum_{\vec{\omega}^{(N)}_{j_N+1}} I_{\{\omega^{(N)}_{j_N}=\omega^{(N-1)}_{l_{N-1}}\}} \mathbb{Q}_\beta^{\vec{\omega}^{(N-1)}_{j_{N-1}+1}}(\vec{\omega}_{j_N} = \vec{\omega}^{(N)}_{j_N}) \leq C, \tag{6.36}$$

where the constant (which depends only on L, d) is a result of summing over $\sum_{\omega^{(N)}_{j_N+1}}$. The second fact is that for each odd $i \in [3, N-1]$, uniformly in $\vec{\omega}^{(i-2)}_{j_{i-2}+1}$, (6.29) holds. The third fact is that

$$\sum_{\vec{\omega}^{(0)}_1} D(\omega^{(0)}_1)\mathbb{Q}^{\vec{\omega}^{(0)}_1}(\omega^{(1)}_{j_1} = \omega^{(0)}_{l_0}) \leq e^{-\mathcal{I}j_1} \tag{6.37}$$

By first applying (6.36), followed by successive applications of (6.29) and lastly (6.37), when N is even we obtain a bound on the contribution to (6.26) from $\vec{j} \in B_1$, of

$$\begin{aligned}(C\beta)^N \prod_{1\leq i\leq N-1,\ \text{odd}} \sum_{\vec{l}} e^{-\mathcal{I}(j_i+(j_{i-1}-l_{i-1}))} &\leq (C\beta)^N e^{-\mathcal{I}m/2} \sum_{\vec{j}\in B_1} \sum_{\vec{l}} \prod_{1\leq i\leq N-1,\ odd} e^{-\mathcal{I}(j_{i-1}-l_{i-1})} \\ &\leq (C\beta)^N e^{-\mathcal{I}m/2} \sum_{\vec{j}} \prod_{i=1}^{N} (j_i+1),\end{aligned} \tag{6.38}$$

which is bounded by (6.35) just as in the previous case.

(c),(d) The bounds for N odd. The bounds for $N \geq 3$ odd are similar to the bounds described above, and we will omit the details. When N is odd, we bound the contribution from $\vec{j} \in B_0$ by using the bound (6.29) successively for each $i \in [3, N]$, and finally (6.37). For $\vec{j} \in B_1 \setminus B_0$, we use (6.36), (6.29) and finally (6.30). In both cases we obtain the same bound (6.35).

To complete the proof of Proposition 6.2(a), we sum (6.35) over $N \geq 2$, giving at most

$$C\beta e^{-(\mathcal{I}/2-(C\beta)^{1/4})m}. \tag{6.39}$$

Choosing β_0 sufficiently small so that $(C\beta)^{1/4} \leq \mathcal{I}/4$ for all $|\beta| \leq \beta_0$, we have that (6.39) is bounded by $C\beta e^{-\mathcal{J}m}$, where $\mathcal{J} = \mathcal{I}/4$ is independent of β. □

6.3 Bounds for excited random walk

In this section we prove Proposition 6.2(b).

In bounding the diagrams arising from the expansion applied to excited random walk, we will make use of the following lemma, in which $\mathbb{Q}^{\vec{\eta}}$ denotes the law of an excited random walk with history $\vec{\eta}$, where $\vec{\eta}$ is a finite path:

Lemma 6.2. *For excited random walk in $d > 2$ dimensions,*

$$\sup_{x,\vec{\eta}} \mathbb{Q}^{\vec{\eta}}(\omega_m = x) \leq \frac{C}{(m+1)^{\frac{d-1}{2}}}. \tag{6.40}$$

Proof. Let $Y_m = \#\{k \le m : \omega_k \notin \{\omega_{k-1} \pm e_1\}\}$ denote the number of steps taken in the dimensions $2, \dots, d$ by the excited random walk up to time m. Note that for excited random walk and simple random walk, Y_n has the same distribution. Then $Y_m \sim Bin(m, q)$ where $q = (d-1)/d > \frac{1}{2}$ for $d > 2$, and standard large deviations estimates give $\mathbb{P}(Y_m < m/2) \le e^{-mI}$ for some $I > 0$.

