Presented at the 334th meeting of the American Institute of Electrical Engineers, New York, November 9, 1917.

AN EXPERIMENTAL METHOD OF OBTAINING THE SOLUTION OF ELECTROSTATIC PROBLEMS WITH NOTES ON HIGH-VOLTAGE BUSHING DESIGN

BY CHESTER W. RICE

ABSTRACT OF PAPER

The electrodynamic method for obtaining the solution of electrostatic and allied problems is developed to a high degree of accuracy. The method is then applied to the study of high-voltage bushings. An experimental high-air-efficiency bushing was built and tested with the result that the arc-over was very materially lower than had been anticipated.

A study was then made to ascertain the reason for this large discrepancy, which was found to be due to an unexpectedly large *surface effect* which varied greatly with different materials.

After obtaining the numerical value for the *surface effect* a reasonably accurate predetermination of the arc-over of structures, in which the stress distribution is known, can be made.

In order to determine the desirability of using artificial equipotential surfaces to increase the efficiency of the use of the supporting dielectric, diagrams were taken and a small bushing of this type constructed and tested.

A study was then made to find out whether the reduction in diameter of condenser bushings is principally due to equalization of potential or due the greater strength of insulation when barriers are used. As a result of this work, it is believed that the barrier effect greatly predominates.

A short discussion follows which shows the difficulties of obtaining a sufficiently exact theory of bushing design to enable us to predetermine the most efficient shape for a practical bushing.

A series of small bushings were made and tested with a view to determining the general shape and characteristics which go to make up a practical all around bushing.

The appendix gives an unexpurgated solution of the following two-flow problems.

I. The distribution of the electrostatic field when any two confocal hyperboloids of revolution of one sheet and of the same family are maintained at given potentials.

II. The distribution of the electrostatic field when any two confocal hyperboloids of revolution of two sheets and of the same family are maintained at given potentials.

INTRODUCTION

IN general, a bushing has to serve two purposes; first, it must support the leading-in wire or conductor and must act as a mechanical unit capable of being removed or replaced in case of damage; secondly, it must insulate the high-potential conductor from the tank which is usually at zero or ground potential. The leading-in wire, or what may be called the high-voltage electrode may have various shapes. The tank with hole in it may be called the grounded electrode and it may be shaped in various ways. A solid supporting dielectric is necessary to make the structure a mechanical unit, that is, to mechanically connect the high-voltage central electrode with the tank or grounded electrode. Between the grounded electrode and the high-voltage electrode, isolated metallic surfaces may be introduced which will form *artificial equipotential surfaces.* The condenser bushing is a familiar example.

The electrical part of the design consists in studying the electrostatic field distribution between the electrodes in order to use the various available insulating materials to their best advantage. It is obvious that there are large numbers of possible electrode shapes and arrangements, the electrostatic field distribution of which should be studied in order to determine the one which is best adapted for commercial use. The following investigation was undertaken with a view of determining the theoretical possibilities of some of the various possible arrangements. The work was started in July 1914, and extended with short interruptions to July 1916. Naturally an investigation of this nature is never complete but I hope that the work to be described will be of interest to engineers and also stimulate others to take up the work so that eventually we will have a better and more complete understanding of this and similar problems.

Preliminary Work

The preliminary work consisted in investigating the possibilities of obtaining a reasonably approximate solution of the bushing problem. The first question which presented itself was the following: Can two-dimensional fields be rotated and used as satisfactory approximations for three-dimensional problems? In order to test out this matter the literature was searched for a solution of the rod and torus problem, so as to be able to compare it with the solution obtained by rotating the field of two parallel wires. If the approximate solution, as obtained by the rotation of the plane figure, did not prove useful as a result of comparison with the exact solution, it was thought that possibly some simple law of distortion or stretching might be arbitrarily imposed upon the rotating-plane diagram which would change the lines of force and equipotential surfaces so as to closely conform with the accurate three-dimensional solution. If such a procedure was found possible, it was hoped that the same method could be used in transforming the many available two-dimensional problems into approximate solutions of the related three-dimensional figures of revolution.

The problem of insulating two parallel wires was studied in order to compare some of the various criterions which might be advanced for the correct and strongest surface along which to place the insulating material. For this preliminary work the question of different inductive capacities was neglected. The following surfaces were studied by graphical construction on a large sized diagram of the field between parallel wires drawn for equal tubes of electrostatic flux and equal differences of potential.

I. —Constant surface flux density.
Equal areas between lines of flux on equipotential surfaces.

II. —Surface of constant potential gradient.
Equal distances along lines of force between equipotential surfaces.

III.—Surface of constant volume energy density.
Surface defined by unit cells of equal volume.

IV.—Surface such that the component of the potential gradient along the surface has a definite limited value, for example say, 50 per cent of the gradient. The object in studying such a surface is apparent if we assume that the surface of the insulation introduces a weakening effect. For in that case the component of the gradient tangent to the surface must not exceed the breakdown strength of the surface, whereas the actual gradient may be equal to the dielectric strength of the surrounding dielectric.

V. —Surfaces of constant creepage.
Equal distances between equipotential surfaces.

The surfaces defined in I, II and III were seen to be identical as would be expected.

The corresponding surfaces were drawn for the solid figure resulting from the rotation of a right section of the plane figure. Of course, such a procedure does not give the solution of the torus problem, but as explained above it was done in order that the result might be compared with the true solution of that problem.

A search of the literature failed to yield a useful solution of the torus problem. C. L. Fortescue and S. W. Farnsworth[1] state that the smooth lines (in Fig. 4 of the paper) show theoretical equipotential surfaces of indicated potential for the given terminals. Inspection of the figure shows that the smooth surfaces are those cut out by revolving a family of circles about the axis of the rod which would yield a set of anchor rings, the orthogonal surfaces would then be a set of spheres. That this cannot be a solution of the torus problem is stated by W. E. Byerly.[2] "Indeed no possible distribution can make our anchor rings or our spheres a set of equipotential surfaces." It is therefore, evident that their solution and the calculations given in their curves Figs. 13, 14, 18 and 19 must have been the result of some sort of an approximation. It seems to me that a discussion would have added to the interest and clearness of their paper.

A study of the solution of the torus problem, as outlined by Byerly and also given by Hicks,[3] convinced me that the difficulties of calculating sufficient points for the construction of a diagram of the field would be very great, and even if the solution were available, it would not be of great assistance in solving the bushing problem because the rod and torus does not constitute a self-supporting structure resembling a bushing. It was realized, at this time, that if the solution of an infinite rod passing perpendicularly through a hole in a plane were available, it would be of considerable value as this would constitute the simplest form of bushing.

As a result of this preliminary work, the difficulty of obtaining even approximate mathematical solutions of such electrostatic problems was brought out.[4]

Various experimental methods were, therefore, looked into with the hope of obtaining a method which would enable us to obtain experimentally the solution of any desired electrostatic problem.

---

1. "Air as an Insulator when in the Presence of Insulating Bodies of Higher Specific Inductive Capacity." A. I. E. E. TRANS., 1913, Vol. XXXII, Part I, p. 893.

2. Fourier's Series and Spherical Harmonics page 265.

3. Toroidal Functions, Philosophical Transactions of the Royal Society, Part III, pages 608–562, 1881.

4. In this connection it is interesting to read what Maxwell has to say about such problems; Electricity and Magnetism. Vol. I, page 177–178.

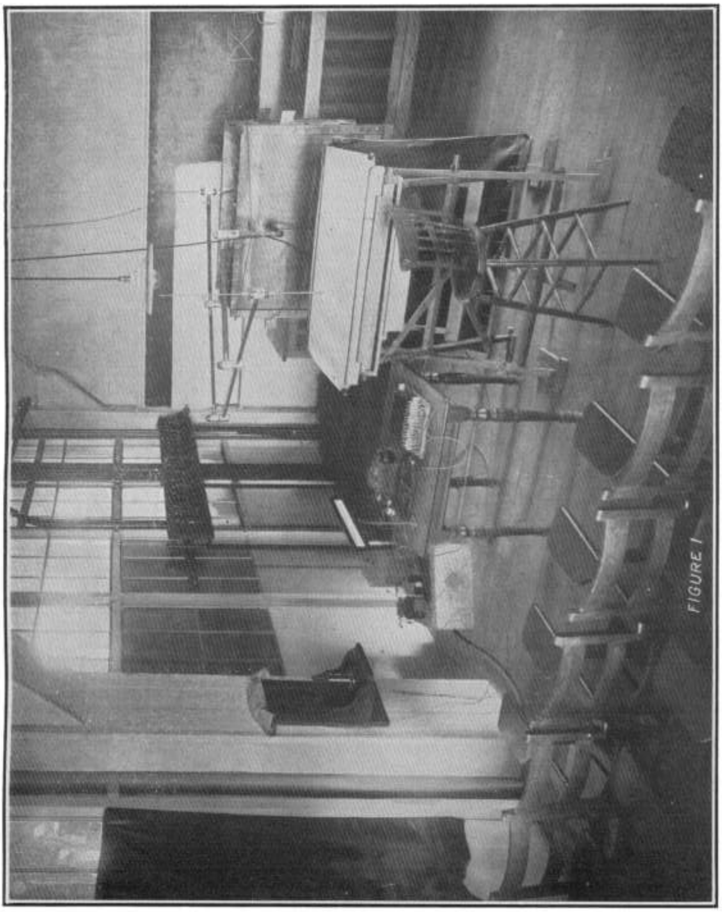

Fig. 1

[RICE]

The desired diagrams can be obtained in a variety of ways[5] some of which are enumerated below:

I. Directly calculated by the process of cut and try. This method is outlined by Karapetoff[6] from which I quote in part; "In order to calculate the permittance (capacity) of a given dielectric, or to find the flux densities and stresses in different parts of it, proceed as follows: The field is mapped out into small cells by lines of force and equipotential surfaces, drawing them to the best of ones judgment, the total permittance is calculated by properly combining the permittances of the cells in series and in parallel. Then the assumed directions are somewhat modified, the permittance is calculated again, and so on, until by successive trials the positions of the lines of force are found with which the permittance becomes a maximum." The method of successive approximations was systematized and used by Lord Rayleigh.[7]

While theoretically possible in all cases this method is very laborious even for problems in two dimensions and for three-dimensional problems it becomes still more exasperating, as will be readily discovered by anyone who tries it. A considerable assistance in the application of this method is rendered by experimentally obtaining the approximate direction of the lines of force by the well known method of using mica filings, or better fine needle-shaped pieces of glass which can be obtained by grinding up glass wool or fabric.

II. Obtain experimentally the isothermal surfaces in the related heat-flow problem. Experimental difficulties such as radiation and conduction, obviously make this method impractical.

III. Obtain the equipotential surfaces in the equivalent electrical conduction problem[8] or what I have termed the electrodynamic method.

---

5. For references to the numerous articles on this subject I will refer the reader to those contained in the excellent article on this same subject by John F. H. Douglas, A. I. E. E., TRANS. 1915, Vol. XXXIV, Part I, page 1067, "The Reluctance of Some Irregular Magnetic Fields."

6. Electric Circuit, pages 160–163.

7. See Phil. Trans. Royal Society, 1871, p. 77, "On the Theory of Resonance," also "Theory of Sound," Vol. II, p. 171.

8. I understand from Mr. Douglas' paper "The Reluctance of Some Irregular Magnetic Fields", A. I. E. E., TRANS., 1915, Vol. XXXIV, Part I, p. 1081, that Kirchoff first proposed such a method in 1845, and was subsequently modified and employed by many investigators. Re-

The possibilities of this method appeared very attractive and it was, therefore, selected as the one best suited for the present purposes.

THE ELECTRODYNAMIC METHOD

The method consists in obtaining the equipotential surfaces for any desired shape of electrodes from the exactly analogous conduction problem in an electrolyte. Thus the chosen electrode shapes are placed in an electrolyte and alternating current passed between them. The equipotential surfaces are then obtained by an exploring point connected through a quadrant electrometer to a definite known potential with regards to that between the electrodes. The locus of the points of zero potential difference as thus read by the electrometer constitutes the desired equipotential surface.

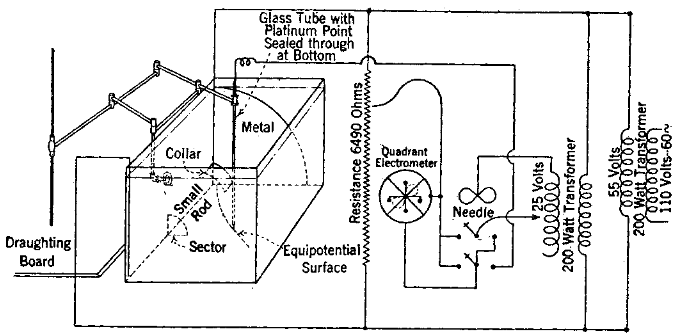

FIG. 2

The electrodes under investigation were placed in a large box made of paraffine treated wood and filled with an electrolyte—ordinary city water being found most convenient. The box was bolted together on the outside as will be seen in Fig. 1, thus eliminating all metal from contact with the electrolyte in the box. A small transformer (200 watt) was employed to step down the 110-volt, 60-cycle lighting circuit to 55 volts which was used as supply for the tests. A non-inductive resistance of 6490 ohms, divided into 50 equal parts, was shunted across the mains between the electrodes. The exploring pointer which was carried by the pantograph consisted of a slender glass tube

cently C. L. Fortescue and S. W. Farnsworth, A. I. E. E., TRANS., 1913, Vol. XXXII, Part I, p. 893, "Air as an Insulator when in the Presence of Insulating Bodies of Higher Specific Inductive Capacity", as well as Mr. J. F. H. Douglas have employed this method.

with a small platinum wire sealed through at the end. The glass tube was slipped over a metal tube carried by the pantograph and the insulated wire brought out from the platinum point through the metal tube. The metal tube was used for mechanical stiffening and the glass tube with platinum point so as to avoid electrical leakage. Any defect in the insulation such as a crack in the glass tube being easily detected.

The pointer was connected to one pair of quadrants of the electrometer, the other pair being connected to a point of known potential on the resistance *R*, Fig. 2. The needle of the electrometer had a metallic suspension and was kept at a definite potential above that of the quadrants by the transformer as shown in the illustration. This method of excitation gives a constant sensitivity regardless of the point at which connection is made to the resistance *R*. It also makes it possible to change the sensitivity of the instrument by varying the potential applied to the needle. For example, in exploring the field between a given pair of electrodes we can apply a certain low potential to the needle when exploring the dense part of the field, and when exploring the weak part of the field we can raise the potential applied to the needle by selecting a tap on the exciting transformer. Thus, we can maintain equal accuracy in all portions of the field.

Some of the advantages of using the arrangement described above are as follows: The use of alternating current eliminates polarization, to a large extent, and automatically gives "reversed" readings. The use of a quadrant electrometer has the advantage that it takes practically no energy to operate, and it is a good zero instrument since when a point has been nearly located the difference in potential between the quadrants is small in comparison with that of the needle and under these conditions the deflection is very nearly proportional to the potential difference being measured.[9]

The fact that these experiments were carried out on a large scale combined with the small energy necessary to operate the electrometer made it possible to use ordinary city water as the electrolyte which is obviously a great convenience.

---

9. For the theory of the quadrant electrometer reference may be made to Maxwell, "Electricity and Magnetism," Vol. I, page 338; Jeans, "Electricity and Magnetism," Vol. I, page 108; J. J. Thomson, "Electricity and Magnetism, Vol. I, p. 97–103.

Method of Procedure

In applying the electrodynamic method to problems having symmetry about an axis of revolution, a quadrant or octant of the actual figure was studied because in this manner the size of the electrodes used could be made larger and, therefore, the accuracy correspondingly improved. Skeleton electrodes were used because of the obvious ease with which they can be cut out of tin or other metal, and for the further reason that if it ever becomes possible to obtain a mathematical solution of such problems the boundary conditions would be of as simple geometry as possible. Of course in applying the resulting diagrams to the design of any piece of apparatus the electrode shape which would actually be built and used would conform to one of the experimentally obtained equipotential surfaces at some distance

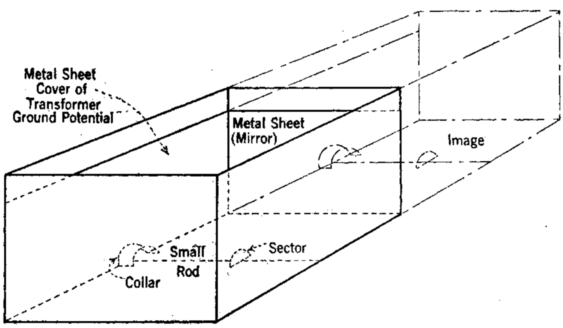

FIG. 3

from the actual skeleton electrodes. Under some circumstances it would be desirable to employ actual models of the desired electrodes.

The method of images is obviously applicable and is of considerable value in improving the accuracy and simplifying many problems. For example, if we wish to study the disturbing influence of one bushing in proximity to another, when they are at opposite potentials, we can set up the experiment as shown in Fig. 3. In this manner we can determine the maximum stress between the two bushings and the minimum stress on the outside of the bushing. Of course a diagram taken in this manner is no longer a plane section through a figure of rotation.

Theoretically there is no difficulty in obtaining the solution of electrostatic problems involving materials of different inductive capacities by this method, as we may employ electrolytes or materials having the proper relative conductances. Practically, however, there are obvious difficulties in obtaining materials having suitable characteristics especially for the case of three-dimensional problems.

In Fig. 4A is illustrated the use of a high-resistance substance such as carbon or a silicon clay composition, etc., as high inductive capacity material immersed in a suitable electrolyte to represent the surrounding air or low inductive capacity material.

Another method is to use an insulating diaphram to separate two electrolytes of different conductivities one inside and the

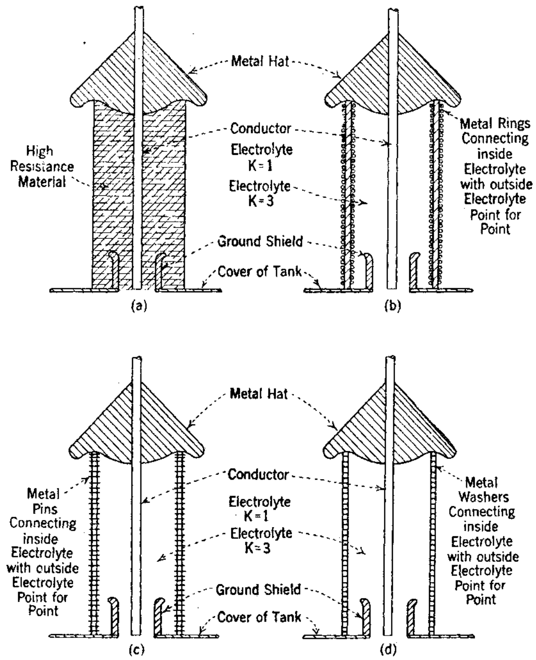

FIG. 4

other outside of the shell. In Fig. 4B a series of wire rings are used which are connected together metallically. In Fig. 4C, the same method is shown except that metal pins are substituted to connect the outside with the inside point for point. Still another and possibly the simplest method of carrying out the same principle is shown in Fig. 4D and was suggseted to me by Mr. G. B. Shanklin. It consists in building up the separating shell of alternate metallic and insulating washers. It will be observed that the methods employing the wire rings or washers are only applicable to problems having circular symmetry.

It is difficult in all of these methods to obtain the equipotential surfaces inside of the containing shell, although this can be accomplished in some instances. For example, in the case of a bushing, by placing the quarter section inverted at the top of the large wooden box somewhat as shown in Fig. 5, the exploring plane would then be at the surface of the electrolyte.

It will be seen that in this simple apparatus we have a most remarkable calculating machine which is able to obtain the solution of Laplace's equation and automatically calculate and plot the results for boundary conditions which as yet have not been obtained by analytical methods. The accuracy merely depends upon the scale and care with which the experiments are carried out. For example, in the experiments here described the potential of any point on the full sized diagram, 3 ft. by 4 ft. (90 cm. by 120 cm.) could be determined inside of a pin head. Of course, the diagrams are not accurate to this extent due to the fact that the electrodes used were large compared with the size of the containing box, and, therefore, the edges of the diagrams show large distortions, as will be readily seen by comparing the experimental diagram Fig. 9 with the mathematical solution for the similar case Fig. 10. The first two diagrams are also subject to slight pantograph errors arising from a lack of rigidity and accuracy of the original pantograph. The later diagrams are more accurate as a new pantograph was constructed of large thin walled steel tubing and provided with ball bearings throughout.

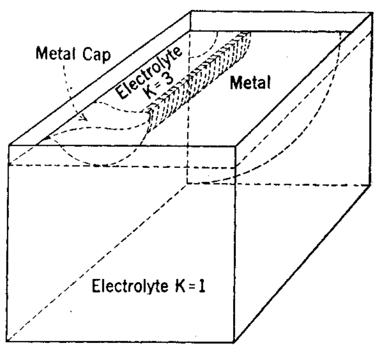

FIG. 5

Before taking a diagram the inside of the paraffine treated wooden box was painted over with a mixture of beeswax and rosin to eliminate or reduce to a minimum any conductivity of the wood.

Completing the Diagrams.

After obtaining an experimental diagram giving the equipotential surfaces for equal or unit differences of potential, in the manner described above, the diagram may be completed by drawing in the family of orthogonal surfaces which divide the field into unit tubes of electrostatic flux. Besides being a great help to the eye in using the diagrams this process furnishes a graphical check on the accuracy of the experimental work. If it is found possible to draw in an orthogonal family of surfaces so that the elementary condensers, cut out between the equipotential surfaces and tubes of electrostatic flux, have the same capacity throughout the field, we know that our diagram is the one and only solution of that particular electrostatic problem. For it is known that there is one and only one solution of any problem in electrostatics. In the second place, the following two conditions must be fulfilled when the field is mapped out in unit tubes of electrostatic flux, and in equipotential surfaces for every unit difference of potential.

I.—The tubes of flux must intersect the lines of force everywhere in the field at right angles, for if this were not true there would be a component of the potential gradient (or electric intensity) along an equipotential surface which is impossible in a static field.

II.—The unit cells, or elementary condensers, which are cut out by the intersection of unit tubes of flux (or force) and unit equipotential surfaces must have the same capacity everywhere in the field. The correctness of this condition can be seen by the following reasoning.

All the elementary condensers between two adjacent flux lines, that is, cut out by a unit tube of flux, are in series and if they were not of equal capacity they would not divide the potential difference equally between them. This is contrary to the assumption that the field is to be divided up into equipotential surfaces for every unit difference of potential.

Again we see that the elementary condensers between any two adjacent equipotential surfaces are in parallel and have unit impressed electromotive force across them. Therefore, if they were not of equal capacity they would not divide the flux equally between them ($\varphi = C\,e$), which would be contrary to the condition that the field is to be divided into equal tubes of flux.