Now for each $\vec{\eta}$, with endpoint u,

$$\mathbb{Q}^{\vec{\eta}}(\omega_m = x) \le \mathbb{Q}^{\vec{\eta}}(\omega_m^{[2,\dots,d]} = x^{[2,\dots,d]}) = \mathbb{P}_u(\omega_m^{[2,\dots,d]} = x^{[2,\dots,d]}), \tag{6.41}$$

where $\mathbb{P}_u$ denotes the law of a simple random walk starting at u. For m even, this is bounded by

$$\begin{aligned}
\mathbb{P}_0(\omega_m^{[2,\dots,d]} = 0^{[2,\dots,d]}) &\le \sum_{k=m/2}^{m} \mathbb{P}_0(\omega_m^{[2,\dots,d]} = 0^{[2,\dots,d]} | Y_m = k)\mathbb{P}(Y_m = k) + \mathbb{P}(Y_m < m/2) \\
&\le \sum_{k=m/2}^{m} \frac{C}{(k+1)^{\frac{d-1}{2}}} \mathbb{P}(Y_m = k) + e^{-Im} \\
&\le \frac{C}{(m+1)^{\frac{d-1}{2}}} \sum_{k=m/2}^{m} \mathbb{P}(Y_m = k) + e^{-Im} \le \frac{C}{(m+1)^{\frac{d-1}{2}}}.
\end{aligned} \tag{6.42}$$

For m odd, (6.41) is bounded by $2d\mathbb{P}_0(\omega_{m+1}^{[2,\dots,d]} = 0^{[2,\dots,d]})$ and we proceed as in (6.42). □

Recall that $\mathcal{A}_{m,N} \equiv \{\vec{j} \in \mathbb{Z}_+^N : \sum j_i = m - N - 1\}$, and that for $N \ge 1$,

$$\begin{aligned}
\sum_{x,y} |\pi_m^{(N)}(x,y)| \le &(C\beta)^N \sup_{\vec{\eta}} \sum_{\vec{j} \in \mathcal{A}_{m,N}} \sum_{\vec{l}} \sum_{\vec{\omega}_1^{(0)}} \sum_{\vec{\omega}_{j_1+1}^{(1)}} \cdots \sum_{\vec{\omega}_{j_N+1}^{(N)}} p^{\vec{\eta}}(u, \omega_1^{(0)}) \prod_{n=1}^{N} I_{\{\omega_{j_n}^{(n)} = \omega_{l_{n-1}}^{(n-1)}\}} \mathbb{Q}_\beta^{\vec{\omega}_{j_{n-1}+1}^{(n-1)}}(\vec{\omega}_{j_n} = \vec{\omega}_{j_n}^{(n)}) \\
= &(C\beta)^N \sup_{\vec{\eta}} \Pi_m^{(N),\vec{\eta}},
\end{aligned} \tag{6.43}$$

where u is the endpoint of the finite path $\vec{\eta}$, and we take this expression as the definition of $\Pi_m^{(N),\vec{\eta}}$. See the top diagram in Figure 3.

Proposition 6.3 (Bounds on the expansion coefficients for ERW)**.** *For excited random walk with $d > 5$, the following bound holds:*

$$\sum_{x,y} |\pi_m^{(N)}(x,y)| \le \frac{(C\beta)^N}{(m+1)^{\frac{d-3}{2}}}. \tag{6.44}$$

In view of (6.43), the conclusion of Proposition 6.3 follows immediately from the following lemma:

Lemma 6.3. *For $d > 5$, there exists C independent of β such that*

$$\sup_{u,\vec{\eta}} \Pi_m^{(N),\vec{\eta}} \le \frac{C^N}{(m+1)^{\frac{d-3}{2}}}. \tag{6.45}$$

Proof. We first prove by induction on $N \ge 1$ that

$$\sum_{\vec{j} \in \mathcal{A}_{m,N}} \sum_{l_1,\dots,l_{N-1}} \sum_{\vec{\omega}_{j_1+1}^{(1)}} \cdots \sum_{\vec{\omega}_{j_N+1}^{(N)}} \prod_{n=1}^{N} I_{\{\omega_{j_n}^{(n)} = \omega_{l_{n-1}}^{(n-1)}\}} \mathbb{Q}_\beta^{\vec{\omega}_{j_{n-1}+1}^{(n-1)}}(\vec{\omega}_{j_n} = \vec{\omega}_{j_n}^{(n)}) \le \frac{C^N}{(m+1)^{\frac{d-3}{2}}}. \tag{6.46}$$

Figure 3: A diagrammatic representation of $\Pi_m^{(N),\vec{\eta}}(u)$ for $N=5$, followed by the decomposition of the diagram when $j_1 > m/2$ and when $j_1 \le m/2$ respectively. In each case, the induction hypothesis is applied to the subdiagram of length $m-(j_1+1)$ that excludes the first walk, and the required decay comes from the part of the diagram with thick lines.

For $N=1$, $j_1 = m-2$ and (6.46) is less than or equal to

$$\sup_{v,\vec{\eta}} \sum_{\vec{\omega}^{(1)}_{m-1}} I_{\{\omega^{(1)}_{m-2}=v\}} \mathbb{Q}^{\vec{\eta}}_{\beta}(\vec{\omega}_{m-2} = \vec{\omega}^{(1)}_{m-2}) = C \sup_{v,\vec{\eta}} \mathbb{Q}^{\vec{\eta}}_{\beta}(\omega_{m-2}=v) \le \frac{C}{(m+1)^{\frac{d-1}{2}}}, \tag{6.47}$$

where the first constant arises from the sum over $\omega^{(1)}_{m-1}$.

For $N \ge 2$, (6.46) is bounded by

$$\begin{aligned}
&\sup_{v,\vec{\eta}} \sum_{j_1 \le m-2} \sum_{l_1 \le j_1} \sum_{\vec{\omega}^{(1)}_{j_1+1}} I_{\{\omega^{(1)}_{j_1}=v\}} \mathbb{Q}^{\vec{\eta}}_{\beta}(\vec{\omega}_{j_1} = \vec{\omega}^{(1)}_{j_1}) \\
&\quad \times \sum_{\vec{j}' \in \mathcal{A}_{m-j_1-1,N-1}} \sum_{l_2,\dots,l_{N-1}} \sum_{\vec{\omega}^{(2)}_{j_2+1}} \cdots \sum_{\vec{\omega}^{(N)}_{j_N+1}} \prod_{n=2}^{N} I_{\{\omega^{(n)}_{j_n}=\omega^{(n-1)}_{l_{n-1}}\}} \mathbb{Q}_{\beta}^{\vec{\omega}^{(n-1)}_{j_{n-1}+1}}(\vec{\omega}_{j_n} = \vec{\omega}^{(n)}_{j_n}) \\
&\le \sup_{v,\vec{\eta}} \sum_{j_1 \le m-2} \frac{C^{N-1}}{(m-j_1-1)^{\frac{d-3}{2}}} \sum_{l_1 \le j_1} \sum_{\vec{\omega}^{(1)}_{j_1+1}} I_{\{\omega^{(1)}_{j_1}=v\}} \mathbb{Q}^{\vec{\eta}}_{\beta}(\vec{\omega}_{j_1} = \vec{\omega}^{(1)}_{j_1}) &(6.48)\\
&\le \sum_{j_1 \le m-2} \frac{C^{N-1}}{(m-j_1)^{\frac{d-3}{2}}} j_1 \frac{C'}{(j_1+1)^{\frac{d-1}{2}}} \le C^{N-1} \sum_{j_1 \le m-2} \frac{1}{(m-j_1)^{\frac{d-3}{2}}} \frac{C'}{(j_1+1)^{\frac{d-3}{2}}}, &(6.49)
\end{aligned}$$

using the induction hypothesis to get (6.48) and (6.47) to get (6.49). The result (6.46) now follows by splitting the sum over j_1 into the cases $j_1 \le m/2+1$ and $j_1 > m/2+1$, taking the term of order $m^{-(d-3)/2}$ outside the sum and performing the remaining finite (since $d > 5$) sum.