The process of drawing in the lines of force was briefly as follows: An appropriate height for the unit cell was selected near the axis of symmetry of the diagram, a perpendicular [10] was

10. A convenient method of erecting the perpendiculars is to employ a small plane mirror held, for example by a block of wood at right angles to the drawing board. When the part of the equipotential line which is seen in the mirror forms a continuous smooth curve with the part of the equipotential line which is viewed directly, perpendicularity exists and a line can be drawn using the mirror as a straight edge.

erected from this point on the equipotential surface half way to the next one. The perpendicular was then drawn to the next equipotential surface which cut the first perpendicular half way between the two equipotential surfaces. The latter perpendicular was extended half way to the next equipotential surface and the process continued,—in this follow your nose fashion,—until an entire flux line was completed. A smooth curve was then drawn through the intersection of the perpendiculars. After completing a pair of lines in this manner the mean heights of the elementary condensers or unit cells were calculated assuming a constant value of capacity. These mean heights were then plotted on the diagram and it was inspected to see whether the flux line, obtained by perpendiculars, was a good mean curve through all these points. At the beginning of a diagram these points were generally high and low in an erratic manner as would be expected. Another flux line was then obtained by the method of perpendiculars, the smooth curve drawn through the intersection and the mean heights calculated as before and compared with the drawn in curve. In this manner the flux lines were built up one upon another. After completing six or seven flux lines it was generally found that the calculated heights of the cells were showing a slight consistent deviation from the curve as obtained by perpendiculars in some part of the field. The constant error was then distributed throughout all of the flux lines so as to make the two criterions, namely, perpendicularity and equal capacities, come into as harmonious agreement as possible. The work of drawing in more flux lines was then continued and the various processes repeated until the entire diagram was completed.

The fact that it was possible to satisfactorily complete the above process is considered a good check on the accuracy of the experimental work.

Determination of Capacities from the Diagrams

After completing the diagrams by dividing up the field into elementary cells, we can calculate the capacity of any desired part of the field by properly combining the elementary cylindrical condensers in series and in parallel. For example, in the diagram, Fig. 12, we may desire to calculate the capacity assuming a single dielectric between equipotential surfaces 0 to 36 and bounded by flux lines 0 to 54. Here we have 54 of our elementary condensers in parallel and 36 in series. Now since the small cylindrical condensers are all of the same capacity C_u we can write as the total capacity of the part of the circuit under consideration

$$C = \frac{54}{36} C_u$$

or,

$$C = \frac{m}{n} C_u$$

where m = number of elementary cells in parallel.
n = number of elementary cells in series.
C_u = the capacity of the unit cell or the capacity of the elementary cylindrical condenser assumed for the particular diagram.

C_u will obviously depend upon the assumed inductive capacity and size of the diagram.

In selecting the mean height of the first cell which determines the capacity of the unit cells no particular value was chosen, the spacing for the flux lines being selected only for convenience in construction. They are not therefore, strictly speaking, unit cells.

It is obviously necessary to assume some limit to the extent of the field in making any calculation on a field of this type for, if we assumed the central rod with cap infinite in extent as well as an infinite plane with hole and collar, the capacity of the entire circuit would be infinite.

A check on the accuracy of the capacity as calculated from any of the diagrams could be obtained from a measurement of the resistance of the circuit as set up in the large box combined with a determination of the specific resistance of the electrolyte used. I am sorry that I neglected to accurately record this data when taking the various diagrams.

In the case of the diagram of Fig. 28, the following resistance measurements were retained.

Artificial Equipotential Surfaces

Voltage applied, 60-cycle	54.2	volts.
Current	1.39	amperes.
Resistance	39.0	ohms.

Artificial Equipotential Surfaces Removed

Voltage applied, 60-cycle	54.2	volts
Current	1.30	amperes.
Resistance	41.7	ohms.

The resistivity of the electrolyte used (Schenectady city water) was determined from a sample taken from the box during the experiment and was found to be approximately 3000 ohm-cm. at 25 deg. cent.

A comparison of the measured resistance of the circuit with and without artificial equipotential surfaces shows that by their introduction we have lowered the resistance of the circuit or, if considering the capacity, we have increased the capacity by the addition of artificial equipotential surfaces.

It will be readily seen that in those cases where we are merely interested in determining the resistance to flow for any of the problems which obey the Fourier-Ohm law, such as the flow of an ideal incompressible fluid, heat, or the so-called electrostatic and magnetic fluxes and current, we merely have to set up the desired electrodes in an electrolyte of known specific resistance and make a determination of resistance. The model used in the tests need not be the same size as the actual piece of apparatus which it is desired to study but may be either a magnified or reduced image. If the linear dimensions are all n times as large as the original then the conductance will be n times larger than it would be for the original model and inversely. A simple and also general method of obtaining the relation between the resistance in ohms as determined from the conduction experiments and the electrostatic capacity of the equivalent electrostatic problem is to compare the two cases for parallel plane electrodes at close spacing when neglecting all edge effect.

The well known expression for the capacity of a parallel plate condenser expressed in practical units is

$$C = \frac{A}{4\pi d} \; k \, \frac{1}{9 \times 10^{11}} \text{ farads}$$

and the resistance between the same electrodes is

$$R = \frac{d}{A} \rho \text{ ohms}$$

where A = area of the dielectric or electrolyte, or what is the same thing, the area of one plane.

d = spacing between the planes.

k = specific inductance capacity (permittivity) of the dielectric.

ρ = specific resistance (resistivity) of the electrolyte in ohms cm.[3]

If we now substitute the value of $\frac{A}{d}$ in terms of R and ρ in the above equation for the capacity we obtain

$$C = \frac{k\,\rho}{4\,\pi\,R}\;\frac{1}{9\times 10^{11}}\text{ farads}$$

as the expression giving the capacity of our electrodes in terms of the resistance measurements[11] R and ρ.

As an illustration, we may calculate the capacity of a bushing built from the diagram of Fig. 28 full size, assuming permittivity unity. The resistance measurement of a quadrant of the bushing gave approximately 40 ohms, and therefore, the resistance for the complete structure would be 10 ohms. Assuming $\rho = 3{,}000$ ohm-cm., we have

$$C = \frac{\rho}{4\,\pi R}\;\frac{1}{9\times 10^{5}}\text{ microfarads}$$

$$= \frac{3000}{4\,\pi\times 10}\;\frac{1}{9\times 10^{5}} = 0.000027\text{ microfarad}$$

Study of Electrode Shapes

The following are some of the questions which presented themselves at the outset, and which it was hoped could be answered by taking various diagrams of the electrostatic field by the electrodynamic method:

I. What is the best form of equipotential surface for the rod or high-potential electrode?

II. What is the best form of surface to select for the cover of the tank with a hole in it?

III. Is there a proper best ratio of rod to hole diameter under various conditions?

IV. Are hats desirable? Of how great an assistance are they in screening the bushing from surrounding influence? Are they useful in acting as a rain shed and thereby reducing the field distortion under rain conditions, etc.?

V. Can artificial equipotential surfaces be of material assist-

11. See A. E. Kennelly, *Electrical World*, December 29, 1906, Vol. XLVIII, page 1239, who has used this method for determining the capacity of wireless antenna.

ance in bringing about a better distribution of stress in the solid or supporting dielectric; and, at the same time, what are their effects upon the distribution of stress in the air and under oil part of the dielectric? Are their effects conflicting?

Application of the Electrostatic-Field Diagrams to Bushing Design

The problem of high-voltage bushing design is in reality a problem involving two or more dielectrics of different inductive capacities. Therefore, in general the diagrams obtained by using a single electrolyte only offer approximate solutions of our problem.

There are two conditions, however, in which the introduction of high inductive capacity material does not result in a distortion of the field by flux refraction.

Case I. High inductive capacity material in parallel with the air dielectric, that is, when the supporting dielectric conforms to a line of flow. In this case the form and distribution of the flux and equipotential surfaces are neither altered in form nor distribution—nothing is changed under these conditions when considering perfect dielectrics.

Case II. High inductive capacity material in series with the air dielectric, that is, when the additional material conforms to an equipotential surface. In this case the form of the equipotential and flux surfaces is not distorted by refraction. There is, however, an increase in the total amount of flux at constant applied potential difference which causes a change in the distribution of the stresses. The stress in the part of the field where the high inductive capacity (permittivity) material has been inserted is reduced, and that in the low capacity (permittivity) material increased. Therefore, in this case allowance has to be made for this effect.

In the above cases it is not necessary to have the surface of discontinuity rigorously conform to the flux or the equipotential surface since a small variation will not greatly disturb the field. This can be easily seen by making simple calculations by the law of flux refraction, which states that the tangent of the angle of incidence θ_1 is to the tangent of the angle of refraction θ_2 as permittivity of the first medium k_1 is to the permittivity of the second k_2.[12]

12. For example see Jeans, "Electricity and Magnetism," p. 335; Webster, "Electricity and Magnetism," p. 327; Karapetoff, "The Electric Circuit," p. 28 or 163.

Thus

$$\frac{\tan \theta_1}{\tan \theta_2} = \frac{k_1}{k_2} \qquad \text{(See Fig. 6)}$$

As an example consider the case in which

$k_1 = 1$ (air)
$k_2 = 3$ (oil)
$\theta_1 = 70$ deg. (the assumed angle of incidence)

Then,

$$\tan \theta_2 = \tan \theta_1 \frac{k_2}{k_1}$$

$$\tan \theta_2 = 2.747 \times 3 = 8.25$$

$\theta_2 = 83$ deg. (the angle of refraction)

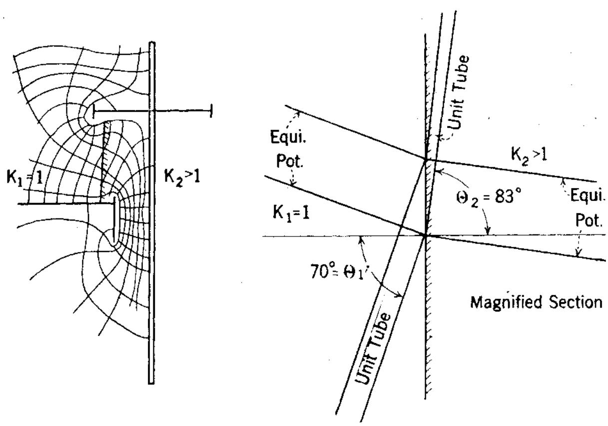

FIG. 6

As θ_1 approaches 90 deg., that is coincidence with the line of force, θ_2 also approaches 90 deg., and in the limit we have the case of two dielectrics of different permittivities in parallel. Under these conditions the form of the electrostatic field is unaltered by the presence of the two dissimilar materials, and the flux density in the air is not increased thereby. If θ_1 approaches zero degree then, in the limit, we have the case of two dissimilar dielectrics in series, and for this case the flux density in the two materials is the same, but due to the increase in total amount of flux the density in the air is increased by the presence of the higher permittivity material.

The above considerations show that there are two cases in which we can apply the diagrams obtained by using a single electrolyte to bushing design without fear of large errors being introduced due to flux refraction between the dissimilar materials which are in practise necessarily used for bushing.

Uniform-Field Type of Bushing

Thus applying Case I we see that it may be possible to so shape the electrode that the *strongest surface of discontinuity* between the supporting or high inductive capacity material and the air conforms closely to a line of flux. In that case the introduction of the solid material will not alter the field distribution and we would thus obtain one of the infinite number of theoretically correct designs. If this design can be obtained without making the electrodes of an impractical shape or size, this particular solution of the problem would be all that we require. For the case of ideal or perfect dielectrics, which assumes that the surface of discontinuity between the air and solid dielectric does not introduce a weakening influence, *the strongest surface would be an equigradient surface which coincides with a line of flux.*[13]

In this type of bushing it will be observed that the surface of discontinuity receives the full value of the potential gradient and therefore any weakening influence due to the surface of discontinuity between dissimilar dielectrics will exert its maximum ill effect. In the following a bushing built along these lines has been referred to as the "High-Air-Efficiency Bushing." It may also be called the "Uniform-Field Type," since the supporting dielectric and air are in parallel in an essentially uniform field.

By the use of artificial equipotential surfaces placed so as to bring about uniform gradient in the supporting dielectric, this type of bushing would, assuming perfect dielectrics, be as small as it is possible to construct, that is, it would use the air and any available supporting dielectric to their maximum efficiencies.

Radial-Field Type of Bushing

If we apply Case II to bushing design we would obtain a condition in which the potential gradient is zero along the surface of discontinuity between the air and supporting dielectric, and, therefore, has its full value normal to the surface. A bushing of this type would not be affected by surface conditions. It is

13. It may be of interest to compare this suggested criterion which assumes ideal dielectrics with that suggested by C. L. Fortescue and S. W. Farnsworth, and J. M. Weed's discussion, A. I. E. E., TRANS., 1913, Vol. XXXII, Part I, p. 893, "Air as an Insulator when in the Presence of Insulating Bodies of Higher Specific Inductive Capacity."

obviously impossible to have the supporting dielectric connecting the grounded and high-potential electrodes actually coincide with an equipotential surface since the two electrodes must be at different potentials, nevertheless the condition can be approximately obtained in practise. This type of bushing has been referred to as the radial-field type. In this case the supporting

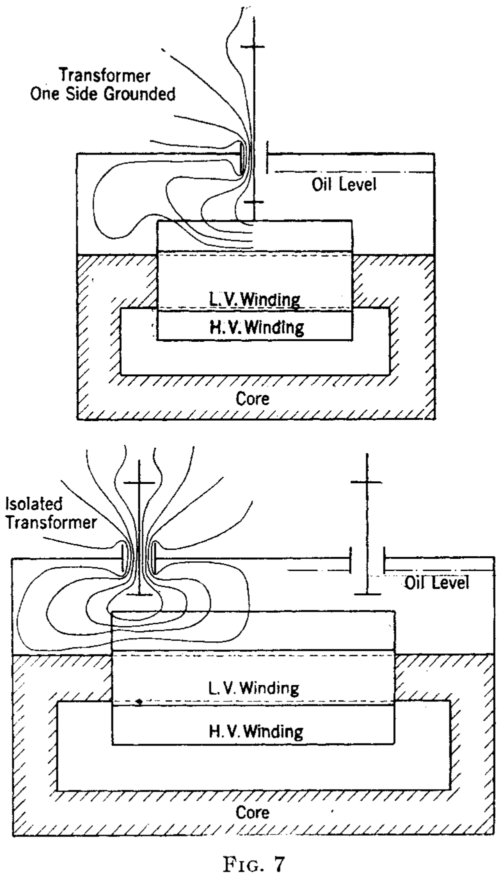

FIG. 7

and air dielectrics are essentially in series in a radial field. If the *surface effect* is very great, that is, if the component of the potential gradient along the surface has to be very small, this type of bushing when provided with artificial equipotential surfaces to obtain uniform stress in the supporting dielectric would result in as small a bushing as it is possible to construct. In this case the potential gradient would be approximately normal to the surface and is, of course, only limited by the strength of air.

Under-Oil End. In the above discussion we have considered the top part of the bushing where two greatly dissimilar dielectrics must generally be employed. The under-oil end of the bushing can, as a fair approximation, be treated as a single dielectric problem since the dielectrics usually employed do not differ very greatly in inductive capacities.

The under-oil end requires careful study due to the proximity of disturbing influences such as the core and windings of the transformer. It may be found desirable to use a rather large cap on the under-oil end to shield the bushing from these disturbing influences.

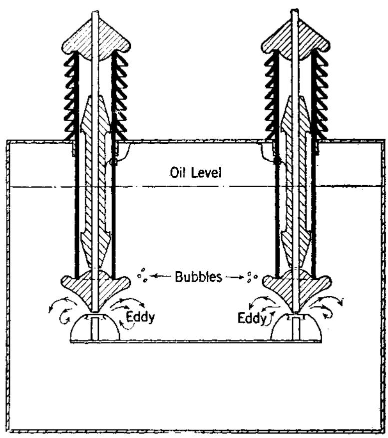

FIG. 8

This part of the problem will have to be carefully considered in conjunction with the whole design of the transformer or switch, etc. Fig. 7 is an illustrative suggestion of transformer design which would assist in an efficient under-oil-bushing design. Fig. 8 shows a switch design in which a large cap is used to reduce the tendency for an arc to re-strike, and, at the same time, provides a sort of deflector which throws the gas bubbles, etc., away from the surface of the dielectric. This could be given a sort of curved or bucket shape so as to produce eddies which might reduce the height of oil necessary over the contacts, etc.

Explanation of Diagrams

The diagrams are drawn for equal tubes of electrostatic flux or force and equal differences of potential between equipotentials.

That is, they are plane sections through the solid figure of revolution. Thus, if we imagine this plane section rotated about the axis of symmetry, the lines in the diagrams will cut out the correct surfaces—the equipotential lines and flux lines will generate equal tubes of flux and equipotential surfaces for equal differences of potential. The elementary cells, or elementary cylindrical condensers, thus cut out will have the same capacity throughout the field.

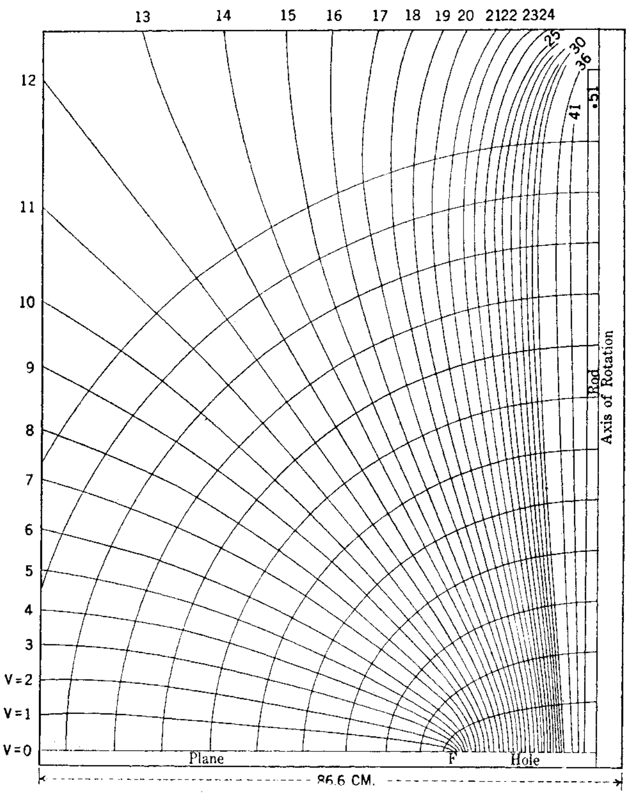

FIG. 9

In the case of diagrams, Figs. 27 and 28 dealing with the use of artificial equipotential surfaces the tubes of flux were not drawn in since they would be discontinuous at the metal surfaces. For this reason it was thought that they would confuse rather than add to the diagrams.

Discussion of Diagrams

Fig. 9. (*Mathematical Solution Fig.* 10). It will be readily seen that if we assume our transformer or switch tank sufficiently large, we can consider the top of the tank as an infinite plane with a circular hole in the middle. As our desired means of metallic connection the simplest would be merely a wire passing perpendicularly through the hole in the tank connecting the inside with the outside. As these electrodes constitute the simplest imaginable form from which to design a bushing their electrostatic field was the first studied.

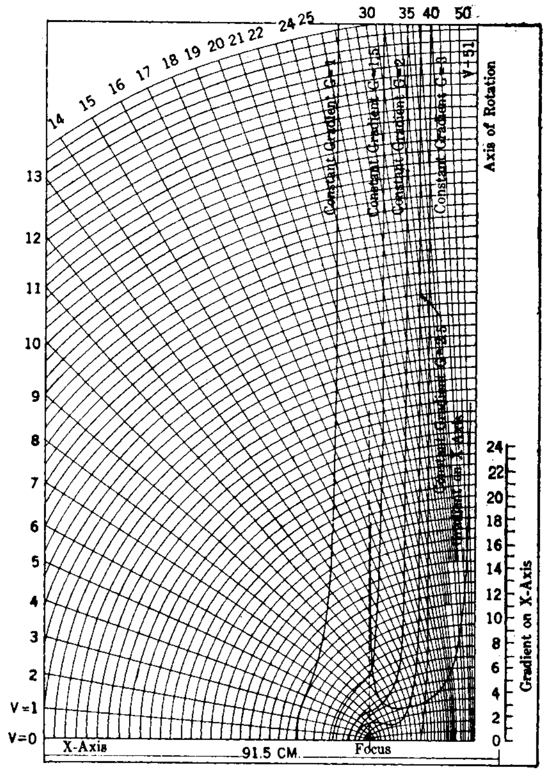

FIG. 10

This diagram shows the characteristics of what has for convenience been termed the radial-field type.

In designing a bushing from this diagram we would probably select equipotential surface 1 or 2 to represent the cover of the tank with hole and rounded edge which makes the stress on the supporting dielectric about three times as high as allowable for the air part of the dielectric. In order to stress the supporting dielectric near the rod to approximately the same value as that at the edge of the hole we would select as our rod an equipotential surface somewhere between No. 25 and No. 30. Under these conditions the ratio of rod diameter to hole diameter would be about 5 to 1. There are, however, many factors (such as the relative strengths of the dielectrics employed and whether the strength is equal in all directions, etc.) which will influence the above ratio and it should, therefore, not be considered as a fixed quantity.

The edge of the hole in the tank must clearly be embedded in the supporting, stronger than air, dielectric to eliminate over stressed air near the edge. That is, the value of a ground shield is clearly shown. This type of bushing will necessarily be quite high in order not to over stress the air at the top.

A study of this diagram suggested that a mathematical solution of this case should be quite easily obtained, due to the apparent simplicity of the equipotential surfaces and their mutual resemblance. Some of the lines of force were drawn in and were seen to conform fairly closely to confocal ellipses, having the edge of the plane as focus (see Fig. 9). This suggested that if there were no disturbing influences, that is, if the rod were very small in diameter and infinitely long and the plane also infinite in extent, the solution of the problem would consist in a confocal system of hyperboloids of one sheet as the equipotential surfaces, and confocal oblate spheroids as the surfaces of force. A plane section through the axis of symmetry of this figure is then a family of confocal hyperbolas and ellipses. The minor axis of the ellipses is the axis of symmetry or the axis about which the plane figure is rotated in order to cut out the solid figure.

In order to test this assumed solution of the problem, a diagram was constructed as follows:

Two large confocal ellipses were constructed having the minor axis along the rod and the radius of the hole in the plane as semi-major axis. The space between these two ellipses was then divided up into cells (see Fig. 11) which obeyed the required geometrical law:

$$\frac{2\pi r \times h}{d} = C \text{ (a constant)}$$

where r = the mean radius of the cell
h = distance between the two confocal ellipses
d = the other dimension of the small rectangular cell (or the distance between the concentric cylinders).