To prove (6.45), for $N=1$, we have from (6.43) and Lemma 6.2 that

$$\begin{aligned}
\Pi^{(1),\vec{\eta}}_m &= \sum_{\vec{\omega}^{(0)}_1} \sum_{\vec{\omega}^{(1)}_{m-2}} p^{\vec{\eta}}(u, \vec{\omega}^{(0)}_1) I_{\{\omega^{(1)}_{m-3}=\omega^{(0)}_0\}} \mathbb{Q}_{\beta}^{\vec{\omega}^{(0)}_1}(\vec{\omega}_{m-3} = \vec{\omega}^{(1)}_{m-3}) \\
&\le \sup_{v,\vec{\eta}'} \mathbb{Q}^{\vec{\eta}}_{\beta}(\omega_{m-3}=v) \sum_{\omega^{(1)}_{m-2}} 1 \le \frac{C}{(m+1)^{\frac{d-1}{2}}}, &(6.50)
\end{aligned}$$

uniformly in $\vec{\eta}$, which initialises the induction hypothesis.

For $N \geq 2$, and for any $\vec{\eta}$, proceeding as in the proof of (6.46),

$$\Pi_m^{(N),\vec{\eta}} \leq \sum_{j_1 \leq m-2} \sum_{l_1 \leq j_1} \sum_{\vec{\omega}_1^{(0)}} \sum_{\vec{\omega}_{j_1+1}^{(1)}} p^{\vec{\eta}}(u, \vec{\omega}_1^{(0)}) I_{\{\omega_{j_1}^{(1)} = \omega_0^{(0)}\}} \mathbb{Q}_\beta^{\vec{\omega}_1^{(0)}}(\vec{\omega}_{j_1} = \vec{\omega}_{j_1}^{(1)})$$

$$\times \sum_{\vec{j}' \in \mathcal{A}_{m-(j_1+1),N-1}} \sum_{\vec{l}'} \sum_{\vec{\omega}_{j_2+1}^{(2)}} \cdots \sum_{\vec{\omega}_{j_N+1}^{(N)}} \prod_{n=2}^{N} I_{\{\omega_{j_n}^{(n)} = \omega_{l_{n-1}}^{(n-1)}\}} \mathbb{Q}_\beta^{\vec{\omega}_{j_{n-1}+1}^{(n-1)}}(\vec{\omega}_{j_n} = \vec{\omega}_{j_n}^{(n)}) \tag{6.51}$$

$$\leq C^{N-1} \sum_{j_1 \leq m-N-1} \frac{1}{(m-j_1)^{\frac{d-3}{2}}} j_1 \frac{C'}{(j_1+1)^{\frac{d-1}{2}}}, \tag{6.52}$$

using (6.50) and (6.46). The result follows as for (6.49).

This completes the proof of Lemma 6.3, and hence also Proposition 6.3. □

6.4 Bounds for random walk in a partially random environment

In this section we prove Proposition 6.2(c), proceeding similarly to the excited random walk case. The main ingredient needed is the analogue of Lemma 6.2 for RWpRE, which is the following Lemma.

Lemma 6.4. *For RWpRE with $d_1 \geq 1$ dimensions,*

$$\sup_{x,\vec{\eta}} \mathbb{Q}^{\vec{\eta}}(\omega_m = x) \leq \frac{C}{(m+1)^{\frac{d_1}{2}}}. \tag{6.53}$$

Proof. Let $Y_m = \#\{k \leq m : \omega_k \notin \{\omega_{k-1} \pm e_i, i = 1, \dots, d_0\}\}$ denote the number of steps taken in the dimensions $d_0+1, \dots d$ by the RWpRE up to time m. Then there exists a sequence of random variables $Y'_m \sim Bin(m, \delta)$ such that $Y'_m \leq Y_m$ for all m, and $\mathbb{P}(Y_m < m\frac{\delta}{2}) \leq \mathbb{P}(Y'_m < m\frac{\delta}{2}) \leq e^{-Im}$ for some $I > 0$, by standard large deviations estimates. Now proceed as in the proof of Lemma 6.2 to get the result. □