By cut and try, the space between the two ellipses was divided up into small cells of constant C. This value, C, is proportional to the capacity of the small concentric cylinder condensers which would be cut out by rotation around the minor axis. The family of hyperbolas which has the same focus as the ellipses was then drawn through the cells. The figure was then completed by drawing in the complete family of con-

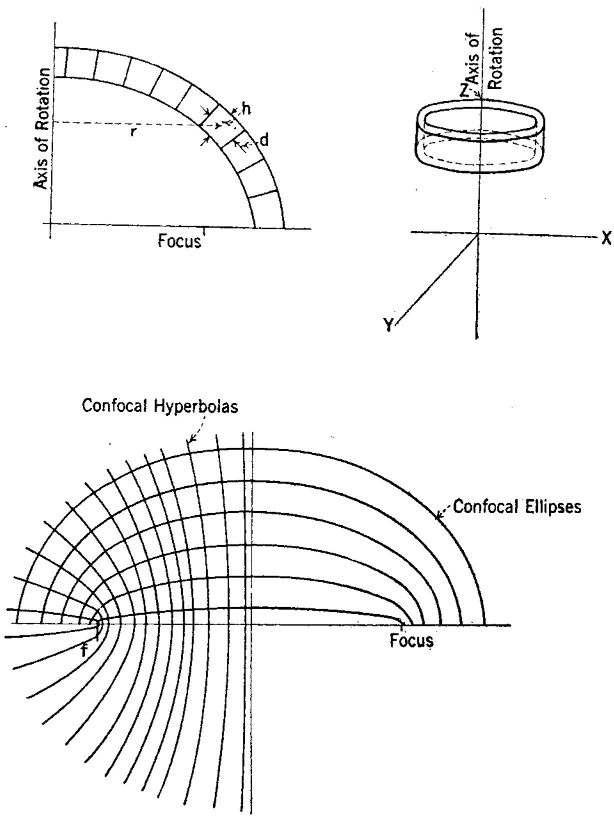

FIG. 11

focal ellipses which divided the whole field into cells having the same capacity or constant C. The fact that it was possible to complete this construction is a graphical proof of the correctness of the assumed solution; namely, that confocal hyperboloids of one sheet are the equipotential surfaces, and confocal oblate spheroids the boundary surfaces of the tubes of flux.

After obtaining the solution of the problem by this method, I found that it was well known; for example Maxwell[14] discusses the mathematical solution of this and similar problems. Other treatments will also be found in many books.[15]

Notwithstanding the numerous treatments of the subject it was not until a lengthy study of these sources, besides correspondence and later the great privilege of an interview with Professor W. E. Byerly, that I was able to obtain what appeared to me to be a clear and useful solution of the problem.

I believe that there are other engineers who find it difficult to readily understand, and therefore use, to the best advantage, the results of the great deal of extremely valuable mathematical work which is available in the various treatises. It seems to me a great pity that the mathematicians do not more frequently reduce their results to a readily utilizable form, and wherever possible sketch out, with examples, some of the applications which must occur to them while working on the subject. In spite of the drudgery which necessarily accompanies any numerical calculations, I believe that a writer would be amply repaid for his trouble by the greatly increased number of people who would study and be able to use his results.

Another difficulty, which I have frequently encountered, is the fact that the writer assumes too great a familiarity with existing mathematical works on the part of his readers. No one is better able to supply page references to what he considers a good and clear treatment of his statements "it has been proved". I also believe that no work would suffer from the inclusion of an appendix of "it can be easily shown" and in some cases the "hences". All of these additions could be skipped by the fluent mathematical readers but would be available as wonderful time savers and often life savers to the ordinary engineer. For these reasons I have included an unexpurgated edition of two electrostatic problems in the appendix to this paper.

Problem I. The distribution of the electrostatic field when any two confocal hyperboloids of revolution of one sheet, and of the same family, are maintained at given potentials.

14. See Maxwell, Electricity and Magnetism, Vol. I, p. 235 and following.

15. See Byerly, Fourier Series and Spherical Harmonics, p. 238–247. J. H. Jeans, Electricity and Magnetism, p. 238–244; A. G. Webster, Electricity and Magnetism, p. 203–242, 273, etc.; I. Todhunter, The Functions of Laplace, Lamé and Bessel, Chapter XXI p. 211.

This problem is useful as an approximation in studying high-voltage bushing design; it also gives us an interesting variety of possible electrode shapes for use in testing insulating materials, where it is desirable to be able to calculate the gradients. When testing a dielectric of given inductive capacity immersed in a dielectric having a different inductive capacity (*i.e.*, hard rubber

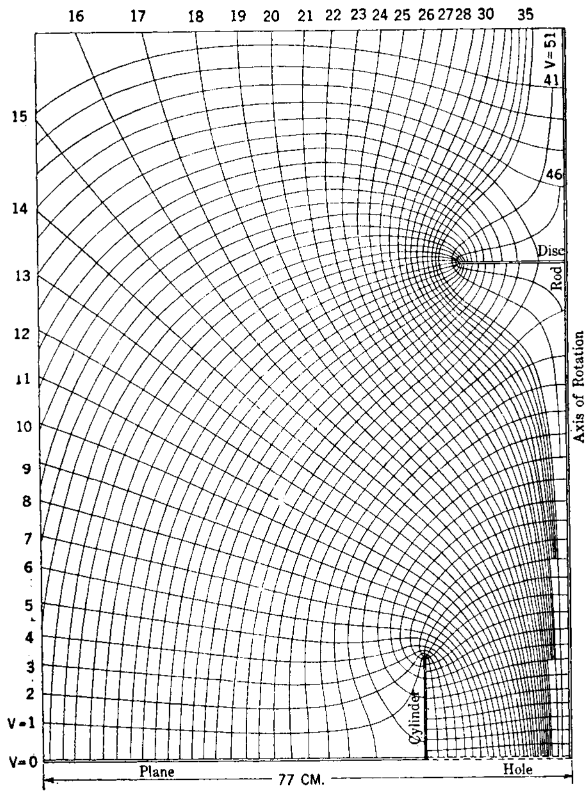

FIG. 12

in air or oil) the edge effect can be eliminated by shaping the test piece so as to follow a line of force or flux.

Problem II. The distribution of the electrostatic field when any two confocal hyperboloids of revolution of two sheets and of the same family are maintained at given potentials. This problem is of interest as an approximation to a group of problems which are of frequent occurrence in engineering. For example, it may be applied to switch electrodes at various spacings; also, as an approximation in vacuum tube designs, such as X-ray tubes, kenotrons, etc. That is, it is an approximation to the problems of two elongated electrodes at various spacings, (*i.e.*, two needles or a needle and a plane) etc.

Fig. 12. This diagram[16] shows a form of field intermediate between the radial and uniform-field types. The effect produced by adding a collar to the edge of the hole in the plane is shown. The addition of a collar or extended ground shield is generally used on the air end to shield the bolts necessary in clamping

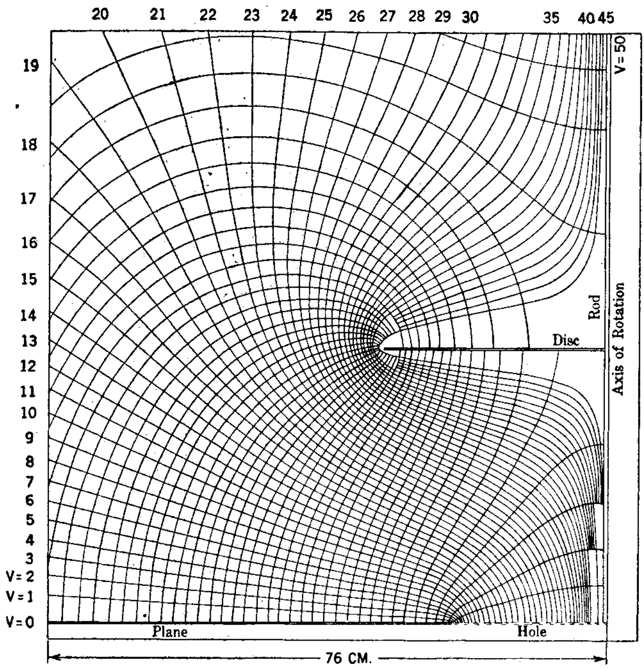

FIG. 13

the bushing to the tank. It may also be used to shield switch mechanism, etc. A ground shield extending below the surface of the oil on the under-oil end is usually necessary in order to remove the stress on the air above the oil.

In designing a bushing from this diagram, we would probably select equipotential surface No. 2 or No. 3 as cover of the tank with hole and collar or ground shield and an equipotential surface

16. Dr. C. P. Steinmetz has published this diagram in the fourth edition of his Electrical Engineering, page 116, which has recently appeared.

resembling No. 20 as high potential electrode. In practise we would probably not build the electrode of the shape of surface No. 20, but would make up the electrode of a straight rod and cap which would be roughly equivalent to surface No. 20. The addition of the cap and collar does not greatly alter the conclusions regarding the ratio of rod to hole diameters as outlined above when discussing this point for the simplest case, that of a rod passing through a hole in a plane. That is, the edge effect of the collar or cylinder is about the same as that for the edge of a plane.

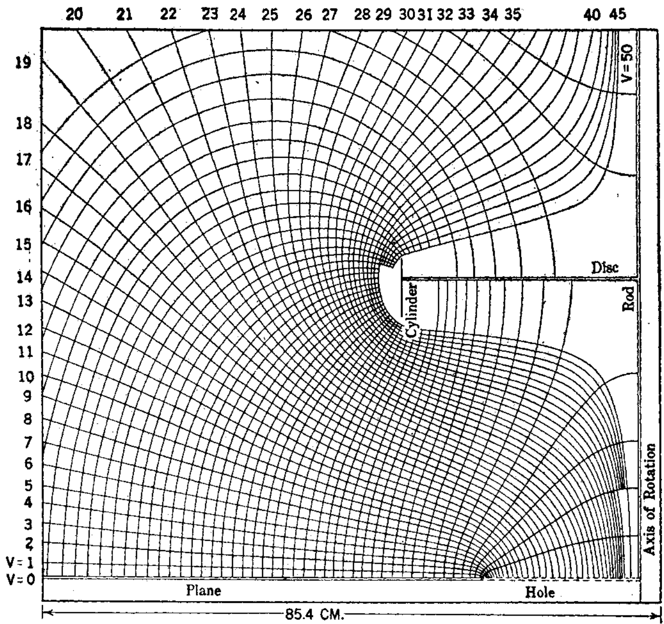

FIG. 14

Figs. 13, 14 *and* 15. These diagrams are essentially of the uniform-field type. A study of diagrams, Figs. 9 and 12, indicated, that by increasing the size of the cap and bringing it closer to the case, the efficiency of the use of the air part of the dielectric could be increased. That is, it appeared practical to shape the electrodes of the bushing in such a way that the distribution of the electrostatic field would approach the condition in which the strongest surface assuming ideal dielectric (constant gradient or flux density) coincides with a line of force in the part of the field where it is desired to introduce the dielectric of higher specific inductive capacity than the rest, which may be air or oil. As previously stated the resulting bushing would use the air at its maximum efficiency and hence the resulting bushing would be as efficient, with respect to the air path, as any of the infinite other solutions which could be obtained by using flux refraction combined with various electrode configurations.

In order to experimentally test out this conclusion, Figs. 13, 14 and 15 were taken. Skeleton electrodes were used, as

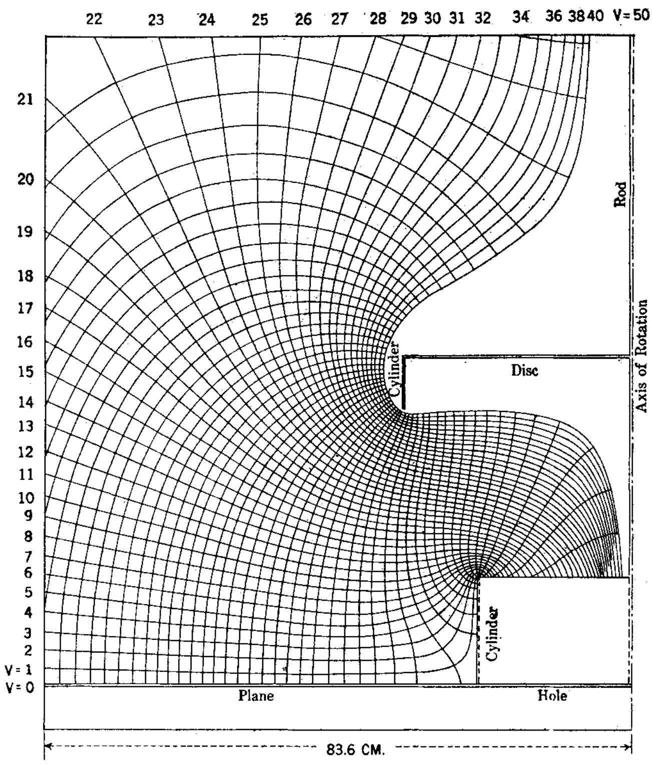

FIG. 15

in the previous experimental work, in order to simplify matters. For instance, one electrode may consist in a thin piece of sheet iron, or tin, with a hole in it; the other electrode a small wire with an attached metal disk. The proper actual shape of the rod with cap and the tank will then be selected from the equipotential surfaces which are obtained by the experiment.

For the new diagrams the size of the cap was increased so as to have a diameter of about 1½ times that of the hole in the plane, and was placed at about one diameter from the plane. The short cylinder was added to the sheet-iron disk for Fig. 14, to form a skeleton electrode which would flatten out the equipotential surfaces near the edge of the disk, and also to bring the lines of force into a more nearly cylindrical form. Briefly, to give the effect of a cap having a greater radius of curvature at its edge.

In applying these diagrams to bushing design we would probably select the equipotential surface 2 as the top of the tank, and an equipotential surface between 28 to 32 as our rod with cap. This selection gives approximately equal gradient at the edge of the hole in the tank and at the rod. Also, the maximum gradient on the cap is not too different from that near the tank. The supporting insulation, stronger than air and of higher specific inductive capacity, would probably be put in somewhere between flux lines number 4 to 7.

It is apparent from the diagram that this bushing would have its maximum air stress near the edge of the cap, and therefore, would probably arc-over clear of the solid insulation, unless the surface introduces appreciable weakening influences. This matter will have to be settled by experiment and then given consideration according to its magnitude. Assuming a surface which does not appreciably affect the arc-over of the bushing, we obtain from the full size diagram, Fig. 14, the following result. Selecting equipotential surface No. 30 as our rod with cap, we see that the maximum potential gradient in the air occurs near the edge of the cap. We must, therefore, select this gradient to be that at which air breaks down or approximately 21 kv. effective per cm. The distance between equipotential surfaces 30 and 29 is about 0.6 cm. (on the full size diagram).

Hence

$$\frac{e\Big|_{30}^{29}}{0.6} = 21 \text{ kv. effective per cm. (assumed strength of air.)}$$

$$e\Big|_{30}^{29} = 12.6 \text{ kv. effective, the potential at which breakdown will occur between these two surfaces.}$$

Between the cap and the tank (equipotential surface No. 2) there are 28 equipotential surfaces drawn for equal differences of potential. Therefore, the arc-over voltage of this terminal would be obtained by multiplying the number of equipotential surfaces between that selected as the rod with cap and the tank, by the voltage which would cause breakdown if applied between the two consecutive surfaces in the densest part of the air field. In this case the number of surfaces is 28, and the voltage at which breakdown first occurs between two surfaces is 12.6 kv. effective.

Hence

$$28 \times 12.6 = 350 \text{ kv. effective arc-over voltage.}$$

The operating voltage, assuming a safety factor of 3, would be 116 kv. effective. This calculation assumes that as soon as the gradient at the edge of the cap reaches the breakdown value, arc-over will occur (unstable condition).

With regards to the stresses on the solid or liquid dielectric which is used in this design, it may be of interest to observe that the maximum gradient, which occurs along the path between rod and edge of tank, is approximately 63 kv. effective per cm. or three times that at which air breaks down.

The efficiency with which this bushing uses the air part of the dielectric may be estimated as follows:

Assuming arc-over to occur from the point of maximum stress on the cap, that is, near the outer edge, to the tank a distance of approximately 30 cm., we see that this air path under uniform gradient, should arc-over at

$$30 \times 21 = 630 \text{ kv. effective.}$$

Hence, taking the efficiency to be the ratio of the arc-over voltage as estimated above from the diagram, to this uniform gradient condition, we have

$$\text{Efficiency} = \frac{350 \text{ kv.}}{630 \text{ kv.}} = 55 \text{ per cent.}$$

Fig. 15, shows the effect of the addition of a collar or cylindrical ground shield to a bushing of this type. The conclusions are similar to those which may be drawn from the previous diagrams.

Tests on Experimental High-Air-Efficiency Bushing

An experimental bushing was constructed from a photographic reduction of Fig. 14, as outlined in Fig. 16. Two supporting dielectrics were tried. In the first case a smooth porcelain piece conforming as closely as possible to the flux line No. 7 was used. In the second case the porcelain surface was corrugated as shown in the figure between flux lines No. 4 and No. 7.

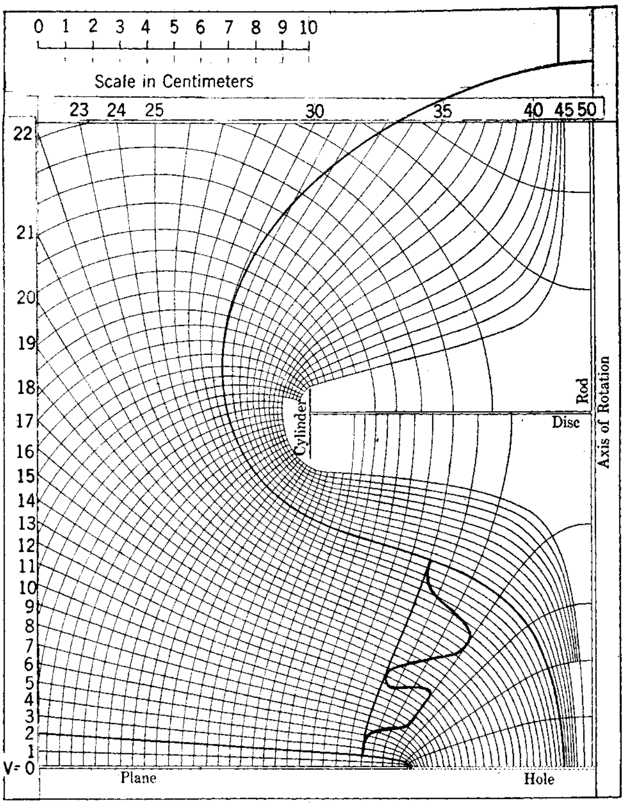

Fig. 16

The electrodes and porcelain pieces were cemented together with a sort of sealing wax compound, considerable effort being made to make good joints at the electrodes. Equipotential surface No. 30 was chosen as the central electrode shape but was modified at the top as shown in the figure as it was not thought important to conform to the diagram at the top part of the electrode where the stress on the air is very low. Equipotential surface

[RICE]

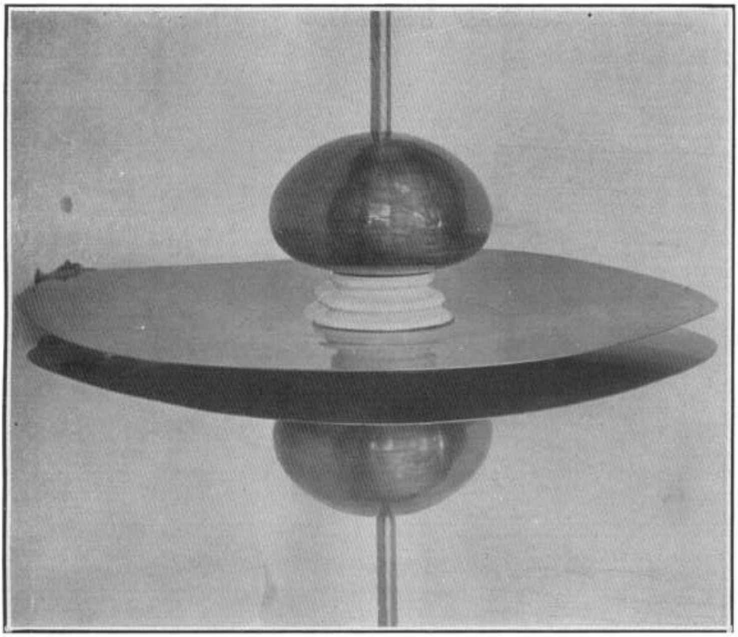

FIG. 18

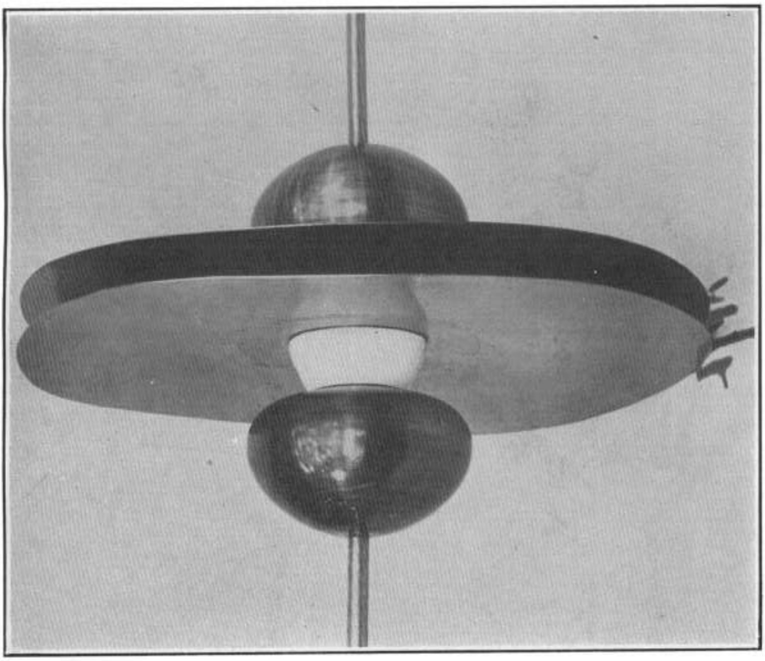

FIG. 17

No. 2 was selected as the ground plane and extended 43 cm. from the axis. The upper and lower parts of the bushing were made as nearly alike as possible; Fig. 17 and 18 show the general appearance and construction of the bushings.

The principal electrical characteristics, as obtained from the experimental diagram, Fig. 14, are as follows: The maximum gradient on the air part of the dielectric occurs on the cap of the central electrodes and well away from the surface of the supporting dielectric. The gradient at this point reaches 21 kv. effective per cm. (or the assumed breakdown strength of air) at about 117 kv. effective. The gradient along flux line No. 7 (7.3 cm. long) is practically uniform and will reach 21 kv. effective per cm. at about 153 kv. The gradient at the edge of the ground electrode and the central rod are approximately equal and are about three times the maximum gradient on the air.

The arc-over voltages of the corrugated and smooth bushings may be estimated as follows:

(1) *Smooth Surface*

If we assume that the surface and joints of the porcelain do not introduce disturbing influences, arc-over should occur clear of the surface at 117 kv. effective. For this potential difference the gradient along the surface of the porcelain would be 16 kv. effective per cm.

(2) *Corrugated Surface.*

If we neglect refraction effects and make the very rough assumption that the corrugations introduce half air and half porcelain in series in a uniform field we would have for the arc-over voltage

$$e = G_1 \left(x_1 + \frac{x_2}{k} \right) \text{ arc-over voltage}$$

Where

$$G_1 = 21 \text{ kv. per cm.}$$

$$x_1 = x_2 = \frac{7.3}{2} \text{ cm.}$$

$$k = 5 \text{ porcelain}$$

Hence,

$$e = 21 \left(3.65 + \frac{3.65}{5} \right)$$

$$e = 92 \text{ kv. effective.}$$

and arc-over would occur by unstable corona formation in the corrugations at 92 kv. effective.

At this voltage the average potential gradient along the arc-over path would be $\frac{92}{7.3} = 12.6$ kv. effective per cm.

Summary of Tests

Sixty-Cycle Arc-over Voltage. (average of ten readings)

Surfaces cleaned with absolute alcohol and wiped dry

I	Smooth Surface	80 kv. ± 10 kv.
	Average gradient	11 kv. per cm.
II	Corrugated Surface	72 kv. ± 10 kv.
	Average gradient	9.9 kv. per cm.

Surfaces cleaned as above and then covered with an oil film.

I	Smooth Surface	75 kv. ± 10 kv.
	Average gradient	10.3 kv. per cm.
II	Corrugated Surface	82 kv. ± 5 kv.
	Average gradient	11.2 kv. per cm.

The arc-over voltages after visible carbonization points have been formed at the sealing wax joints and on the porcelain is given below. This condition is reached after about 15 arc-overs, when dry clean surfaces are used and in 5 to 10 arc-overs when the surface is oiled. Handling the surface results in about the same arc-over voltages.

I	Smooth Surface	50 kv. ± 2 kv.
	Average gradient	6.85 kv. per cm.
II.	Corrugated Surface	5 kv. ± 2 kv.
	Average gradient	7.25 kv. per cm.

Surface of electrodes and porcelain covered with city water, that is, the surfaces were sprayed after all the oil had been removed.

I	Smooth Surface	23 kv. ± 3 kv.
	Average gradient	3.15 kv. per cm.
II	Corrugated Surface	27 kv. ± 3 kv.
	Average gradient	3.7 kv. per cm.

Surface oiled and a needle point ¼ in. long placed upright at the corrugated surface on the ground plate.

First arc-over	50 kv.
Average gradient	6.85 kv. per cm.

After about three arc-overs a carbonized point could be seen to form at the arcing point on the upper electrode above the needle, also considerable carbonization took place around the needle.

Under these conditions the arc-over was............................35 kv.
An average gradient of.............. 4.8 kv. per cm.

60 *Cycle Under-Oil Tests.* After the above mentioned tests were completed the bushing was immersed in oil (40 kv. oil as measured by standard gap of 0.2 in. (0.5 cm.) between 0.5 in. (1.27 cm.) terminals) in order to determine the under-oil characteristics.

In both cases arc-over took place over the porcelain surfaces and in the corrugated bushing followed in and out of the corrugations. The results were

I	Smooth Surface.................	122 kv.
	Average gradient.............	16.7 kv. per cm.
II	Corrugated Surface.............	144 kv.
	Average gradient.............	19.7 kv. per cm.

Impulse Arc-over Tests. Impulse tests were made using F. W. Peek's impulse generator.[17] Impulses having various frequencies and wave shapes were used but as the results were in good agreement in all cases only a typical set of readings will be given. The calculated frequency of the impulse wave is given as 500 kilocycles with a maximum voltage corresponding to 188 kv. effective.

The arc-over voltages as measured by 12.5-cm. spheres placed in parallel with the bushing were as follows:

I	Smooth Surface............	111.5	spheres only
		117.	half and half
		121.	bushing only
	Average gradient........	16.	kv. per cm.
II	Corrugated Surface........	104.	spheres only
		111.	half and half
		117.	bushing only
	Average gradient........	15.2	kv. per cm.

The results were practically identical with clean dry or oiled surfaces.

17. "The Effect of Transient Voltages on Dielectrics." A. I. E. E., TRANS., 1915, Vol. XXXIV, Part II, p. 1857.

Observations and Discussion

In all of the tests on the smooth bushing the arc-over appeared to follow close to the surface. In the corrugated case arc-over appeared to take place along flux line No. 7, that is, did not follow in and out of the corrugations. (The only exception was in the case mentioned of the under-oil arc-over of the corrugated bushing where the path followed the corrugations in and out).

The previous calculations indicated that the smooth porcelain bushing should arc-over clear of the surface at about 117 kv. whereas in test it arced-over along the surface at about 80 kv. ± 10 kv. (oily and dry surfaces do not seem to be greatly different) or say approximately 70 per cent of the calculated arc-over voltage. However, this comparison is not a truly correct one since the arc-over occurred at the surface and not at the point of maximum gradient. A better comparison of calculated and observed arc-overs is obtained by comparing the calculated surface arc-over with the actual surface arc-over.

Sixty-Cycle Tests. (A) Averaging the clean and oily surfaces we have

$$\frac{\text{calc. surface arc-over}}{\text{measured surface arc-over}} \ldots\ldots = \frac{153}{77} = 2.0$$

(B) Carbonization points formed after a number of arc-overs

$$\frac{\text{calc. surface arc-over}}{\text{measured surface arc-over}} \ldots\ldots = \frac{153}{50} = 3.1$$

(C) Surfaces and electrodes sprayed with water . . $= \frac{153}{23} = 6.6$

Impulse Tests. (D) Clean or oily surfaces..... $= \frac{153}{117} = 1.3$

Many explanations may be advanced for the above results; at first I was inclined to account for them in the following manner.

Case A. Here breakdown occurred over the surface of the bushing at approximately one-half the calculated voltage. To account for this we might assume that a conducting corona ring forms prematurely due to an imperfect joint between the porcelain and electrodes, since at the joint a high stress might exist on the air due to the high inductive capacity material being in series with the air at this point. If such a conducting corona ring was formed the maximum stress upon it would be approximately twice that existing in the uniform part of the field. For if we insert a cylindrical conductor in a uniform field the maximum gradient on the cylinder will be twice the gradient of the uniform field in which it is placed or if we have two infinite plane electrodes having hemicylindrical bosses upon them, then for spacings between the planes such that the field between is essentially uniform in the middle the gradient in the vicinity of the cylindrical bosses will be a maximum on the crest of the boss and will be twice the gradient existing in the uniform part of the field.

The solution of this problem is obtained by the method of images; by superimposing a uniform field upon the field of a linear doublet.[18] A complete diagram of the electrostatic field is given by Maxwell in Plate XV at the end of Vol. II of "Electricity and Magnetism."

If we make the further assumption that the corona ring is unstable then we see that arc-over should occur at approximately one-half the calculated arc-over voltage assuming perfect joints and surfaces, which is about the value observed from the tests. It is of course probable that the gradient at the surface of the boss will have to be somewhat above 21 kv. effective per cm. at breakdown, or what is the same thing, the breakdown strength of air (assumed 21 kv. eff. per cm.) must be reached at a finite though small distance from the surface of the boss. This same effect is well known in the case of wires or concentric cylinders, etc., for example, it has been discussed by F. W. Peek, Jr.[19]

Case B. Carbonization points formed after a number of arc-overs. In this case arc-over took place over the surface at about one third the calculated value. A method of approximately explaining this case may be based upon the assumption that a small hemispherical corona boss forms on the surface of the electrodes. The solution of the electrostatic problem of two infinite planes having hemispherical bosses is obtained by superimposing a uniform field upon the field of a spherical doublet. The gradient is a maximum on the crest of the boss and is three times that in the uniform part of the field.[20]

---

18. See "Electricity and Magnetism". A. G. Webster, p. 202, also p. 88. Notes on Electric Field Distribution. Journal of the Franklin Institute, July, 1913. W. S. Franklin, p. 72–75.

19. A. I. E. E., TRANS., 1911, Vol. XXX, Part III, p. 1889, "The Law of Corona and the Dielectric Strength of Air."

20. J. H. Jeans, "Electricity and Magnetism," p. 188–191; A. G. Webster, "Electricity and Magnetism", p. 371–373; Lord Kelvin Papers on "Electrostatics and Magnetism," p. 492; J. J. Thomson, "Electricity and Magnetism," p. 158–160.

Diagrams of the lines of force or flux will be found in the various references given below and a somewhat similar and instructive case is shown in Plate III at the end of Vol. I—Maxwell, "Electricity and Magnetism."

Case C. Water particles on the surface. In this case arc-over occurs at about one-half the needle-gap arc-over at this spacing. This low arc-over under these conditions may be a sort of progressive breakdown between adjacent water particles or distortion of the field by erratic conduction, etc.

Case D. The impulse test shows an arc-over voltage which is very close to the calculated arc-over assuming that it takes place clear of the surface. To the eye, however, the arc-over seems to take place along or very close to the surface and therefore there seems to be some disturbing influence at work which the above explanations do not take into account.

The previous calculations for the corrugated porcelain bushing indicated that the arc-over should take place along the edge of the corrugations due to unstable corona formation in the valleys of the corrugations. The calculation is necessarily only approximate as the flux refraction produced at the corrugations is neglected.

A comparison of the various cases may, however, be of interest.

Sixty-cycle Tests.

Case A.	Oily and clean surfaces ratio of calculated to test arc-over........................	$\frac{92}{77} = 1.2$
Case B.	Carbonization points formed after a number of arc-overs.......................	$\frac{92}{53} = 1.74$
Case C.	Surfaces sprayed with water.............	$\frac{92}{27} = 3.4$

Impulse Test

Case D.	Clean and Oily Surface................	$\frac{92}{111} = 0.83$

Explanations similar to those offered for the smooth surface might be advanced but cannot be quantitatively applied as the field conditions are not as definitely known.

Under-Oil Tests (60 *Cycles*). The under-oil arc-over tests also show a lower value than would be expected if we assume perfect joints and ideal dielectrics.

The strength of oil at these large spacings (7.3 cm.) is about 33 kv. effective per cm., and therefore the surface arc-over voltage should be $7.3 \times 33 = 240$ kv. effective for the smooth porcelain surface

$$\frac{\text{calculated surface arc-over}}{\text{measured surface arc-over}} = \frac{240}{122} = 2.0$$

It will be observed that the explanation above (corona ring or joint effect) could be applied to these tests and give what would appear to be a satisfactory explanation of the observed discrepancy between calculated surface arc-over voltage and that obtained from the tests. The corrugated porcelain surface could be discussed qualitatively in a similar manner.

ADDITIONAL TESTS ON "HIGH-AIR-EFFICIENCY BUSHINGS.

The object of the following tests was to determine the disturbing influence of the walls and floor of the room or ground upon the arc-over voltage of this type of bushing. The following tests were made using the *corrugated supporting dielectric* as it was still assembled with the electrodes. The previous under-oil tests had damaged the porcelain surface on one side, and therefore, necessitated submerging the injured end under oil. For this purpose, a wooden oil tank was used, the tests being made on the uninjured end. The surface of the porcelain was oiled for all tests.

Summary of Tests

Sixty-Cycle Arc-Over Voltage. (average of ten readings). Corrugated Surface.

(A) Plane grounded
  Arc-over voltage.................67 kv. ± 5 kv. eff.
  Average gradient................ 9.2 kv. per cm.

(B) Plane isolated, central electrode grounded
  Arc-over voltage.................67 kv. ± 4 kv.
  Average gradient................ 9.2 kv. per cm.

(C) Bushing entirely isolated.
  Arc-over voltage.................80 kv. ± 10 kv.
  Average gradient................11 kv. per cm.

A comparison of these tests with the previous ones show a reduction of the arc-over voltage for the grounded cases to about 80 per cent of the previous values. The isolated case agrees closely with the previous tests where the plane was grounded.

The results might be expected from the following consideration: In the previous tests the bushing was suspended about 10 ft. above the floor of the room, and therefore was not appreciably influenced by the floor and walls, whereas the present tests were made with the bushing about 2 to 3 ft. from the floor of the room. The conditions of the test had to be changed, due to the fact that it was necessary to have the damaged end of the bushing under oil. Under these conditions the proximity of the floor of the room or ground would have a greater effect upon the grounded cases, A and B, than for case C where both plane and central electrode were isolated.

It may be interesting to note, at this point, that the type of electrostatic field, which we are here concerned with, for example, an infinite rod passing through a hole in an infinite plane is essentially different from the case of two spheres at different potentials in space. In our case the rod and the plane both are considered as going to infinity and are at different potentials. Thus, it is evident that the potential at infinity is indeterminate and the electrostatic field distribution merely depends upon the relative potential between the two electrodes. This can be seen analytically from the mathematical solution for this simplest case (see appendix). The case of two spheres in infinite space is quite different. For example, assume them to have equal and opposite potentials, in that case, the equipotential surfaces are approximately spherical at great distances from the spheres and at an infinite distance the equipotential surfaces around each sphere are two infinite spheres made up of the infinite plane which passes midway between them. By reason of symmetry this infinite plane is seen to be at zero potential since it forms the equipotential surface which lies midway between the two spheres and since this plane goes to infinity the potential at infinity is zero.

If we now assume one sphere at zero potential and the other at a positive potential of such a value as to make the relative potential between the two spheres the same as before, the electrostatic field distribution is no longer symmetrical about the plane midway between the spheres, that is, the field has been changed though the relative potential between the spheres is the same. A diagram of the electrostatic field, under these conditions, is shown in Fig. II at the end of Vol. I, Electricity and Magnetism, by Clerk Maxwell. In this diagram, Q, is the spherical surface at zero potential and, A, the other sphere at a positive potential. The concentration of the field about, A, will be observed. This is quite a different matter from the effect of floor, walls, etc., of the room. I have taken the liberty of calling attention to this fact because it has puzzled me considerably and may have bothered others also. The above remarks, of course, do not apply to the case of two bushings in proximity, in which case the fields of the two bushings would have to be treated as one.

Conclusions

In general, the results of tests on the above high-air-efficiency bushings were not very encouraging from the point of view of building a satisfactory commercial bushing of this type. However, if the above suggested explanation of the discrepancy between theory and test were found to be correct, that is, if the disturbing influence is due to corona at the joints between the porcelain and electrodes, it should be possible to eliminate it by electrostatic shielding of these joints. This might eliminate all the lowering of arc-over except that due to rain or sprayed surfaces, and, therefore, might give us a useful bushing for indoor use, especially on testing transformers, where the *impulse safety factor* is not important.

Investigation of the Discrepancy Between Calculated and Test Arc-Over Voltage of Experimental High-Air-Efficiency Bushing

Joint Effect

The following tests were carried out in order to obtain an experimental check upon the explanation offered above which was based upon the assumption that a corona ring formed prematurely at the joint between the supporting dielectric and the electrodes.

Large plane electrodes with smoothly rounded edges were spun up of brass of the form shown in Fig. 19. The object of the design was to obtain as nearly as possible, a uniform field between the electrodes without disturbing edge effects. The sixty-cycle arc-over voltage was then obtained at various spacings, with planes alone, and when provided with small rings, hemispherical bosses and points. The results are given in the form of curves in Fig. 19.

It will be observed from the data given on the planes alone that the field between them is not accurately uniform since the breakdown gradient changes with the spacing and is in general lower than 21 kv. effective per cm. the approximate strength of air at these large spacings. The arc-overs were, however, well distributed and did not show a tendency to take place at the rounded edges. The 200 kilocycle impulses arc-over of the planes is given on the curve, and approximately represents the true strength of air under uniform field conditions, as the edge effect does not seem to greatly lower the impulse arc-over.

To show roughly the effect of hemicylindrical bosses small copper rings $\frac{1}{8}$ in. (0.318 cm.) cross-section were used. Two diameters for the rings were tried first, one-inch (0.254 cm.)

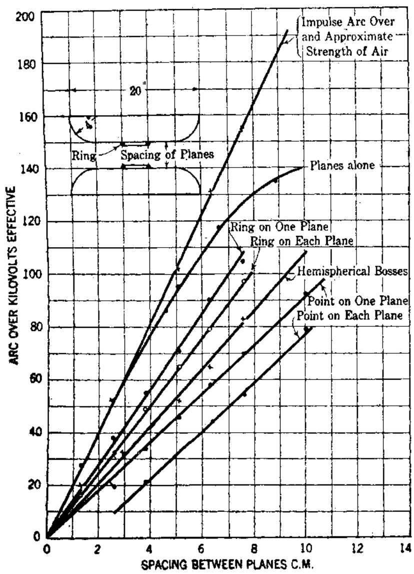

FIG. 19—SIXTY-CYCLE TESTS IN AIR

diameter and secondly, four-inch (10.15 cm.). The results were, however, not appreciably different and, therefore, the averages are shown, first for the case of a single ring on the upper plane, and secondly for a ring on each plane. The rings were not made half-sections because it was not thought necessary for such rough tests. Of course the effect produced by the ring is not rigorously equivalent to a straight hemicylindrical boss except where the cross section of the ring is exceedingly small and the diameter of the ring extremely large in comparison.

The hemispherical bosses used had a radius of $\frac{1}{8}$ in. (0.318 cm.) No appreciable difference was observed between the case in which a boss was placed on one or on both planes, and, therefore, a single curve is shown.

The points or cones which were employed were approximately 30 deg. and $\frac{1}{4}$ in. (0.635 cm.) high and had rather dull points.

From the previous discussion we saw that the apparent arc-over gradient when hemicylindrical bosses were placed upon the planes should be, assuming any corona formation to be unstable somewhat above 10.5 kv. eff. per cm. due to the so-called "energy distance effect" which requires that the surface gradient exceed the breakdown strength of air. The rough tests given above indicate an apparent surface gradient of about 13 kv. eff. per cm. Observation in the dark indicated that arc-over takes place without previously showing corona.

A similar explanation probably suffices to explain the difference between the apparent arc-over gradient, as discussed above, for the case of a hemispherical boss which indicated that on the assumption of unstable corona formation the arc-over would take place at an apparent gradient somewhat above 7 kv. eff. per cm. In test the apparent gradient was found to be about 10 kv. eff. per cm. No corona could be observed in the dark before arc-over.

The case of the small points on the planes is not so easily calculated but is interesting in showing the extreme condition. Observation in the dark showed streamers from the point to the plane just below the arc-over voltage, sometimes resulting in arc-over and sometimes going out, the appearance being like very fine so-called "static sparks" and not a corona like glow.

Surface Effect

Tests in Air and Oil. We have seen that the explanation of the discrepancy between calculated and test arc-over voltages of the high-air-efficiency bushing can be fairly satisfactorily explained on the assumption that the joint between the porcelain and metal of the electrodes was the disturbing influence which brought about the reduction in arc-over voltage. It has also been pointed out that, if this were the true explanation, it should be possible to eliminate the effect by proper electrostatic shielding of the joints. The following preliminary tests were, therefore, carried out in order to roughly check the assumption experimentally.

For this purpose the two brass electrodes used in the previous tests were available, and in addition a second pair of planes were spun up on exactly the same form, except that the annular grooves were added as shown in Fig. 20. The object of the grooves was to supply an electrostatic seal for the edge of the test piece (a glass cylinder in these tests). It is believed that in this manner we have shielded the joint between the test piece and the electrodes, and, therefore, should have eliminated any disturbing influence which might arise from the joints. An imaginary sketch of the electrostatic field about the grooves has been drawn which is also a detail drawing of the groove. Reference may also be made to Figs. XI and XIII, at the back of Vol. I, Electricity and Magnetism, by Clerk Maxwell, which show the screening

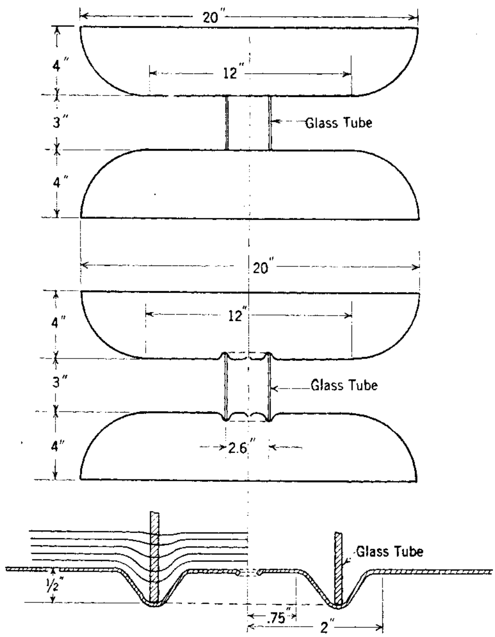

FIG. 20

effect of plates and gratings, which are somewhat analogous cases. Under these conditions the introduction of the glass cylinder of high specific inductive capacity in parallel with the air dielectric should have no effect upon the arc-over voltage, providing the strength of air is not different when in contact with the test piece.

The data obtained using various test pieces between the smooth and grooved planes in air and oil are summarized by the curves, Figs. 21 to 25. During the tests the barometer varied between 74.5—76. cm. Hg.; the temperature between 23 deg. cent. to 27 deg. cent. and humidity between 34 per cent to

53 per cent. The effect of these variations did not show themselves above the experimental errors involved in the tests and therefore all the data were averaged together. The glass cylinders used as test pieces were ordinary soda glass approximately 2.5 inches (6.25 cm.) diameter with a wall thickness of about 0.07 inches (0.18 cm). In all cases minute longitudinal flaws were visible along the grain of the glass. The edges of the cylinders were roughly ground but the process left appreciable irregularities which probably would be sufficient to introduce corona disturbances, if they are appreciable.