Since for RWpRE as defined in Section 2.3, the Δ factors satisfy the same bounds (3.29) as excited random walk (3.27), the analysis continues exactly as in Section 6.3 except that the exponents have changed in Proposition 6.3 and Lemma 6.3 from $\frac{d-3}{2} = \frac{d-1}{2} - 1$ to $\frac{d_1}{2} - 1$. In the inductive analysis we then use the fact that when $\frac{d_1}{2} - 1 > 1$ (i.e. $d_1 > 4$),

$$\sum_{j \leq m-2} \frac{1}{(m-j)^{\frac{d_1}{2}-1}} j \frac{1}{(j+1)^{\frac{d_1}{2}}} \leq \frac{C}{(m+1)^{\frac{d_1}{2}-1}}.$$

6.5 Discussion of the bounds

In the examples given in this paper, an estimate of the form

$$\sup_{\vec{\eta},x} \mathbb{Q}^{\vec{\eta}}(\omega_m = x) \leq A(m) \tag{6.54}$$

is crucially used in bounding the diagrams, where $A(m)$ is decreasing sufficiently rapidly in m. In the case of the reinforced random walk with drift, Cramér's Theorem enabled such a result with $A(m)$ exponentially small in m. For excited random walk, the simple random walk behaviour in all but the first dimension gave such a result with $A(m) = (m+1)^{-(d-1)/2}$. Similarly for random walk in partially random environment with $A(m) = (m+1)^{-d_1/2}$. In these examples, we ignore considerable information contained in the expansion in order to bound certain quantities arising from the expansion in terms of diagrams. In the case of excited random walk, we bounded these diagrams using very simple, but non-optimal estimates. The diagrammatic estimates are used to verify a set of non-optimal assumptions under which the central limit theorem holds. Improvements in any of these areas could lead to a reduction in the dimension above which our methods imply a central limit theorem for excited random walk. We note that different bounds, valid for *all* $\beta \in [0,1]$, are proved in [19] for ERW in order to prove monotonicity of $\beta \mapsto \theta(\beta, d)$ when $d \geq 9$.

The approach taken above works more generally. We can obtain a LLN and CLT for any translation invariant self-interacting random walk model that has the properties that

(1) $p^{\vec{\eta}_m \circ \vec{x}_n}(x_n, x_{n+1}) - p^{\vec{x}_n}(x_n, x_{n+1}) \neq 0 \Rightarrow x_n \in \vec{\eta}_m$,

(2) this difference in transition probabilities is small (uniformly) for all possible histories, and

(2) the walker is "sufficiently transient" (uniformly) for all possible histories,

can be handled in the same way as we have handled the models above. For an explicit example, one can take an (annealed) multi-cookie random walk in an i.i.d. random cookie environment with multi-dimensional excitement, provided that there are $d_1 > 4$ (sufficiently transient) coordinates where the walker is behaving as a simple random walk.

It would require a great advance in our understanding and analysis of the recursion equation, in order for us to apply this methodology to a "non-repulsive" model such as the once reinforced random walk. Inductive arguments as in [18, 22] have been used rather successfully for oriented percolation [23], the contact process [20], and various related problems. However all of these made crucial use of the self-repellent nature of the problems involved.

7 Proof of the variance formula in Theorem 3.2

Multiplying both sides of (2.29) by $x^{[i]}x^{[j]} = (x^{[i]} - y^{[i]} + y^{[i]})(x^{[j]} - y^{[j]} + y^{[j]})$ and summing over x we obtain,

$$\begin{aligned}
\mathbb{E}[\omega^{[i]}_{n+1}\omega^{[j]}_{n+1}] =& \sum_y y^{[i]}y^{[j]}D(y)\sum_x c_n(x-y) + \sum_y y^{[i]}D(y)\sum_x (x^{[j]}-y^{[j]})c_n(x-y) \\
&+ \sum_y y^{[j]}D(y)\sum_x (x^{[i]}-y^{[i]})c_n(x-y) + \sum_y D(y)\sum_x (x^{[i]}-y^{[i]})(x^{[j]}-y^{[j]})c_n(x-y) \\
&+ \sum_{m=2}^{n+1}\sum_y y^{[i]}y^{[j]}\pi_m(y)\sum_x c_{n+1-m}(x-y) + \sum_{m=2}^{n+1}\sum_y y^{[i]}\pi_m(y)\sum_x (x^{[j]}-y^{[j]})c_{n+1-m}(x-y) \\
&+ \sum_{m=2}^{n+1}\sum_y y^{[j]}\pi_m(y)\sum_x (x^{[i]}-y^{[i]})c_{n+1-m}(x-y) \\
&+ \sum_{m=2}^{n+1}\sum_y \pi_m(y)\sum_x (x^{[i]}-y^{[i]})(x^{[j]}-y^{[j]})c_{n+1-m}(x-y). \qquad (7.1)
\end{aligned}$$