Three conditions of the glass surface were investigated: I—

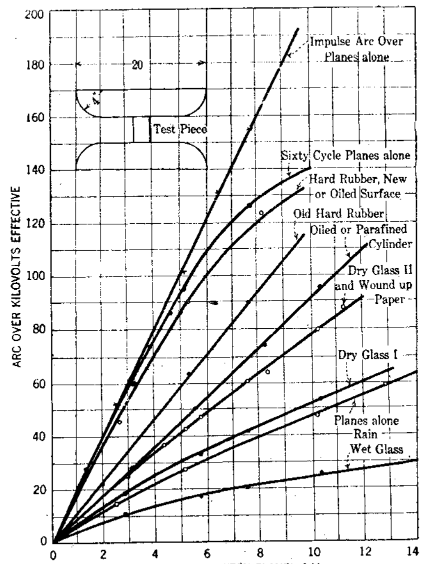

FIG. 21—SMOOTH PLANES—SIXTY CYCLE TESTS IN AIR

Glass cylinders cleaned with absolute alcohol and dried with a cloth. This case is represented on the curves as dry glass (I).

II—Glass cylinders cleaned as in (I) and then heated for about 3 hours at 180 deg. cent. The cylinders were then removed from the oven, handling being done by means of dried fibre strips, and set up between the planes for test. Arc-overs were taken while the cylinders were still hot and also after they had reached room temperature. The later condition is that referred to on the curves as dry glass (II). The hot arc-overs differed from the room temperature ones merely in proportion to the change in air density on the supposition of a heated film of air in the vicinity of the hot cylinders.

III—Wet glass cylinders. For these tests the cylinders were held under the water faucet just before tests were made.

For the tests in which the cylinders were coated with paraffine, or oil, etc., the test piece was soaked in the hot material and then tested after reaching room temperature.

The tests with hard rubber varied considerably as shown by the two curves. The upper curve is for hard rubber cylinders which had been re-surfaced with fine emery just before test. The lower curve shows the arc-over after the test piece had stood around the laboratory for a week or so. The tests with oiled

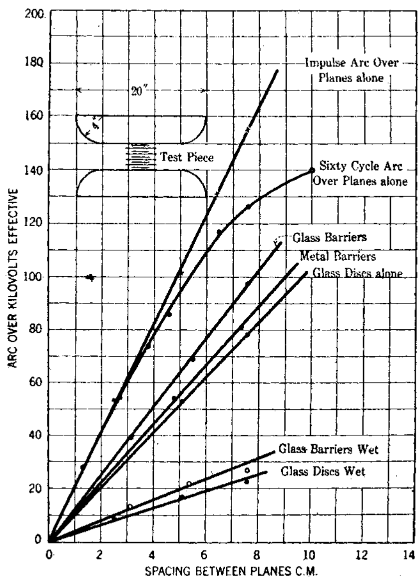

Fig. 22—Smooth Planes—Sixty-Cycle Tests in Air

hard rubber refer to cylinders which had been soaked in No. 6 transil oil for a few days.

Tests were also made on wound-up paper cylinders impregnated with compound.

The tests using glass disks given on curve sheet Fig. 22 were intended to show the effect of breaking up the surface first by the natural irregularity of the piled up disks (see Fig. 26), secondly with larger glass disks inserted at intervals, and thirdly with metal disks inserted at intervals. The glass disks were 0.075 in. (0.19 cm.) thick and 2 in. (5 cm.) and 4 in. (10 cm.) diameter and had roughly ground edges. The metal barriers were 5 in. (12.7 cm.) in diameter and 0.019 in. (0.048 cm.) thick.

The under-oil tests were carried out in a large wooden oil tank containing No. 6 transil oil, testing 40 kv. between 0.5-in. (1.27 cm.) terminals at 0.2 inches (0.51 cm.) spacing.

The impulse tests were made using F. W. Peek's impulse generator [21] with an impulse resembling half a cycle of a 200 kilocycle sine wave. The voltages given are on the assumption that arc-over occurs at the top of the sine wave impulse.

A few tests were made using glass tubes and hard rubber cylinders having different diameters, so as to see whether the arc-over was affected by changes in the surface resistance be-

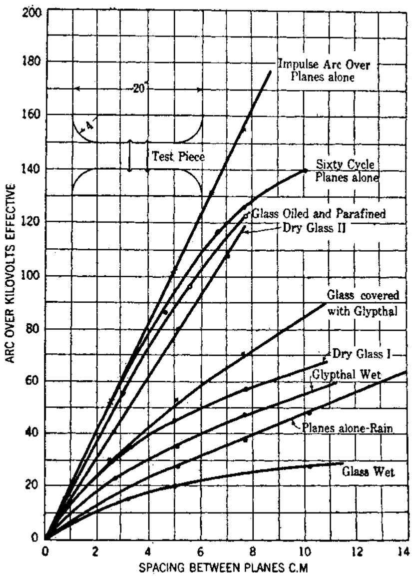

FIG. 23—GROOVED PLANES—SIXTY-CYCLE TESTS IN AIR

tween the planes. The tests did not, however, show any variation from the previous ones.

The incompleteness of the data and the irregularities which occurred between different samples of the same material as well as the rather crude methods of test make any conclusions rather questionable. I am, however, inclined at present to the following conclusions.

If all precautions were taken to absolutely free the test samples from moisture, and furthermore, if practically perfect joints were made between the test samples and the planes, either by electro-

21. See footnote 17.

statistically shielding the joints or by mechanical fit, I believe that the presence of insulation in parallel with air or oil, etc., in a uniform field would not result in a breakdown lower than that of the weakest material. When, however, the joint is not carefully made, I believe that premature breakdown occurs at the joint when a high inductive capacity material is placed in parallel with a lower inductive capacity material.

When dealing with dielectrics, which have been cleaned and dried, with what would be considered great care in the household sense of the word, there appears to be a very large reduction in arc-over under the conditions here discussed, namely, under uniform field conditions, depending upon the character of the material which cannot be accounted for by joint effect. In general, it may be further pointed out that materials such as clean new hard rubber, oiled, or paraffined surfaces, which Mr. Harvey L. Curtis has shown have a high surface resistivity under varying humidity conditions, also show a high breakdown under the test conditions here described. From a purely electrostatic point of view it is, of course, obvious that if the surface conduction took place in a perfectly regular and uniform manner it should not result in any reduction in the arc-over of the test pieces as the equipotential surfaces of conduction would coincide

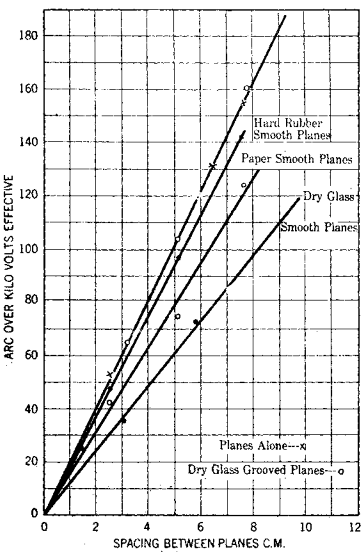

FIG. 24—IMPULSE TESTS IN AIR—200 KILO-CYCLES

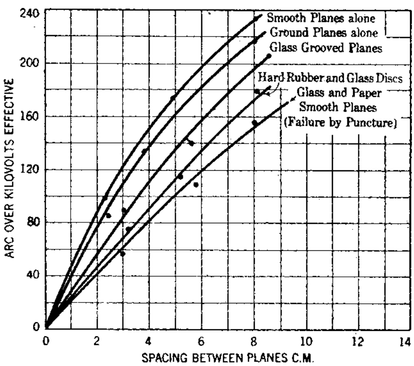

FIG. 25—SIXTY-CYCLE TESTS IN OIL

22. The Volume Resistivity and Surface Resistivity of Insulating Materials—*General Electric Review*, October 1915, p. 996.

with electrostatic ones and no distortion or increase in stress would thereby result. If, therefore, the reduction is to be attributed to surface leakage or conduction, this effect must be of an erratic and discontinuous nature.

It would be of considerable theoretical interest to test this conclusion experimentally. For example, I believe that a suitably high resistance and homogeneous metallic film could be obtained on a glass cylinder by subjecting it either to a cathode spray or by volatilization in a high vacuum. Some tests were tried in which ground-glass cylinders were coated with graphite, etc., but without satisfactory results.

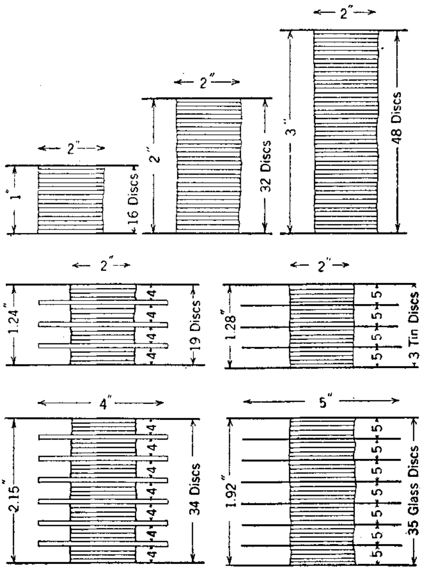

FIG. 26

I believe that a thorough and accurate investigation of this phenomena would be of considerable practical as well as theoretical interest.

REVISED ARC-OVER CALCULATION OF HIGH-AIR-EFFICIENCY BUSHING

A revised calculation of the arc-over voltage of the high-air-efficiency bushing may now be made which takes into account the surface and joint effects. We have seen previously that the surface arc-over of the bushing in air should take place under ideal conditions at about 150 kv. effective. Now, since the electrostatic conditions along the surface are practically identical for the bushing and for the test pieces between the smooth planes, a direct comparison is legitimate except for the fact that a glass surface is probably not identical with a glazed porcelain surface. Nevertheless the comparison is of interest.

Arc-over of High-Air-Efficiency Bushing	Arc-over of Smooth Planes 7.3 cm. Spacing
Tests in Air	
Average of dry and oiled smooth porcelain surfaces	Average of dry and oiled glass
60 cycle, 73 kv. effective	60 cycle, 67 kv. effective
Impulse, 117 kv. effective	Impulse, 90 kv. effective
Surface wet with city water	Surface wet with city water
60 cycle, 23 kv. effective	60 cycle, 21 kv. effective
Tests in Oil.	
Smooth porcelain surface	Glass cylinder,
60 cycle, 122 kv. effective	140 kv. effective

In conclusion it should be observed that if we are forced to use supporting dielectrics, which show large surface effects, we must consider this fact when we form any criterion for the best use of air or oil in combination with the supporting dielectric. If we assume that our surface of discontinuity between the air part of the dielectric and the solid or supporting dielectric, for instance glass, is greatly weaker than air alone, for example—only stands a potential gradient of 7.3 kv. effective per cm. instead of 21 kv. effective as would probably be the case if perfectly dry and clean; then considering this feature alone, we see that in order to use both materials most efficiently, the component of the potential gradient along the surface should not exceed 7.3 kv. effective per cm., whereas the air itself should be used to its full strength of 21 kv. effective per cm. Thus, the component of the potential gradient normal to the surface should be $G = \sqrt{21.^2 - 7.3^2} =$ 19.7 kv. effective per cm. or the flux lines should make an angle of approximately 70 deg. from the surface. This is practically the condition existing in the so-called "Radial Type of Field" in which the supporting dielectric and air are approximately in series in a radial field. It will be seen that when considering ordinary materials from which bushings are usually made that this surface effect practically requires us to go to the radial type of field in constructing a bushing.

CALCULATION OF THE ARC-OVER OF THE TEST PIECE, DESCRIBED AND TESTED BY C. L. FORTESCUE AND S. W. FARNSWORTH.[23]

The electrodes consisted of confocal hyperboloids of revolution of two sheets. The equation of the hyperbolas which constitute the generating curves were obtained from the description and Fig. 20 of the paper as follows: [24]

$$\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1$$

Where

$f = \sqrt{a^2 + c^2} = 1$ in. the semi-focal distance.

and $a = 0.875$ in.

$c = 0.485$ in.

from which we have

$$\frac{x^2}{(0.875)^2} - \frac{z^2}{(0.485)^2} = 1$$

The ellipsoidal hard-rubber supporting dielectric is cut out by rotating the ellipse whose equation was determined in a similar manner to be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Where $f = \sqrt{a^2 - b^2} = 1$ in.

and $b = 3.1$ in.

$a = 3.26$ in.

and

$$\frac{x^2}{(3.26)^2} + \frac{y^2}{(3.1)^2} = 1$$

Reference to Fig. 18 or 19 of the appendix shows that the hypobola No. 27 is very approximately the one corresponding to that used for the electrodes. The ellipse bounding the hard rubber is No. 22 counting out from the focal ellipse as 0.

23. See footnote 1.

24. See p. 863 and Figs. 1 and 2 of the appendix.

Assuming the breakdown strength of air as 21 kv. effective per cm. and perfect dielectrics, corona would start at the junction between the hard rubber and electrodes at 205 kv. eff. If we further assume that the corona formation is unstable arc-over should occur assuming perfect dielectrics without surface or joint effects at about this value. The factor which takes into account the reduction in arc-over, due to what has been called surface and joint effect, for the case of hard rubber in parallel with air in a uniform field, in which the full value of the gradient is along the surface, is applicable in this case. The tests have shown that a factor of about 73 per cent should be applied to the value as calculated neglecting this effect. Thus, the calculated arc-over of this structure is about 150 kv. effective, the test results reported by Fortescue and Farnsworth were given as 160 kv. effective.

Outline of Some of the Possible Methods for Increasing the Efficiency of the Use of the Supporting Dielectric

In the previous discussion on bushing design attention has been principally directed towards studying the possibilities of an efficient use of the air dielectric. Theoretically, the ideal bushing would be that in which the whole dielectric is used to its maximum efficiency, that is, it would be on the point of breakdown simultaneously at all points. Therefore, it is interesting to study the possibilities of increasing the efficiency of the use of the solid or supporting dielectric. For example, inspection of the diagram, Fig. 14 shows that the supporting dielectric in a bushing built along these lines is not used to its maximum strength except right near the rod and edge of the hole in the tank. Therefore, if it is possible by some means to redistribute the stress in the solid dielectric so as to bring about a more uniform condition, it would be possible to greatly reduce the diameter of the bushing as a whole. The following are some of the possible methods which might be employed.

I—Theoretically we could bring about the desired condition provided we had suitable dielectrics of various inductive capacities and dielectric strengths. We could then place the high inductive capacity materials near the rod and edge of the hole in the tank putting the lower inductive capacity materials in the less dense parts of the field and in some such manner as this bring about the condition in which all the dielectrics used were stressed to their maximum strengths. This is the well known principle of graded insulation. Unfortunately there is not a very wide variation in the inductive capacities of the materials which are available in practise.

II—If there were available extremely high resistance materials, or dielectrics, we could construct a bushing which would maintain a proper potential distribution by conduction through the dielectric. This might be called a resistance bushing. A graded resistance bushing would also be possible.

III—Another method is to insert artificial equipotential surfaces and thereby control the stresses in the manner which we desire. In order to effect the desired distribution of stresses the artificial equipotential surfaces must be maintained at the proper potentials. Some of the methods which can theoretically be used for this purpose are as follows:

A. Metallic connection to a proper source of potential. For example, to transformer taps or external balancing resistances, inductances or capacities, etc. A balancing resistance might even be embedded in the dielectric.

B. By conduction through the supporting dielectric. This would be similar to the resistance bushing mentioned above except that the distribution of the conduction current could be modified, in various ways, by the insertion of artificial equipotential surfaces.

C. Electrostatically (or the well known condenser principle)[25] in this case we have the balancing condensers embedded in the supporting dielectric, and, at the same time, forming the artificial equipotential surfaces. Of course graded insulation is also applicable to this type of bushing.

Theoretically where artificial equipotential surfaces are used they could be extended through the surface of the solid or supporting dielectric into the air part of the field where they could be used to assist in bringing about the desired field distribution, and at the same time, even be made to act as petticoats to shield the surface from rain, etc.

Obviously the possibility exists of making various combinations of the above principles.

When the practical difficulties of utilizing the above methods

25. R. Nagel *Elektrische Bahnen und Betriebe*, 1906, p. 278; A. B. Reynders, A. I. E. E., TRANS., 1909, Vol. XXVIII, Part I, p. 209; C. L. Fortescue, *Electrical Journal*, August 1913, p. 718; W. S. Franklin, *Journal of the Franklin Institute*, July 1913.

are considered it appears that case III alone, or possibly in combination with grading by inductive capacities, is the most feasible at the present time. Therefore, a brief study of this case was undertaken making use of the electrodynamic method.

Preliminary Study of the Use of Artificial Equipotential Surfaces or Potential Equalizers

Diagrams of Figs. 27 and 28, were taken as a preliminary study of the effects produced by forcing a uniform distribution of

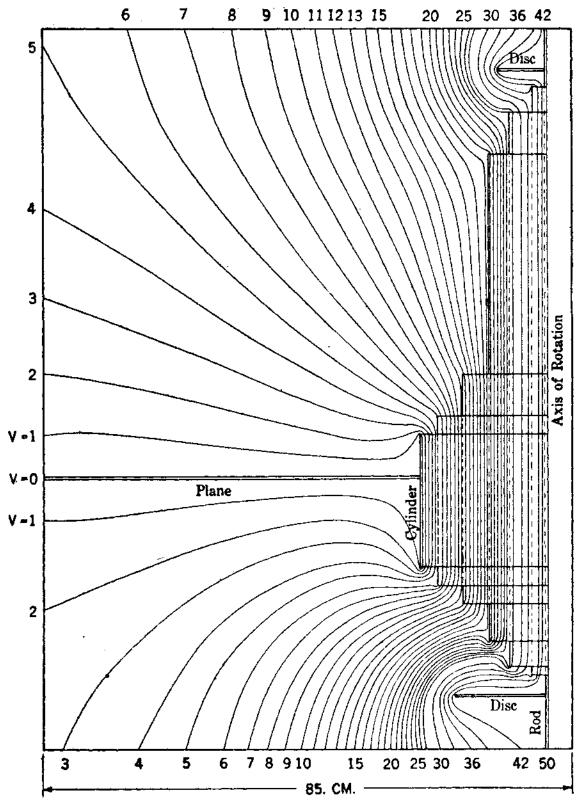

Fig. 27

potential gradient in the solid or supporting dielectric by the use of cylindrical artificial equipotential surfaces. Diagram, Fig. 27, shows the effect when the radial-field type is used on the air end and diagram, Fig. 28, for the uniform-field type. The under-oil ends are what might be termed semi-radial.

At the outset, it should be noted that it is necessary to study both the air and oil ends simultaneously if they are not symmetrical, otherwise the problem is indeterminate. Briefly the method of experiment was as follows: The skeleton electrodes, for Fig. 27 consisted of a plane with cylinder (ground shield) and a central rod provided with a tin disk at both ends. The disk on the under-oil end was put in so as to make a definite field in which to end the artificial equipotential surfaces. If this were not done the bushing would be greatly affected by the piece of apparatus in

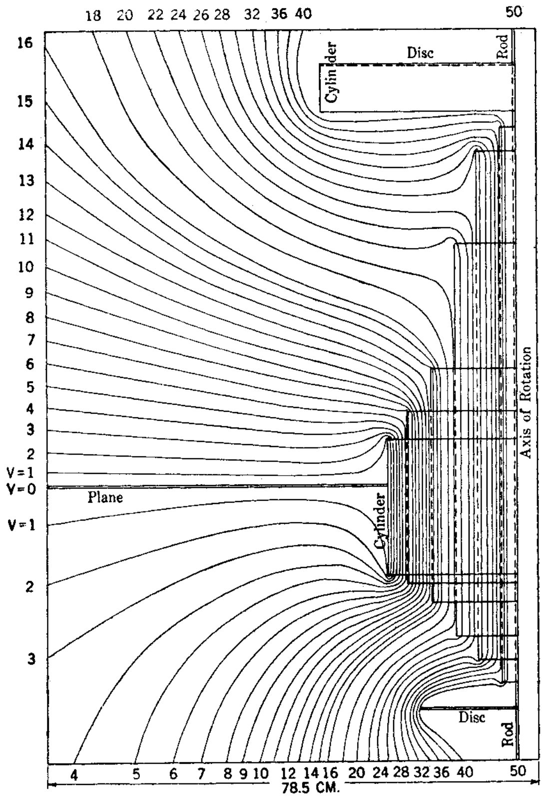

FIG. 28

which it was afterwards used. I believe that this feature is quite important. The ground shield was used in order to relieve the stress on the air above the oil level, as usual, and was projected into the air end so as to shield the joint which would be necessary in construction. Five tin quarter-cylinders were then added as the artificial equipotential surfaces. These were constructed of two pieces so that they could be telescoped, thereby making it possible to change their length and the position of their ends both in the air and oil parts of the field. The adjustment was accomplished by a system of strings so that different positions could be tried while experimenting with water in the box. The apparatus was set up in this manner and the position of the cylinders in the air and oil ends adjusted until the potential gradient between the cylinders was as nearly equal as could be easily predetermined. It was during the course of this work that it was found necessary to add the tin disk at the top of the bushing, (See Fig. 27) in order to have a dense field in which to end the two inner cylinders. If this were not done they would have had to be either too close to the under-oil disk which would mean too great a stress at the bottom of the under-oil end, or greatly extended in the air end. In the latter case the resulting bushing would have been much higher than seemed desirable for the preliminary work where an exaggeration of effects was desirable. Naturally, the position of the potential equalizing cylinders may be almost anything, that is, the potential of any of the cylinders could be brought to the desired value either by adjustment of the ends of the cylinder in the air or oil end. Such an arbitrary arrangement, without regard to the stresses brought about on other parts of the bushing, would obviously result in an impractical design. For instance, it might result in bad stresses at the ends of the cylinders and outside of them at the expense of the uniform gradient thus obtained within. In both diagrams the object was to obtain, as nearly as possible, equal gradient within the cylinders, and at the same time, obtain the best external field consistent with this requirement. Naturally, in practise it would probably be best to compromise between the internal and external parts of the field but for a preliminary study it seemed best to take one object to start with and study the consequences.

When the diagram was completed, it was found that the preliminary adjustment had not actually obtained the uniform gradient in the solid material as had been attempted. For example, the distributions between the various artificial equipotential surfaces starting from the ground shield were as follows: (See Figure 27).

I —Number of lines.................. 6.
Average gradient.............. $\dfrac{6 \text{ lines}}{2.92 \text{ cm.}} = 2.05$

II —Number of lines.................. 6.4
Average gradient.............. $\dfrac{6.4 \text{ lines}}{3.94 \text{ cm.}} = 1.62$

III —Number of lines.................. 7.
Average gradient.............. $\frac{7 \text{ lines}}{4.2 \text{ cm.}} = 1.66$

IV —Number of lines.................. 6.5
Average gradient.............. $\frac{6.5 \text{ lines}}{3.3 \text{ cm.}} = 1.97$

V —Number of lines.................. 10
Average gradient.............. $\frac{10 \text{ lines}}{3.8 \text{ cm.}} = 2.64$

VI —Number of lines.................. 14
Average gradient.............. $\frac{14 \text{ lines}}{2.16 \text{ cm.}} = 6.5$

No attempt was made to obtain a low gradient in the last case, as it was contemplated to select the equipotential surface of the inner cylinder as our rod with cap.