Since $\sum_x c_n(x) = 1 = \sum_y D(y)$ and $\sum_y \pi_m(y) = 0$, many terms simplify, so that (7.1) becomes

$$\begin{aligned}\mathbb{E}[\omega_{n+1}^{[i]}\omega_{n+1}^{[j]}] = &\sum_y y^{[i]}y^{[j]}D(y) + \sum_y y^{[i]}D(y)\sum_x x^{[j]}c_n(x) + \sum_y y^{[j]}D(y)\sum_x x^{[i]}c_n(x) \\ &+ \sum_x x^{[i]}x^{[j]}c_n(x) + \sum_{m=2}^{n+1}\sum_y y^{[i]}y^{[j]}\pi_m(y) + \sum_{m=2}^{n+1}\sum_y y^{[i]}\pi_m(y)\sum_x x^{[j]}c_{n+1-m}(x) \\ &+ \sum_{m=2}^{n+1}\sum_y y^{[j]}\pi_m(y)\sum_x x^{[i]}c_{n+1-m}(x) \\ =&\mathbb{E}[\omega_1^{[i]}\omega_1^{[j]}] + \mathbb{E}[\omega_1^{[i]}]\mathbb{E}[\omega_n^{[j]}] + \mathbb{E}[\omega_1^{[j]}]\mathbb{E}[\omega_n^{[i]}] + \mathbb{E}[\omega_n^{[i]}\omega_n^{[j]}] \\ &+ \sum_{m=2}^{n+1}\sum_y y^{[i]}y^{[j]}\pi_m(y) + \sum_{m=2}^{n+1} a_m^{[i]}\mathbb{E}[\omega_{n+1-m}^{[j]}] + \sum_{m=2}^{n+1} a_m^{[j]}\mathbb{E}[\omega_{n+1-m}^{[i]}]. \end{aligned} \tag{7.2}$$

Turning this into a statement about covariances we have, with $C(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$ denoting the covariance between the random variables X and Y,

$$\begin{aligned} C(\omega_{n+1}^{[i]},\omega_{n+1}^{[j]}) - C(\omega_n^{[i]},\omega_n^{[j]}) =&\mathbb{E}[\omega_1^{[i]}\omega_1^{[j]}] + \sum_{m=2}^{n+1}\sum_y y^{[i]}y^{[j]}\pi_m(y) \\ &+ \mathbb{E}[\omega_1^{[i]}]\mathbb{E}[\omega_n^{[j]}] + \mathbb{E}[\omega_1^{[j]}]\mathbb{E}[\omega_n^{[i]}] - \mathbb{E}[\omega_{n+1}^{[i]}]\mathbb{E}[\omega_{n+1}^{[j]}] + \mathbb{E}[\omega_n^{[i]}]\mathbb{E}[\omega_n^{[j]}] \\ &+ \sum_{m=2}^{n+1} a_m^{[i]}\mathbb{E}[\omega_{n+1-m}^{[j]}] + \sum_{m=2}^{n+1} a_m^{[j]}\mathbb{E}[\omega_{n+1-m}^{[i]}]. \end{aligned} \tag{7.3}$$

If the right hand side converges then by the first condition of (3.43) it must converge to Σ_{ij}, since the left hand side summed from $n=0$ to $k-1$ is $C(\omega_k^{[i]},\omega_k^{[j]})$. Note that if $\mathbb{E}[\omega_n] = 0$ for each n then the last two lines of (7.3) are zero and the claimed result then follows immediately (with $\theta^{[i]} = 0$ for all i). Otherwise we need to show that under the conditions of (3.43) and (ii) the right hand side of (7.3) converges to that of (3.44).

Use the relationship

$$\mathbb{E}[\omega_{n+1}^{[i]}] = \mathbb{E}[\omega_1^{[i]}] + \sum_{m=2}^{n+1} a_m^{[i]} + \mathbb{E}[\omega_n^{[i]}] \equiv \theta_{n+1}^{[i]} + \mathbb{E}[\omega_n^{[i]}],$$

to see that

$$\begin{aligned}\mathbb{E}[\omega_{n+1}^{[i]}]\mathbb{E}[\omega_{n+1}^{[j]}] =& \left(\theta_{n+1}^{[i]} + \mathbb{E}[\omega_n^{[i]}]\right)\left(\theta_{n+1}^{[j]} + \mathbb{E}[\omega_n^{[j]}]\right) \\ =&\theta_{n+1}^{[i]}\theta_{n+1}^{[j]} + \theta_{n+1}^{[i]}\mathbb{E}[\omega_n^{[j]}] + \theta_{n+1}^{[j]}\mathbb{E}[\omega_n^{[i]}] + \mathbb{E}[\omega_n^{[i]}]\mathbb{E}[\omega_n^{[j]}]. \end{aligned} \tag{7.4}$$

Thus the right hand side of (7.3) is

$$\begin{aligned} &\mathbb{E}[\omega_1^{[i]}\omega_1^{[j]}] + \sum_{m=2}^{n+1}\sum_y y^{[i]}y^{[j]}\pi_m(y) + \sum_{m=2}^{n+1} a_m^{[i]}\mathbb{E}[\omega_{n+1-m}^{[j]}] + \sum_{m=2}^{n+1} a_m^{[j]}\mathbb{E}[\omega_{n+1-m}^{[i]}] \\ &+ \mathbb{E}[\omega_1^{[i]}]\mathbb{E}[\omega_n^{[j]}] + \mathbb{E}[\omega_1^{[j]}]\mathbb{E}[\omega_n^{[i]}] - \theta_{n+1}^{[i]}\theta_{n+1}^{[j]} - \theta_{n+1}^{[i]}\mathbb{E}[\omega_n^{[j]}] - \theta_{n+1}^{[j]}\mathbb{E}[\omega_n^{[i]}] \\ =& \mathbb{E}[\omega_1^{[i]}\omega_1^{[j]}] + \sum_{m=2}^{n+1}\sum_y y^{[i]}y^{[j]}\pi_m(y) + \sum_{m=2}^{n+1} a_m^{[i]}\mathbb{E}[\omega_{n+1-m}^{[j]}] + \sum_{m=2}^{n+1} a_m^{[j]}\mathbb{E}[\omega_{n+1-m}^{[i]}] \\ &- \mathbb{E}[\omega_n^{[j]}]\sum_{m=2}^{n+1} a_m^{[i]} - \mathbb{E}[\omega_n^{[i]}]\sum_{m=2}^{n+1} a_m^{[j]} - \theta_{n+1}^{[i]}\theta_{n+1}^{[j]}. \end{aligned} \tag{7.5}$$

Collecting terms, we can rewrite (7.3) as

$$
\begin{aligned}
C(\omega_{n+1}^{[i]}, \omega_{n+1}^{[j]}) - C(\omega_n^{[i]}, \omega_n^{[j]}) =& \mathbb{E}[\omega_1^{[i]}\omega_1^{[j]}] - \theta_{n+1}^{[i]}\theta_{n+1}^{[j]} + \sum_{m=2}^{n+1} a_m^{[i]} \left(\mathbb{E}[\omega_{n+1-m}^{[j]}] - \mathbb{E}[\omega_n^{[j]}]\right) \\
&+ \sum_{m=2}^{n+1}\sum_y y^{[i]}y^{[j]}\pi_m(y) + \sum_{m=2}^{n+1} a_m^{[j]} \left(\mathbb{E}[\omega_{n+1-m}^{[i]}] - \mathbb{E}[\omega_n^{[i]}]\right) \\
=& \mathbb{E}[\omega_1^{[i]}\omega_1^{[j]}] - \theta_{n+1}^{[i]}\theta_{n+1}^{[j]} - \sum_{m=2}^{n+1} a_m^{[i]} \sum_{r=n+2-m}^{n} \theta_r^{[j]} \\
&+ \sum_{m=2}^{n+1}\sum_y y^{[i]}y^{[j]}\pi_m(y) - \sum_{m=2}^{n+1} a_m^{[j]} \sum_{r=n+2-m}^{n} \theta_r^{[i]}.
\end{aligned}
$$