Inspection of Fig. 27 will show that a concentration of the field at the top and bottom ends of the bushing has resulted from forcing the nearly uniform gradient between the cylinders. It should also be observed that the capacities of the concentric cylinder condensers formed by the artificial equipotential surfaces have no definite relation, under these conditions, where the effect of the caps and assymmetry of the problem introduce large effects. For example, assuming as the length of the condensers the mean height of the two adjacent cylinders we have for the capacities starting from the ground shield.

I $C = \frac{\text{mean height}}{\log_\epsilon \frac{R}{r}} = \frac{25.2 \text{ cm.}}{\log_\epsilon 1.16} = \frac{25.2}{0.148} = 170.$

II C proportional to............ $\frac{33.3 \text{ cm.}}{\log_\epsilon 1.29} = \frac{33.3}{0.255} = 131.$

III C " " .... $\frac{59.4 \text{ cm.}}{\log_\epsilon 1.43} = \frac{59.4}{0.358} = 166.$

IV C " " .... $\frac{86.0 \text{ cm.}}{\log_\epsilon 1.52} = \frac{86}{0.419} = 205.$

V C " " .... $\frac{94.2 \text{ cm.}}{\log_\epsilon 2.5} = \frac{94.2}{0.916} = 103.$

VI C " " .... $\frac{100 \text{ cm.}}{\log_\epsilon 6.7} = \frac{100}{1.90} = 53.5$

I wish to emphasize caution against any conclusions which might be drawn from the above values for the capacities; for an entirely different adjustment of the lengths of the cylinders could obviously be obtained which would still give equal potential gradient between the successive cylinders and in which the capacities calculated as above would have entirely different values and interrelations.

Before leaving the discussion of this diagram it may be interesting to note that if we built a bushing along these lines; for example, if we selected equipotential surface No. 36 as our central rod with cap and equipotential surface No. 0 as plane with ground shield, corona would start at the top of the bushing at about 238 kv. effective and would probably arc-over slightly above this value. The average voltage per cm. of height of this bushing would be $\frac{238 \text{ kv.}}{70 \text{ cm.}} = 3.4$ kv. effective per cm.

Diagram, Fig. 28 was taken using the same electrode configuration except that the large skeleton cap was put in place of the small one used for Fig. 27. The position of the cylinders was adjusted with the same objects in view as for the previous diagram.

The distribution of potential between the various artificial equipotential surfaces starting from the ground shield will be seen to be as follows:

(See Diagram, Fig. 28).

I. —Number of lines.................. 6.5

Average gradient.............. $\frac{6.5 \text{ lines}}{3.05 \text{ cm.}} = 2.13$

II. —Number of lines..................7.

Average gradient.............. $\frac{7. \text{ lines}}{3.94 \text{ cm.}} = 1.77$

III.—Number of lines.................. 7.5

Average gradient.............. $\frac{7.5 \text{ lines}}{3.94 \text{ cm.}} = 1.90$

IV.—Number of lines.................. 6.

Average gradient.............. $\frac{6 \text{ lines}}{3.55 \text{ cm.}} = 1.69$

V. —Number of lines.................. 10

Average gradient.............. $\frac{10 \text{ lines}}{3.82 \text{ cm.}} = 2.62$

VI.—Number of lines.................. 13

Average gradient.............. $\frac{13 \text{ lines}}{2.16 \text{ cm.}} = 6.0$

As before, no attempt was made to obtain a low gradient inside of the small last cylinder as it was contemplated using this equipotential surface for the rod with cap, as in the previous case.

The capacities of the adjacent cylinders considered as concentric cylinder condensers, starting from the ground shield, are as follows:

I.— $C = \ldots\ldots\ldots \frac{\text{mean height}}{\log_\epsilon \frac{R}{r}} = \frac{24.9 \text{ cm.}}{\log_\epsilon 1.17} = \frac{24.9}{0.157} = 158$

II.— C proportional to .... $\frac{32.8 \text{ cm.}}{\log_\epsilon 1.27} = \frac{32.8}{0.239} = 138$

III.— " " " .... $\frac{51 \text{ cm.}}{\log_\epsilon 1.41} = \frac{51.}{0.344} = 148$

IV.— " " " .... $\frac{73 \text{ cm.}}{\log_\epsilon 1.56} = \frac{73}{0.445} = 164$

V.— " " " .... $\frac{86 \text{ cm.}}{\log_\epsilon 2.5} = \frac{86}{0.916} = 94$

VI.— " " " .... $\frac{99 \text{ cm.}}{\log_\epsilon 6.7} = \frac{99}{1.90} = 52$

As in the previous case, caution should be urged against hasty conclusions drawn from the above capacity values, for in this case also an entirely different adjustment would be possible in which the magnitudes and interrelations between the capacities would have very different values.

In designing a bushing from this diagram we might select equipotential surface No. 38 for the rod with cap and equipotential surface No. 0 for the plane with ground shield. A straight cylindrical shell would be added enclosing the central core and the ground shield. The resulting distribution of the potential on the air end of such a bushing would be approximately uniform. Under these conditions corona would start at the cap on the air end at about 440 kv. effective which gives an average gradient along the surface of 7.75 kv. effective per cm.

The tests made on glass cylinders in a uniform field in which the surface breakdown gradient was found to be about 9.2 kv.

per cm. indicate that the air end of this bushing is about as short as practical assuming good surface conditions. A shell would also be added to the under-oil end of the bushing enclosing the ground shield. Inspection of the under-oil end shows a concentration of the field about the cap. Also in view of the under oil arc-over tests on glass cylinders and hard rubber it looks as if the under-oil end is about as short as possible. The diameter of this bushing is evidently larger than necessary to withstand puncture as will be readily seen by comparing the thickness of insulation used with that found in practise to be necessary.[26] It will also be observed that the ratio of external diameter to rod diameter as used in practise is considerably less than that used for these preliminary experiments where it was about 8, if we take the inner equalizer as our rod with caps. For example, the ratio between outside diameter to rod diameter varies in practise between 2.3 and 3.7. With these relatively small diameter ratios the gradient without artificial equipotential surfaces would not be very far from uniform to start with (see diagram, Fig. 10). There is, therefore, very little potential equalizing to do. In our tests, however, we have taken a ratio of about 8, which means that we have an exaggerated condition and considerable equalizing to do.

This immediately brings up the question as to whether the small diameters of the condenser bushings is due to the equalizing effect of the tin-foil layers in bringing about uniform potential gradient, or whether the effect is not largely due to the subdivision of the solid material by metallic barriers as well as the laminated structure of the dielectric itself, thereby increasing the apparent strength of the structure,(an effect analogous to the use of solid barriers, pressboard, etc.) in liquid dielectrics. This matter will be briefly discussed before leaving the subject.

It should be observed that there is no definite proper ratio between the diameters of the central rod and the hole when artificial equipotential surfaces are used in this manner. It is, however, undesirable to use too small a central electrode because this means that more redistribution of stress has to be brought about by the artificial equipotential surfaces. This in general necessitates either too long a bushing or an excessive stress at the ends. If we select a ratio of rod to hole diameter which gives more nearly equal gradient at the

26. See "Construction and Application of the Condenser Terminal", by J. E. Mateer, *Electrical Journal*, August, 1913.

edge of the hole and rod without artificial equipotential surfaces, we will have less potential equalizing to do which will probably be found advantageous.

Experimental Bushing with Artificial Equipotential Surfaces, Uniform-Field Type

A bushing of the uniform-field type was built from a photographic reduction of Fig. 28. It has been pointed out that

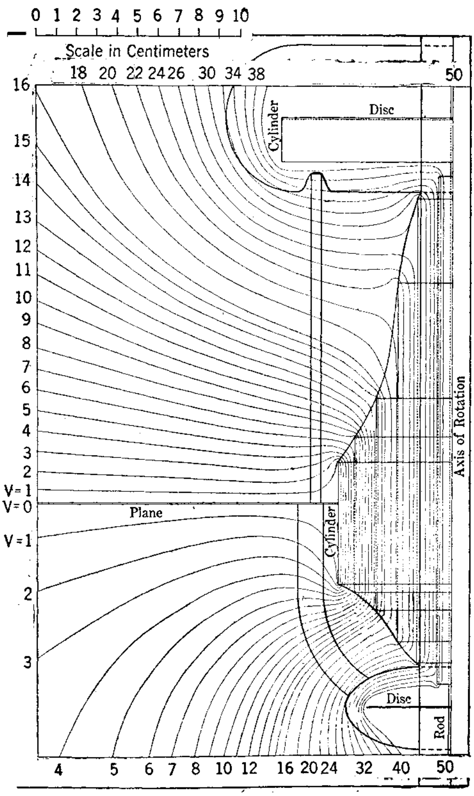

Fig. 29

in taking the diagram the artificial equipotential surfaces were adjusted so as to give an approximately uniform distribution of stress in the solid or supporting dielectric, while at the same time we attempted to produce as good an external field distribution as could be foreseen before the diagram was actually completed.

Fig. 29 shows the outline of the bushing drawn on the diagram of Fig. 28. The electrodes were not made to rigorously conform to any of the equipotential surfaces in the diagram as it did not seem necessary to do so, since the field will probably not be greatly changed by the deviations made. A groove was put in the cap on the air end to electrostatically shield the joint between the supporting dielectric and the air. The condenser core of the bushing was made of wound-up shellac paper with tin-foil layers at intervals of 1/16 in. It was turned down to the shape shown in Fig. 29 and then stepped off as will be seen from the photograph Fig. 30.

A shellac paper cylinder enclosed the air end of the bushing and the shell for the under-oil end was built up of hard-rubber rings. Sarco, an asphaltic compound, was used as filler.

Arc-Over Voltage as Calculated from the Diagram

An inspection of Fig. 29 will show that the gradient on the air end is very nearly parallel to the surface of the containing cylinder and is approximately uniformly distributed being somewhat higher near the top. The maximum gradient on the air end is seen to be on the cap away from the surface. Under these conditions corona should start on the cap at about 90 kv. effective as calculated from the diagram making allowance for the fact that the cap used did not rigorously conform to any single equipotential surface of the diagram. The bushing is 15 cm. high and therefore the average gradient along the surface would be 6.7 kv. eff. per cm.

Inspection of the under-oil end shows that a concentration of stress has resulted on the cap from forcing the uniform internal stress by the artificial equipotential surfaces. The distance between lines No. 22 and 28 (6 lines) where the under-oil shell meets the cap is 0.8 cm. thus we have 7.5 lines per cm. Tests on the under-oil arc-over of rubber cylinders between parallel planes gave a surface arc-over gradient of about 23 kv. effective per cm. Assuming this data to apply in this case, arc-over of the under-oil end of the bushing should take place at

about $\frac{30}{7.5} \times 23 = 92$ kv. effective.

It will be observed from these calculations that the air and under-oil ends are about equally strong. Before the tests were made it was therefore doubtful which end would arc-over first as the calculations are necessarily only approximate, since we have not actually followed the equipotential surfaces of the diagram.

Sixty-Cycle Dry Arc-Over Tests

These tests were made with the plane grounded:

Arc-over voltage.....................86 kv. effective
Average gradient along surface........ 5.7 kv. per cm.

Arc-over took place clear of the surface from the cap to the plane. Upon applying the voltage for the third time the bushing apparently punctured either inside of the shell or under oil at 76 kv. effective. Inspection did not indicate an under-oil arc-over and therefore the bushing had to be taken to pieces in order to determine the cause of failure. This process necessitated destroying the bushing as it is difficult to get the compound out in any other way. The investigation showed that the result of failure was due to an arc-over of the bushing on the air end along the inside surface of the containing shell. A large blister about 1½ in. (4 cm.) long was found at this point which had held the compound away from the surface and thus left an air pocket which weakened the inside surface and resulted in the failure.

I was disappointed not to be able to obtain wet arc-over, impulse arc-over and under-oil arc-over, but did not believe that these results would warrant rebuilding a bushing of this sort for it is quite certain that the results would be low, for the same reasons which were met with in the tests on the high-air-efficiency bushing and from which it was concluded that a more nearly radial type of field is generally desirable in practise.

A reference to Fig. 10, will show that in the radial-field type of bushing the natural equipotential surfaces are approximately cylindrical in form and therefore it will be impossible to produce any considerable change in the potential distribution by means of cylindrical artificial equipotential surfaces, unless they extend to a great distance in the air end, or unless a cap or hat, or the equivalent, is provided on the under-oil end, which changes the natural equipotential surfaces from the cylindrical form, and therefore, allows us to bring about a change in potential distribution by inserting cylindrical artificial equipotential surfaces. That is, if our inserted metal equipotential surfaces do not differ in form from the naturally existing equipotential surfaces no effect in the potential distribution results from their insertion, regardless of number, spacing, etc. It is, therefore, useless to employ them unless they produce an effective increase in the apparent strength of the solid or supporting dielectric by virtue of a subdivision of the dielectric into elements. The following preliminary tests were, therefore, undertaken to check up this point.

Effect of Barriers on Strength of Insulation

It is well known that the strength of dielectrics decrease with increase in thickness. I believe that Lord Kelvin was the first to observe this effect for the case of air[27] in 1860.

For example, assume that we have two parallel plane electrodes with such a gradual curvature at the edges that the gradient is always less at the edges than that in the uniform part of the field. If we then take measurements of the voltage required to breakdown various thicknesses of air, we find that the apparent strength of air as determined by the potential gradient necessary to cause breakdown decreases as we increase the spacing of the planes. At exceedingly small spacings the potential gradient required to produce breakdown may be very great, for example, tests recently made by F. W. Peek, Jr.,[28] using 2.54 cm. diam. spheres at 0.0035 cm. spacing gave as the disruptive gradient 150 kv. effective per cm. As we increase the spacing the breakdown gradient for air apparently approaches a constant value of approximately 21 kv. effective per cm. at normal temperature and pressure.

De La Rue and Müller investigated this effect[29] for various spacings of parallel plane electrodes at constant pressure and temperature as well as the variation under constant spacing and varying pressure and concluded that, "The law of the hyperbola holds equally well for a constant pressure and varying distance as it does for a constant distance and varying pressure; the obstacle in the way of a discharge being as the number of molecules intervening between the terminals up to a certain point." Harris[30] had previously found that a change in air density pro-

24. Measurement of the electromotive force required to produce a spark in air between parallel metal plates at different distances. Proceedings Royal Society, Feb. 23 and April 12, 1860, or *Philosophical Magazine*, 1860, or Papers on Electrostatics and Magnetism, p. 247–259 Lord Kelvin. An account of these tests will also be found in, Electricity in Gases, J. S. Townsend, p. 346.

28. Law of Corona and Dielectric Strength of Air, III., F. W. Peek, Jr., A. I. E. E., TRANS., 1913, Vol. XXXII, Part II, p. 1767.

29. Experimental Researches on the Electric Discharge with the Chloride of Silver Battery. Phil. Trans. Royal Society, Part I, 1880, p. 79–83.

30. W. S. Harris, Phil. Trans. Royal Society, 1834, p. 230.

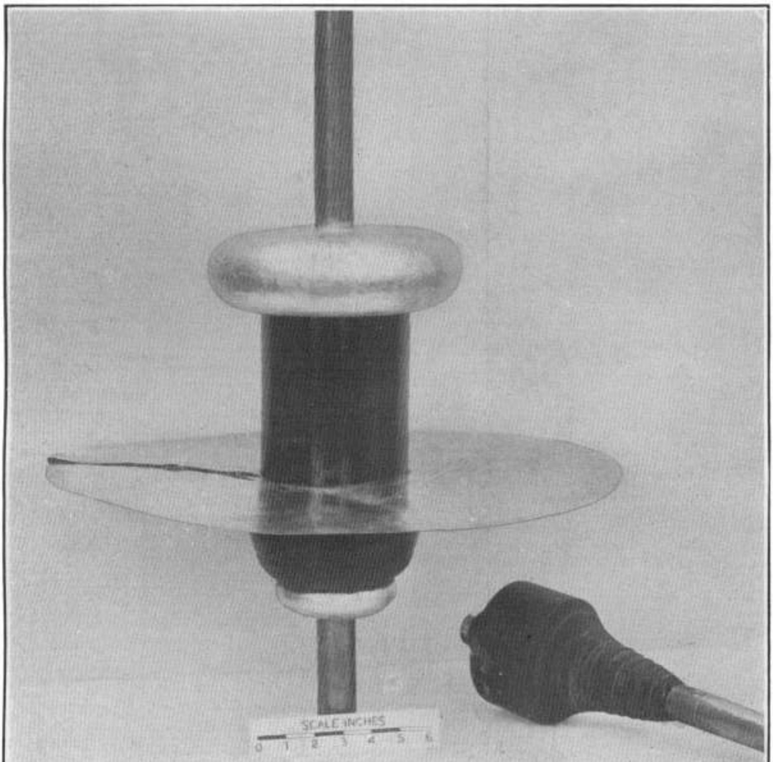

FIG. 30 [RICE]

duced the same effect whether due to temperature or pressure, that is, when air is contained in an air-tight receiver, so that the density remains constant the potential difference between two electrodes required to produce a discharge was unaltered when the temperature varied from 50 deg. fahr. to 300 deg. fahr.

If we immersed our electrodes in oil we would obtain a similarly shaped curve showing the relation between the disruptive gradient and spacing. The constants in the equations for air and oil would probably, however, be different. Again if we imagine our parallel plane electrodes embedded in some solid dielectric, for example glass, we would obtain a similarly shaped curve. For very great and very small spacings it is quite possible that all materials have the same apparent strength.

Now, if we assume that our dielectric is homogeneous and isotropic and is a perfect dielectric in every respect (no conduction, dielectric losses, etc.,) the equipotential surfaces for equal differences of potential would be equally spaced between our parallel planes. If we desired we could put in infinitely thin conducting sheets at any desired intervals without in any way changing the problem from a purely electrostatic point of view. For convenience we could space them at equal intervals so that the dielectric between any two adjacent metal equipotential surfaces between the planes was subjected to the same potential gradient. If we wished we could also metallically connect the inserted metal sheets to equal intervals on a balancing resistance shunted across our parallel plane electrodes, that is, connect the equipotential surfaces to a source of potential having the value naturally existing. Obviously nothing has been changed electrostatically by any of these processes and therefore it is evident that we can confine our attention to the dielectric included between any two equipotential surfaces, and if we now raise the potential between our parallel plane electrodes the potential gradient on the dielectric between the two equipotential surfaces which we have selected to watch will increase until finally breakdown will occur when the gradient exceeds a certain value called the strength of the material. It is observed that nothing has been said about the actual thickness of the section of the dielectric between our two metal equipotential surfaces under observation. In other words, we might in one case insert a great number of equipotential surfaces dividing the dielectric into exceedingly thin layers and in another case use fewer metallic surfaces and have thicker insulation under observation. When considering ideal dielectrics from a purely electrostatic point of view, as here described, I think that it is clear that the actual thickness of the dielectric should have no influence upon the breakdown strength. For it should not make any difference whether we inserted isolated thin metal equipotential surfaces or left them out, or inserted them and connected them to a proper source of potential. Nothing that we could detect from a purely electrostatic view point has been changed. It is also interesting to observe that a uniform leakage current should not affect these conclusions provided the heating effects were taken care of, since the equipotential surfaces for current flow would coincide with the electrostatic equipotential surfaces and no potential distortion would result.

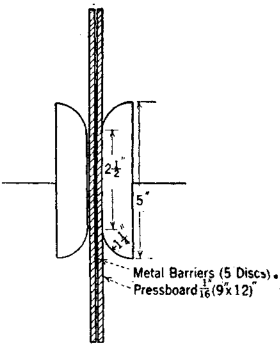

FIG. 31—ARRANGEMENT OF PRESSBOARD AND BARRIERS

The fact that we observe a difference in the apparent strength of dielectrics with different thicknesses, which is contrary to the above electrostatic reasoning, based on the assumption that the metal equipotential surfaces could be inserted without changing the breakdown strength, makes it interesting to determine what does take place in practise when a dielectric is divided up in this manner. Obviously when we consider the case of the metallic equipotential surfaces connected to the proper sources of potential, the breakdown strength must be the same for a given spacing between the surfaces or thickness of dielectric, no matter how many are connected in series, or what is the same thing, regardless of the total thickness of insulation under test, for each element forms a separate test piece isolated between the two metal surfaces and connected to a definite potential. The interesting thing therefore is a comparison of the dielectric strength of a given thickness of insulation with and without isolated metallic barriers placed at various spacings.

I am sorry that the experimental results, which I have to offer, are so crude and incomplete, but if they will serve the purpose of starting some one on a more complete investigation of this subject I will regard them as having served a good purpose. I believe that a complete investigation of this subject would lead to a great deal better understanding of the true mechanism of the breakdown of insulations, or the theory of ionization by collision which of course offers a qualitative explanation of the phenomena.

Tests to Determine the Effect of Barriers on Solid and Gaseous Insulation Under Uniform-Field Conditions

The first set of tests recorded below in Table I were made on sheets of oiled pressboard 1/16 in. (0.16 cm.) thick. The barriers and pressboard were sealed together under No. 6 transil oil and then placed between the parallel plane electrodes as shown in Fig. 31, the whole then being immersed under No. 6 transil oil at 25 deg. cent. for test.

The object of using solid metal barriers and comparing their effect with wire-gauze barriers was to see whether the mean velocity of the particles or ions in the solid dielectric, which must precede rupture, was sufficiently high to allow an appreciable number, to pass through the openings in the gauze, whereas they should all be stopped by a metal sheet. Electrostatically the gauze should not be essentially different from the solid metal sheet.

TABLE I

Instantaneous 60-cycle puncture tests in No. 6 transil oil at 25 deg. cent. on dried 1/16 in. oiled pressboard. The average of 5 readings are given. A maximum variation of ± 2 kv. existed between individual raedings.

No. of sheets of pressboard each 0.16 cm. thick	60-cycle puncture kv. effective	Average gradient kv. eff. per cm.	Number of isolated barriers.
1	70.5	445.0	no barriers
2	113.0	355.0	" "
3	164.5	345.0	" "
2	117.0	368.0	1 tin barrier
2	124.0	390.0	Fine copper gauze
2	121.0	380.0	Coarse copper gauze

The points of puncture were well distributed over the surface of the insulation and in no cases occurred at the edge of the barriers. Contrary to the expectation the gauze barrier seems to be more effective than the tin barrier but the tests are so fragmentary that conclusions are not safely drawn. If, we attempted to use larger thicknesses arc-over occurred around the edge of the pressboard before puncture. A more satisfactory method of test was, therefore, sought. The method decided upon consisted in setting up a series of parallel plane electrodes as shown in Fig. 32, the sheet insulation being placed between the several electrodes in series. Thus, each pair of electrodes which are connected together may be considered as a barrier. It is also possible, under these conditions, to have the barriers isolated or connected across a proper balancing resistance. An additional barrier can be placed midway between each pair of electrodes without fear of distorting the field.