The right hand side is equal to

$$
\begin{aligned}
&\mathbb{E}[\omega_1^{[i]}\omega_1^{[j]}] - \theta_{n+1}^{[i]}\theta_{n+1}^{[j]} - \sum_{m=2}^{n+1} a_m^{[i]} \sum_{r=n+2-m}^{n} \left(\theta^{[j]} - \sum_{k=r+1}^{\infty} a_m^{[j]}\right) \\
&+ \sum_{m=2}^{n+1}\sum_y y^{[i]}y^{[j]}\pi_m(y) - \sum_{m=2}^{n+1} a_m^{[j]} \sum_{r=n+2-m}^{n} \left(\theta^{[i]} - \sum_{k=r+1}^{\infty} a_k^{[i]}\right) \\
&= \mathbb{E}[\omega_1^{[i]}\omega_1^{[j]}] - \theta_{n+1}^{[i]}\theta_{n+1}^{[j]} - \theta^{[j]} \sum_{m=2}^{n+1} a_m^{[i]}(m-1) + \sum_{m=2}^{n+1}\sum_y y^{[i]}y^{[j]}\pi_m(y) - \theta^{[i]} \sum_{m=2}^{n+1} a_m^{[j]}(m-1) \qquad (7.6) \\
&+ \sum_{m=2}^{n+1} a_m^{[i]} \sum_{r=n+2-m}^{n} \sum_{k=r+1}^{\infty} a_k^{[j]} + \sum_{m=2}^{n+1} a_m^{[j]} \sum_{r=n+2-m}^{n} \sum_{k=r+1}^{\infty} a_k^{[j]}.
\end{aligned}
$$

The first line of the last equality of (7.6) converges to (3.44). It therefore remains to show that the terms on the second line of the last equality of (7.6) converge to zero. Since i and j are arbitrary, it suffices to verify the result for the first term on the second line of the last equality of (7.6). For $n \geq 4$ this term is equal to

$$
\sum_{m=2}^{n+1} a_m^{[i]} \sum_{r=n+2-m}^{n} \sum_{k=r+1}^{\infty} a_k^{[j]} = \sum_{k=2}^{\infty} a_k^{[j]} \sum_{m=(n+3-k)\vee 2}^{n+1} a_m^{[i]}((k-1) + (m-n-1)). \qquad (7.7)
$$

This is bounded in absolute value by

$$
\sum_{k=2}^{\infty} |a_k^{[j]}|(k-1) \sum_{m=(n+3-k)\vee 2}^{n+1} |a_m^{[i]}| + \sum_{k=2}^{\infty} |a_k^{[j]}| \sum_{m=(n+3-k)\vee 2}^{n+1} |a_m^{[i]}|(m-1) \qquad (7.8)
$$

The first term of (7.8) is

$$
\begin{aligned}
&\sum_{k=2}^{\lfloor n/2 \rfloor} |a_k^{[j]}|(k-1) \sum_{m=(n+3-k)\vee 2}^{n+1} |a_m^{[i]}| + \sum_{k=\lfloor n/2 \rfloor+1}^{\infty} |a_k^{[j]}|(k-1) \sum_{m=(n+3-k)\vee 2}^{n+1} |a_m^{[i]}| \\
&\leq \sum_{k=2}^{\infty} |a_k^{[j]}|(k-1) \sum_{m=n/2}^{n+1} |a_m^{[i]}| + \sum_{k=\lfloor n/2 \rfloor+1}^{\infty} |a_k^{[j]}|(k-1) \sum_{m=2}^{\infty} |a_m^{[i]}|, \qquad (7.9)
\end{aligned}
$$

which converges to 0 as $n \to \infty$, since each of these is the tail of a convergent series multiplied by a convergent series. Similarly the second term of (7.7) converges to 0. □

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