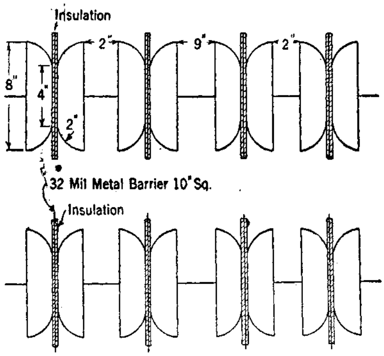

FIG. 32

A great difficulty with this method is the unbalancing of stress which the capacity or electrostatic flux to ground of the various electrodes will introduce. It was hoped that possibly this effect would not be appreciable when using fairly high inductive capacity material for the tests, as the capacity between the

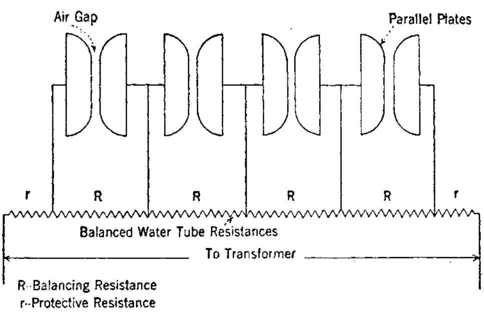

FIG. 33

electrodes proper would then be large in comparison with that to ground. A few tests made with one end of the series grounded and compared with the case in which the electrodes were isolated, in which case the neutral or ground potential would be in the middle of the series, shows a rather large effect which means that the isolated tests are subject to voltage distortion from this cause though to a less extent than when one end was grounded.

Fig. 32 shows the arrangement of barriers and insulation for four and eight isolated barriers respectively. When it was desired to metallically connect the barriers to the proper potential, water tube resistances were used of such a value as to allow 7 to 10 times the capacity current to flow (see Fig. 33). The maximum variation of the individual resistance tubes was about 2 per cent. The apparatus was placed at about 42 in. above the floor of the room and the gap line parallel to the ground. The tests were made in No. 6 transil oil (40 kv. flat electrodes 0.2 in. (0.51 cm.) gap, 0.5 in. (1.27 cm.) diameter) at 25 deg. cent.

The insulation used was 12-mil black varnished cambric the individual sheets being sealed to each other under oil. When the balancing or shunted resistance was connected the voltage readings were corrected for the drop in the protecting resistances. The results are tabulated in Table II.

TABLE II

Instantaneous 60-cycle tests on 12-mil black varnished cambric immersed in No. 6 transil oil at 25 deg. cent. Terminals 4 in. plane with 2 in. radius at edge. Individual readings differed by ± 1 kv.

Variation of Insulation Strength with thickness or number of layers.

Average of five readings

Using one pair of planes

No. of sheets	Total thickness insulation cm.	60-cycle kv. effective	Kv. per cm. effective
1	0.031	16.5	540.0
2	0.061	29.7	480.0
4	0.122	51.5	422.0
8	0.244	92.5	378.0
16	0.488	138.3	284.0

Data repeated using four pair of planes connected in parallel to obtain the effective strength when larger areas are used and therefore greater chances for weak spots.

Average of two readings.

1	0.031	14.7	482.0
2	0.061	28.5	451.0
4	0.122	51.3	421.0
8	0.244	91.7	376.0
16	0.488	136.5	280.0

Effect of Isolated Barriers on Puncture Strength of Insulation.

Condition of test	Total thickness insulation under test cm.	Thickness between barriers cm.	Non-Grounded			One End Grounded	
			Puncture kv. eff.	Kv./cm.	Kv./cm. roughly corrected	Puncture kv. eff.	Kv./cm.
2 sheets between each of 4 planes in series	0.244	0.061	106.8	438.0	521.0	86.8	355.0
1 sheet between each of 8 barriers in series	0.244	0.031	105.8	434.0	516.0	....	....
4 sheets between each of 4 barriers in series.	0.488	0.122	168.5	345.0	411.0	....	....

Effect of Barriers when Shunted by High Resistance on the Puncture Strength of Sheet Insulation

2 sheets between each of 4 barriers in series. Each unit shunted by 8 × 10^5 ohms.	0.244	0.061	112.6	462.0	433.0	98.1	403.0
1 sheet between each of 8 barriers in series. Each unit shunted 4 × 10^5 ohms	0.244	0.031	110.0	452.0		....	....
4 sheets between each of 4 barriers in series. Each unit shunted 8 × 10^5 ohms	0.488	0.122	195.8	402.0		....	....

In order to obtain an estimate of the unbalancing due to the capacity of the parallel planes to ground in this method of determining the effect of barriers on the strength of insulation the following tests were carried out using air as the dielectric. Two sets of tests were tried, the first using ½-in. (1.27 cm.) spacing between the individual planes and for the second a ¼-in. (0.635 cm.) gap was used. The gaps between the individual pairs of planes were set as accurately as feasible and individual spark-over readings taken non-grounded and with one end grounded. The gaps were then all connected in multiple to check the strength of this combination as compared with the weakest gap. After doing this they were connected in series and arc-over taken both grounded and isolated with and without shunted balancing resistance. A summary of the data is given in Table III.

From other sources we know that the strength of air for large spacings (*i. e.*, 2 in. or 5.1 cm.) between parallel planes is about 21 kv. eff. per cm. Whereas these data show an average of 18.1 kv. eff. per cm. when the tests were made non-grounded and 14.2 kv. eff. per cm. for the grounded case. That is, when using air as our dielectric with this arrangement we have an increase in stress of $1 - \frac{18.1}{21.} = 14$ per cent brought about by unbalance of voltage due to capacity to ground, etc. When operating with one side grounded we have $1 - \frac{14.2}{21} = 32$ per cent or approximately twice the ill effect.

This immediately gives us a rough method of correcting the previous tests for this unbalancing effect where no balancing resistance was used. We may write the following from the data given in the table for isolated barriers.

$1 - \frac{438}{x}$ = per cent apparent decrease in strength of insulation assumed due to capacity to ground for the non-grounded tests.

$1 - \frac{355}{x}$ = per cent apparent decrease in strength of insulation for grounded tests.

Where

x = assumed true strength of the material.

If we further assume that the apparent decrease for the grounded tests is twice the decrease for the non-grounded tests as shown to be approximately the case in air, we can write

$$2\left(1 - \frac{438}{x}\right) = \left(1 - \frac{355}{x}\right)$$

$x = 521$ assumed true strength of the material.

TABLE III

½-INCH GAP

Relative Humidity 40 per cent. Bar. 76 cm. Temp. 25 deg. cent.

Condition of test	Non-Grounded		Grounded	
	60-cycle kv. eff.	Kv./cm. eff.	60-cycle kv. eff.	Kv./cm. eff.
Gap 1 ½ in.	26.3		26.3	
2	26.3		26.3	
3	26.5		26.4	
4	26.4		26.4	
Ave.	26.4	20.8	26.4	20.8
Four gaps in multiple..	26.3	20.7	26.3	20.7
Four gaps in series.	87.5	17.2	69.5	13.5
Four gaps in series each shunted by 7×10^5 ohms......	106.0	20.8	106.0	20.8

¼-INCH GAP

Temp. 20 deg. cent. Bar. 76 cm. = 1.00

Condition of test	Non-Grounded		Grounded	
	60-cycle kv. eff.	Kv./cm. eff.	60-cycle kv. eff.	Kv./cm. eff.
Gap 1 ¼ in.	13.9	22.0		
2	14.1	22.2		
3	13.8	21.8		
4	13.9	22.0	14.0	22.1
Average	13.9	22.0		
Four gaps in multiple (arc-over No. 3)...	13.8	21.8		
Four gaps in series.	48.5	19.0	38.0	15.0
Four gaps in series each shunted by 8.5×10^5 ohms with 1.4×10^5 protective resistance.....	52.0	20.5	51.0	20.1
Averaging the data for ½-in. and ¼-in. spacings.				
Four gaps in multiple		21.2		
Four gaps in series.		18.1		14.2
Four gaps in series each shunted by resistance.........		20.7		20.4

The strength as indicated from the tests was 438 kv. effective per cm. or only 84 per cent of the value obtained after applying this rough method of correction. I have also applied the same correction factor to the rest of the values in this part of the table. We might consider the corrected column as an upper limit and the data as observed from test as a lower limit to the true value. In that case the average of these two columns would give another estimate of the strength as affected by barriers.

When considering the tests using air dielectric in which balancing resistances were used, we see that the stress due to capacity to ground is practically eliminated. It should also be noted from these tests that there is no appreciable increase in stress analogous to that which has been discussed for the case of two spheres in space where it was seen that even though there is no potential unbalance between the spheres, there is nevertheless considerable increase in stress when one sphere is at zero potential and the other at a given high potential, over the case in which they are both isolated and have the same difference of potential between them.

Therefore, it would seem that the best method of estimating the true strength of the insulation when balancing resistances are used is to average directly the non-grounded tests with the grounded tests. The data obtained are not sufficiently extensive to enable us to see whether there is a real difference between these two cases or not.

TABLE IV

AVERAGE OF DATA

Thickness of insulation No barriers. cm.	Thickness of insulation between barriers. cm.	Average strength of material No barriers. Kv. eff./cm.	Strength of material with barriers average isolated and shunted kv. eff./cm.
0.031	0.031	511	467
0.061	0.061	466	464
0.122	0.122	422	386

In conclusion it appears probable from a theoretical point of view that the strength of insulation under uniform-field conditions is the same whether the barriers are isolated or connected metallically to the proper sources of potential and also that the strength merely depends upon the thickness of the material between the barriers. The averages, as given in the above table, do not warrant the above conclusions, but I am inclined to believe that the discrepancy is due to insufficient data and errors in the method. I hope, therefore, that someone will take up the subject and make a careful study of this important theoretical and also practical problem.

Another interesting question would be to determine the limiting value of the electrostatic flux density which would result in excessive heating, and therefore, breakdown irrespective of the so-called "instantaneous strength" of the structure. The effect of metal barriers in distributing the heating, eddy-current losses in the metal barriers due to electromagnetic flux accompanying the charging current, etc., would also be of interest.

General Discussion of the Problem of High-Voltage-Bushing Design

So far we have considered the problem of high-voltage-bushing design mainly from an ideal and purely electrostatic point of view, that is, we have greatly simplified the problem by assuming ideal or perfect dielectrics and a constant or static condition of the equipotential surfaces. These conditions do not however, actually exist in practise, and therefore, the effects brought about by rain, dirt, snow, steep wave front impulses must be considered in a complete theory of the correct and most effcient design.

For example, under rain conditions the equipotential surfaces are in a general constantly changing and will be different from the dry condition. There is probably also a certain space or volume charge effect due to the rain drops becoming charged from contact with the high potential electrodes and then falling in the vicinity of the bushing to the grounded electrode. Furthermore, a considerable change in the electrostatic field may be brought about by an effective increase in the permittivity of the air part of the dielectric due to the presence of small rain drops. The problem is further complicated by the presence of conduction over wet or dirty surfaces with the consequent change in the potential distribution as well as the possibility of an arc-over resulting from what might be called fuse action, that is, the conducting material may become over-heated by conduction and blow like a fuse and thus precipitate an arc-over along the path of the hot gases.

There are a great many other disturbing factors and possibilities which have not been mentioned, but I believe that the above are sufficient to show the multitude of factors over which the designer can have no control and which, therefore, must be included in the theoretically correct and most efficient bushing design.

We have already observed the large surface effect which exists even with relatively clean dry surfaces, also the change in the dielectric strength of materials when subdivided into thin layers by metal barriers and probably, to some extent, merely by laminating the material.

It might appear from what has been said about the complexity of the problem that it is beyond the power of analysis to try to obtain a complete theory of bushing design. Strictly speaking, I believe that this is the case, nevertheless, I believe that the incomplete theory which we have is of great value in determining the proper lines along which to experiment.

We will now briefly describe the construction and tests on a series of small experimental bushings which were built for the purpose of determing the rain and dry characteristics of various types.

The object was to determine the general type of bushing which would best meet the following conditions.

I. Rain and dry arc-over should have as nearly as possible the same value.

II. The arc-over of the under-oil end should be considerably above either the rain or dry-air end arc-over.

III. Puncture should occur considerably above the under-oil arc-over.

IV. The impulse safety factor or ratio should be as high as possible, both under wet and dry conditions and application of a large number of impulses should not result in puncture.

The materials used in the construction of these bushings were selected from the point of view of ease of construction and alterations. The caps used on the air and oil ends were turned up of wood and metal covered by the Schoop metal spraying process. After completing the tests with a certain shape of cap it could be put in a lathe and altered in the desired manner and then re-metallized. Hard rubber was selected as the material for the core since it can be readily machined to any desired form; for example, the necessary protection of the edges of the ground shield was easily obtained (see Fig. 34). This material also has other desirable qualities such as homogeneity, high dielectric strength, small surface effect.

A single glass tube was used for the containing shell in order to eliminate the difficulty of making oil tight joints. The only joints being that at the lower cap which was satisfactorily sealed with ceresine. A glass seal was provided by which connection was easily made between the ground shield and the tin disk which

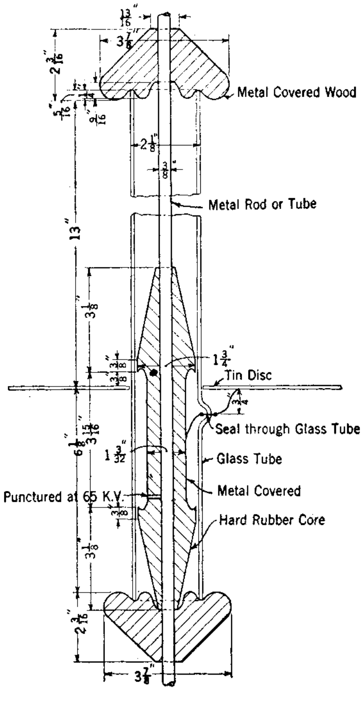

FIG. 34

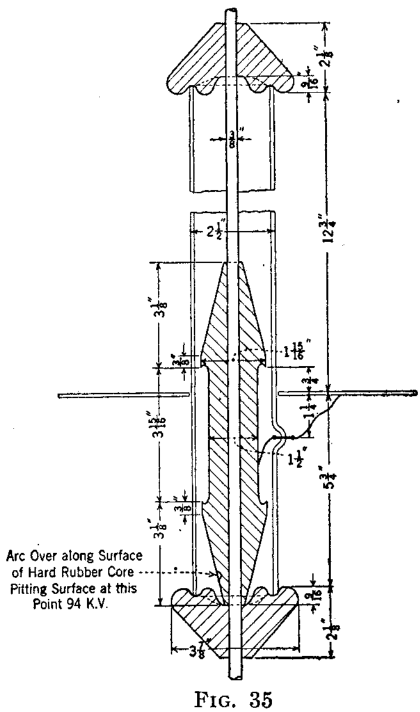

FIG. 35

represents the transformer tank cover. For convenience, a transparent Russian mineral oil was used as filler. During tests the under-oil end was placed in a 5-gallon tin can filled with oil, having dimensions 9 in. by 9 in. by 14 in. high. All tests were made at 60 cycles and with the tin can and tin disk connected together and grounded.

Determinatoin of Puncture and Under-Oil Arc-Over

This series of tests were made to determine the under-oil and puncture characteristics of the bushings.

Bushing No. 1. A bushing was built up as shown to scale in Fig. 34. The joints between the glass containing cylinder and the metal caps were shielded by the grooves as shown. The joint between the hard-rubber core and cap on the under-oil end was likewise protected. The electrostatic field of the under-oil end is practically of the uniform-field type (Fig. 14), the desirability of which has been pointed out. The air end is essentially the radial-field type.

In test this bushing punctured at 65 kv. effective at the point shown in Fig. 34. After sawing the core open longitudinally, inspection showed that the hole had not been smoothly drilled which may have accounted for the puncture, as the space thus left between the rod and core may not have been properly filled with oil.

Bushing No. 2. (See Fig. 35). The core was increased to 1½ in. (3.8 cm.) external diameter so as to insure a safe margin against puncture. As an additional precaution a brass tube closed at the bottom end and having small holes drilled radially at intervals inside the core was used in place of the brass rod used in Bushing No. 1. The object being to enable oil poured in at the top of the tube to fill up any irregularities resulting from imperfect fit between the core and central rod.

In test this bushing arced over along the surface of the hard rubber and jumped to the inner guard ring on the cap at 94 kv. effective. Small chips were taken out of the rubber at the point shown in Fig. No. 35, apparently where the surface arc turned to jump to the guard ring.

Bushing No. 2-A. Bushing No. 2 was disassembled and the inner guard ring cut down to the shape shown by the dotted lines in Fig. 35. In assembly the core was inverted putting the damaged part toward the air end of the bushing, bringing the new end under test.

Arc-over occurred over the surface of the hard rubber core on the under-oil end without pitting the surface at 93.5 kv. effective. The voltage was then brought up again and arc-over occurred over the surface of the hard-rubber core on the air end to the previously damaged spots and there punctured to the rod at 82.5 kv.

Bushing No. 3. (See Fig. 36). The hard-rubber core, which had been damaged by the previous tests, was turned down to the form shown in Fig. 36, thereby removing the damaged material. The under-oil end cap was also altered to a form approximately as shown. Practically the only electrostatic change made by this alteration was to somewhat reduce the shielding effect of the cap on the hard-rubber core. The air end was also materially shortened.

This bushing arced-over the air end at 81 kv. Upon bringing up the voltage a second time arc-over occurred over the surface of the hard-rubber core on the under-oil end without damaging the core at 86.5 kv. effective.

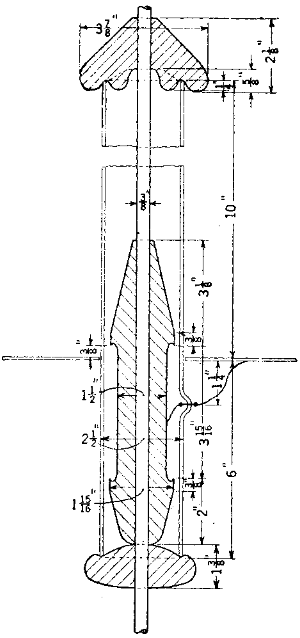

FIG. 36

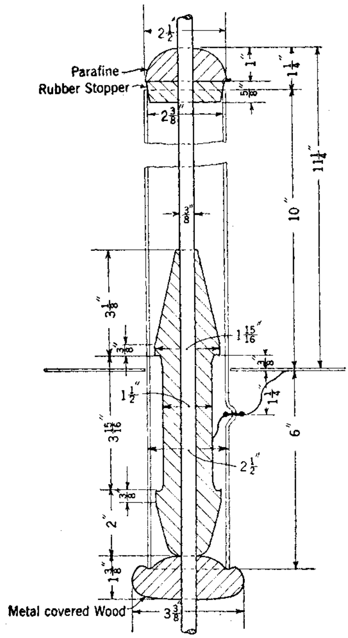

FIG. 37

Determination of the Air-End Characteristics

Having secured a fairly efficient and satisfactory under-oil end the work was then directed towards a study of the rain and dry characteristics of various air-end combinations.

Further Tests on Bushing No. 3. (See Fig. 36.)Temp. 24 deg. cent. Barometer 76 cm. Humidity 57 per cent.

Bushing Dry

Corona visible air end near ground shield.......... 41 kv.
Corona visible on cap of air end.................. 59 kv.
Bushing outlined air end......................... 87 kv.

Arc-over was not taken.

Rain about 0.2 in. per minute at 45 deg.

Corona starts top and bottom.................... 10 kv.
First streamers.................................. 16 kv.
Whole tube outlined.............................. 20 kv.
Arc-over... 52 kv.

This bushing is obviously unsatisfactory because of the low corona starting points both wet and dry and the very low rain arc-over.

Bushing No. 4. (See Fig. 37). This bushing was similar to No. 3 except that the metal cap at the top was replaced by a rubber stopper covered with paraffine. The object being to make the field on the air end as nearly radial as possible.

Summary of Tests on Bushing No. 4

Temp. 23.5 deg. cent. Barometer 76 cm. Hum. 50 per cent.

Bushing Dry

Corona starts near ground shield................. 35 kv.
Corona at top and bottom......................... 60 kv.
Voltage raised without arc-over to................ 85 kv.

Rain Test about 0.2 in. per minute 45 deg.

Corona starts top and bottom..................... 15 kv.
Tube outlined.................................... 17 kv.
Vicious streamers over surface.................... 34 kv.
Arc-over... 57 kv.
Arc-over (paraffine removed)...................... 55 kv.

This bushing is unsatisfactory for the same reasons as stated for No. 3.

Bushing No. 5. (See Fig. 38). This bushing was exactly similar to No. 4 except that the internal brass shield was added as shown in Fig. 38. The object of adding this element was to somewhat relieve the stress on the rubber stopper and paraffine and to throw it away from the rod at the top of the bushing.

Summary of Tests on Bushing No. 5

Temp. 23.5 deg. cent. Barometer 76 cm. Hum. 50 per cent

Bushing Dry

Corona starts at ground shield.................... 32 kv.
Corona top and bottom............................ 68 kv.
Voltage raised without arc-over to................ 80 kv.

Rain Test about 0.2 in per minute 45 deg.

Corona starts top and bottom..................... 18 kv.
Arc-over... 42 kv.

This bushing is obviously unsatisfactory.

Bushing No. 6. (See Fig. 39). This bushing was constructed as shown in Fig. 39. The air end was shortened and the glass tube drawn down so as to get rid of the rubber stopper and paraffine, which gave trouble in the previous two bushings by carbonization and burning. The internal shield shown was used for the same reasons as in Bushing No. 5.

Summary of Tests on Bushing No. 6.

Bushing Dry

Corona starts at ground shield and extends up the surface of glass tube.......................... 50 kv.
Corona appears opposite guard ring on the air end... 77 kv.
Arc-over.................................... 79 kv.

Rain Test about 0.2 in. per minute 45 deg.

Corona starts at top near tin foil................. 5 kv.
Corona streamers run out from tin foil over the glass surface.................................... 10 kv.
Bushing outlined.............................. 12 kv.
Arc-over.................................... 36 kv.

Bushing No. 6-A. A metal cone similar to that shown in Fig. 41 was slipped over the top of bushing No. 6 and the following results obtained.

Bushing Dry

Corona on surface of glass opposite guard ring...... 30 kv.
Bushing outlined.............................. 60 kv.
Arc-over.................................... 78 kv.

Rain Tests approximately 0.2 in. per minute 45 deg.

Corona starts at edge of metal cone............... 14 kv.
Bushing outlined.............................. 20 kv.
Arc-over.................................... 29 kv.

These bushings No. 6 and No. 6-A, are seen to be unsatisfactory from the point of view of corona starting point and wet arc-over.

Bushing No. 7. (See Fig. 40). The previous tests showed the necessity of breaking up the flow of water which adheres to the smooth glass surface and thereby produces great field distortion, or a sort of short circuit, or conducting sheet over the bushing. In order to accomplish this result the usual method of adding petticoats was resorted to. These were made by belling out glass tubing as will be seen from the figures. The petticoats were then slipped over the glass shell and the intervening space filled in with ceresine. In other respects, the bushing was exactly similar to No. 6.

From an electrostatic point of view, it will be observed that in the radial-field type of bushing the corrugations approximately coincide with the direction of the flux lines. Therefore, the potential gradient is along the surface of the corrugations and will tend to break down any surface layer of dust or dirt. For this reason it would appear that their effectiveness is somewhat limited. We may further observe from Fig. 9 that when dry the presence of the corrugations will not appreciably affect the field distribution in this type of bushing, except those which are close to the tank. In this latter case they are seen to introduce an

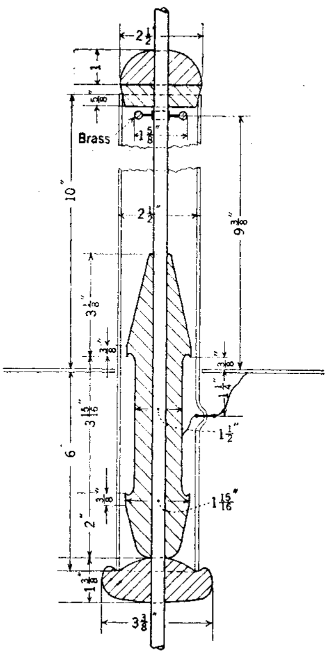

FIG. 38

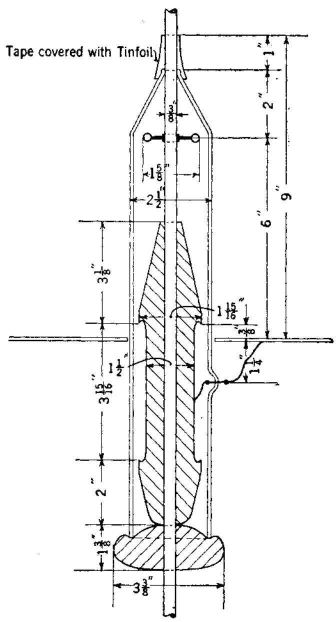

FIG. 39

appreciable percentage of high inductive capacity material in series with the air. Therefore, if petticoats of this kind are used it may be best not to extend them too near the tank.

In the uniform-field type of bushing it will be observed that the petticoats introduce high inductive capacity material in series with the air (except in case of a vacuum bushing). This means that when dry the stress on the air will be increased by their addition. The surfaces of the corrugations will lie practically along the equipotential surfaces; hence, an accumulation of dust, water, etc., would not result in so great a field distortion.

Summary of Tests on Bushing No. 7.

Humidity 60 per cent. Barometer 76 cm. Temp. 24 deg. cent.

Bushing Dry

Corona on tin-foil at top of bushing	24 kv.
Corona near ground shield	38 kv.
Bushing outlined	60 kv.
Streamers over the entire surface of glass	70 kv.

Rain Tests approximately 0.2 in. per minute 45 deg.

Corona on tin-foil at top	26 kv.
Corona on edge of upper petticoat	34 kv.

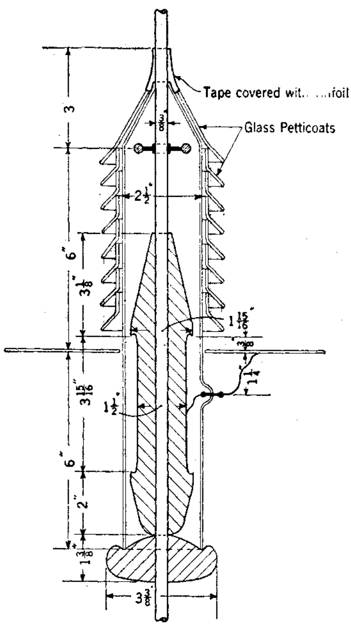

FIG. 40

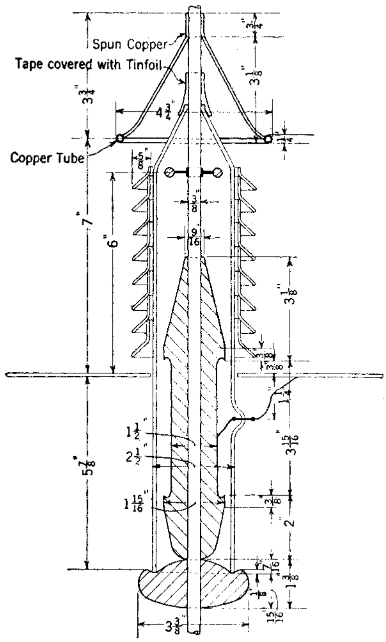

FIG. 41

Corona near ground shield	50 kv.
Corona streamers over surface	70 kv.
Arc-over	75 kv.

It will be observed that the addition of the petticoats has greatly improved the wet arc-over of the bushing.

Bushing No. 8. (See Fig. 41 and Fig. 42). This bushing is identical to No. 7 except that the spun copper cone was added in order to act as rain shed and thereby eliminate the formation of streamers which went out over the top surface of the glass in the previous bushing No. 7.

Summary of Tests on Bushing No. 8.

Humidity 60 per cent. Barometer 76 cm. Temp. 24 deg. cent.

Dry Tests

- Corona near ground shield........................ 46 kv.
- Arc-over.. 69 kv.

Rain Test approximately 0.2 in. per minute at 45 deg.

- Corona on edge of cone and near ground shield..... 36 kv.
- Bushing outlined............................... 43 kv.
- Arc-over.. 69 kv.

This bushing is seen to be quite satisfactory from the point of view of wet and dry arc.

Bushing No. 9. (See Figs. 43 and 44). For this bushing a new core and caps were turned up as shown in Fig. 43; also an idea of the appearance can be obtained from Fig. 44 which is of a similar bushing. The air and oil ends were made identical except for the addition of petticoats to the air end. The method of assembly of the petticoats previously used namely to fill the space between the petticoats and glass shell with ceresine was given up. Instead the individual petticoats were cemented together with a sealing wax composition. The resulting petticoat shell was slipped over the inner containing glass cylinder and the space filled with a viscous compound. This made a more satisfactory scheme for assembly and disassembly.

Summary of Tests on Bushing No. 9

Temp. 24 deg. cent. Barometer 76 cm. Humidity 60 per cent.

60-cycle Tests

Bushing Dry

- Corona on edge of upper cap.................. 51 ± 2 kv.
- Arc-over..................................... 59 ± 2 kv.

Rain Test approximately 0.2 in. per minute 45 deg.

- Corona on edge of upper cap.................. 24 ± 2 kv.
- Bushing outlined............................. 42 ± 4 kv.
- Arc-over (clear of surface).................. 48± 5 kv.

$$\frac{\text{wet arc-over}}{\text{dry arc-over}} = 0.813$$

Impulse Tests

200-kilocycle "B" wave

Bushing Dry

- Arc-over (1 out of 10 impulses).............. 99 kv. eff.
- Arc-over (5 " " " ")....................... 103 kv. eff.

Rain Test 0.2 in. per minute 45 deg.

Arc-over (1 out of 10 impulses).............. 93 kv. eff.

Arc-over (5 " " " ")...............95.3 kv. eff.

Impulse ratio or safety factor

$$\text{Dry} = \frac{99}{59} = 1.68$$

$$\text{Rain} = \frac{93}{48} = 1.93$$

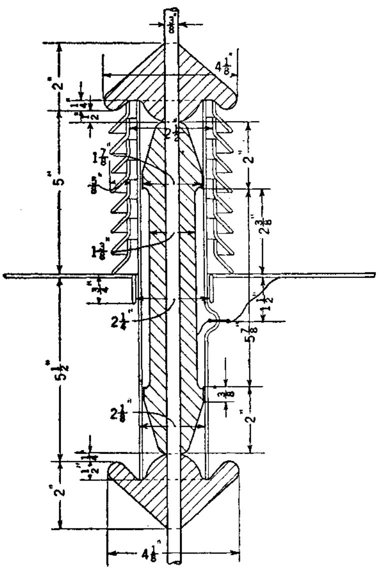

Fig. 43

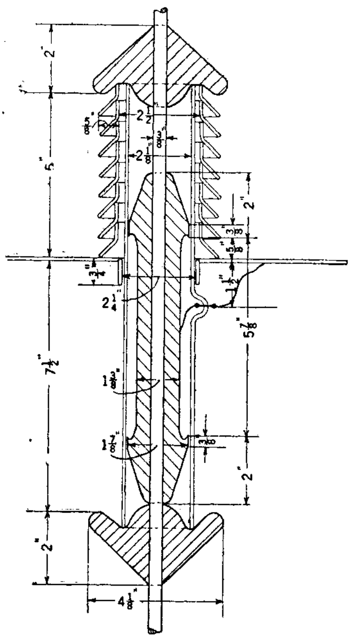

Fig. 45

Bushing No. 10. (See Figs. 45 and 46). This bushing was identical to No. 9 except that the hard-rubber core was moved down to the position shown in Fig. 45, thereby changing the type of field used on the upper part of the air end from the parallel field type to a more radial type. This bushing is shown in the photograph Fig. 46.

Summary of Tests on Bushing No. 10

60-cycle Tests

Bushing Dry

Corona on edge of upper cap.................. 54 ± 4 kv.

Arc-over (clear of surface)................... 64 ± 1 kv.

[RICE]

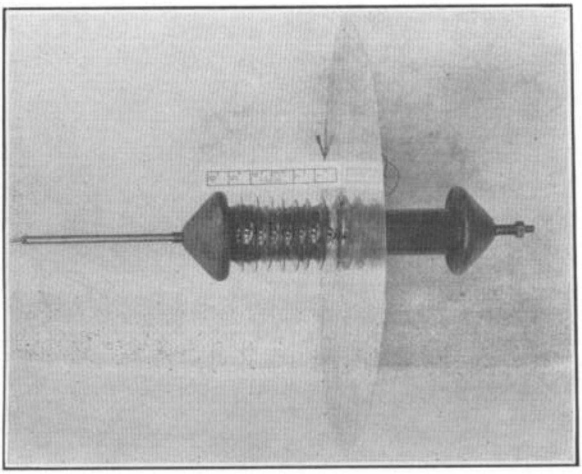

FIG. 16

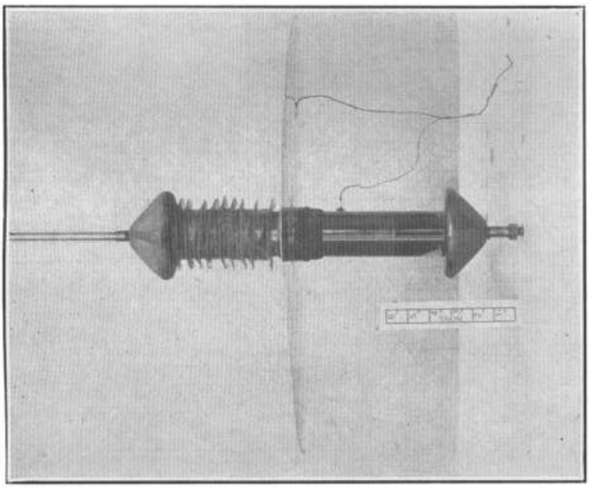

FIG. 44

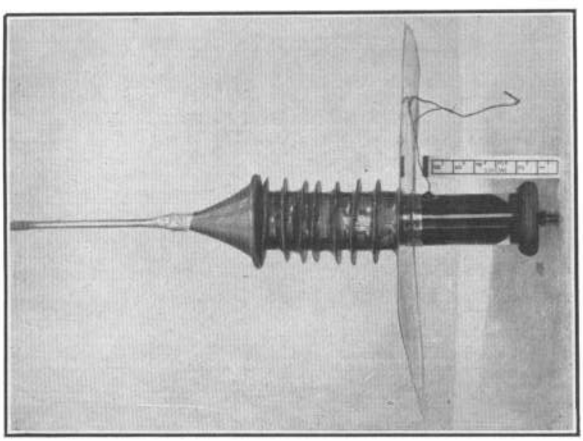

FIG. 42

Rain Test approximately 0.2 in. per minute 45 deg.
Bushing outlined.............................. 40 ± 3 kv.
Arc-over (clear of surface)................... 46 ± 2 kv.

$$\frac{\text{wet arc-over}}{\text{dry arc-over}} = 0.72$$

Impulse Tests
200-kilocycle "B" Wave

Bushing Dry
Arc-over (1 out of 10 impulses)............ 111 kv. eff.
Arc-over (5 " " " ")............ 114 kv. eff.
Bushing Rain approximately 0.2 in. per minute at 45 deg.
Arc-over (1 out of 10 impulses)............ 104 kv. eff.
Arc-over (5 " " " ").............107.5 kv. eff.

Impulse ratio or safety factor

$$\text{Dry} = \frac{111}{64} = 1.73$$

$$\text{Rain} = \frac{104}{46} = 2.26$$

Of this series of bushings it appears that the general form shown in Figs. 43 to 46 best fulfills the various requirements which an all around bushing has to meet.

Acknowledgments

In conclusion I wish to express my appreciation of the interest which was shown by Dr. C. P. Steinmetz, Mr. G. Faccioli, and Dr. E. J. Berg in connection with the experimental part of the work and to Professor W. E. Byerly for the great assistance which he so kindly gave me in the course of the mathematical work contained in the appendix. I wish, furthermore, to acknowledge the help which Mr. B. L. Stemmons has given in making the various tests.

APPENDIX

Mathematical Solution of Two Electrostatic Problems

It is not in general possible, by known mathematical methods, to obtain the solution of Laplace's equation so as to fulfil arbitrarily given boundary conditions.[31] If, however, we take the simplest conceivable electrode arrangement which would form the skeleton of a bushing, we can obtain a mathematical solution for this case. The most simple electrode arrangement will be seen to be a fine wire passing perpendicularly through a hole in an infinite plane. The mathematical formulation of these skeleton electrodes can be obtained. Thus, for instance, we can formulate the equation of an infinite plane with a hole in its centre as the focal or limiting hyperboloid of revolution of one sheet. The edge of the hole will be at the focus of the hyperboloid. We can now formulate the equation of a fine wire passing perpendicularly through the centre of the hole in our plane. It will be represented by the other limiting confocal hyperboloid, that is, the one which degenerates into the axis of revolution of the confocal family. You will readily see that we have thus reduced the problem to finding the distribution of the electrostatic field between two confocal surfaces of the same family maintained at given potentials. In this case the equipotential surfaces will be hyperboloids of revolution.

We have thus reduced our problem to a form which can be treated quite simply mathematically and for which the form of the solution is known.[32] Here Maxwell states that the equipotential surfaces will be confocal hyperboloids of one sheet and the surfaces of flow will be the confocal oblate spheroids.

The simplicity of the mathematical solution rests upon a wise selection of the coordinates as it does in many other problems of this type. For example, in dealing with cylindrical distributions we would employ cylindrical co-ordinates; in spherical distributions we would use spherical co-ordinates.

In this case where we have to deal with surfaces formed by the revolution of hyperbolas about their conjugate axis thus forming hyperboloids of one sheet; a system of cur-

31. An extremely interesting discussion of this subject will be found in Maxwell's Electricity and Magnetism, Vol. I Chap. VII, page 175.

32. See—Maxwell's Electricity and Magnetism, Vol. I, page 235.

vilinear co-ordinates of a proper form are made use of. Since most engineers are not very familiar with their meaning and use I have started from the beginning and gone through the development of the system of curvilinear co-ordinates which are used in solving our problem. This makes the work appear rather long and clumsy, but will, I believe, make the whole subject quite simple and clear, and is therefore justified. Assuming a familiarity with the system of co-ordinates, the solution is exceedingly simple, in fact it is practically as simple as the familiar solution for parallel wires at large spacing where the diameter of the wires is neglected.

Curvilinear Co-ordinates; Conicoids or Quadric Surfaces

A surface whose equation is of the second degree in x, y and z, is called a quadric surface or conicoid. The sphere is a special case of such a surface.

It is possible, by a suitable transformation of co-ordinates, to reduce the general equation of the second degree in x, y, and z, namely;

$$Ax^2 + By^2 + Cz^2 + 2\,Dyz + 2\,Ezx + 2\,Fxy.$$
$$+ 2\,Gx + 2\,Hy + 2\,Iz + K = 0 \qquad (\mathbf{1})$$

to the following form in which the terms yz, zx and xy are all absent. That is, the equation can be put into the following simpler form;

$$A'x^2 + B'y^2 + C'z^2 + 2\,G'x + 2\,H'y + 2\,I'z + K' = 0 \qquad (\mathbf{2})$$

If in this equation the constants A', B', C' are all finite, we can further simplify it by making a change in the origin of co-ordinates and obtain an equation which when referred to its new axis is of the form

$$A'x^2 + B'y^2 + C'z^2 = D' \qquad (\mathbf{3})$$

The locus of this equation is evidently symmetrical with respect to each of the co-ordinate planes, and hence with respect to the origin. Such surfaces are therefore called central *quadric surfaces.*

We may now divide equation (**3**) through by D' and obtain

$$\frac{x^2}{\frac{D'}{A'}} + \frac{y^2}{\frac{D'}{B'}} + \frac{z^2}{\frac{D'}{C'}} = 1 \qquad (\mathbf{4})$$

If in (**4**) we substitute

$$a_1 = \frac{D'}{A'}, \; a_2 = \frac{D'}{B'}, \; a_3 = \frac{D'}{C'}$$

We have the familiar equation of a central quadric surface referred to, its principal axes, namely;[33]

$$\frac{x^2}{a_1} + \frac{y^2}{a_2} + \frac{z^2}{a_3} = 1 \qquad (5)$$

where a_1, a_2 and a_3 may be positive or negative. If they are all negative, the surface is imaginary. We will now consider the nature of the surfaces under the various other conditions.

First. Suppose one of the constants a_1, a_2 or a_3 is negative, while the remaining two are positive.

Thus, let $a_3 = -c^2$
while $a_1 = a^2$
$a_2 = b^2$

Also assume that, a, is numerically greater than b, and, b, greater than c,
or $a > b > c$

Substituting these values in (**5**) we obtain

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \qquad (6)$$

This particular central quadric surface will be recognized to be that of a hyperboloid of one sheet with conjugate axis along the z axis.

This can be seen by considering the shape of the plane curves which results as the intersection of this solid figure with the reference planes, or what are generally called the principal sections of the surface. First take the X-Y plane, that is, the plane whose equation is $z = 0$. Substituting this value in (**6**) we obtain.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

33. Proofs of the statements made above may be found in any good book on Solid Analytic Geometry, for example; Solid Geometry by Charles Smith,—Macmillan and Company.

That is, the surface is cut by the X-Y plane in the ellipse whose semi-axes are a and b. The foci are at a distance from the origin.

$$f = \sqrt{a^2 - b^2} = \sqrt{a_1 - a_2}$$

which is seen by inspection from Fig. 1.

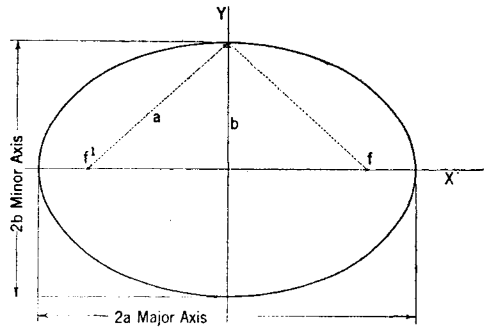

FIG. 1—ELLIPSE

The section cut out by the Z-X plane ($y = 0$) is seen to be the hyperbola

$$\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1$$

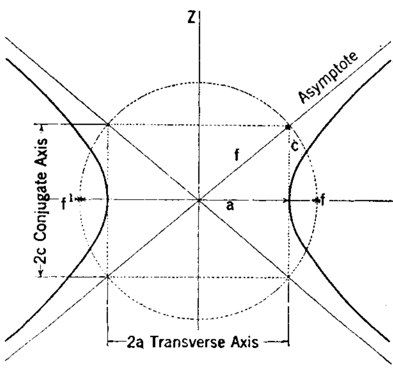

FIG. 2—HYPERBOLA

with semi-axes a and c. The foci are at the distance

$$f = \sqrt{a^2 + c^2} = \sqrt{a_1 - a_3}$$

on the X axis which can be seen by inspection from the values of the various quantities as given in Fig. 2.

The section by the Y-Z plane ($x = 0$) is the hyperbola

$$\frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

with semi-axes b and c and foci at the distance

$$f = \sqrt{b^2 + c^2} = \sqrt{a_2 - a_3}$$

on the Y axis.

Thus the surface defined by equation (**6**) is seen to be a hyperboloid of one sheet with the conjugate axis on the Z axis.

If in the above equation (**6**) we let $a = b$ the section of the surface by any plane parallel to the X-Y plane (plane $z = K$) is a circle. Hence, the surface would be formed by the revolution of the hyperbola $\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1$ about its conjugate axis. Fig. 3 shows the form of the surface.

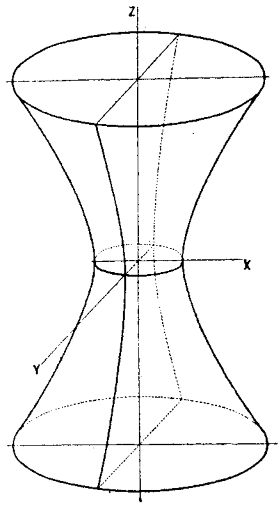

FIG. 3—HYPERBOLOID OF ONE SHEET

Second. Let two of the constants a_1, a_2, a_3 in equation (**5**) be negative.

For example, let

$$a_2 = -b^2$$
$$a_3 = -c^2$$
$$a_1 = a^2$$

Also assume as before that a is numerically greater than b and b greater than c,

$$\text{or} \quad a > b > c$$

Substituting these values in (**5**) we obtain,

$$\frac{x^2}{a^2} - \frac{y_2}{b^2} - \frac{z^2}{c^2} = 1 \qquad (\mathbf{7})$$

and the surface is called a hyperboloid of two sheets.

The sections by the co-ordinate planes and their focal distances are found as before.

The section by the X-Y plane ($z = 0$) is the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

with semi-axes a and b and focal distance

$$\sqrt{a^2 + b^2} = \sqrt{a_1 - a_2}$$

The section by the ZX plane ($y = 0$) is the hyperbola

$$\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1$$

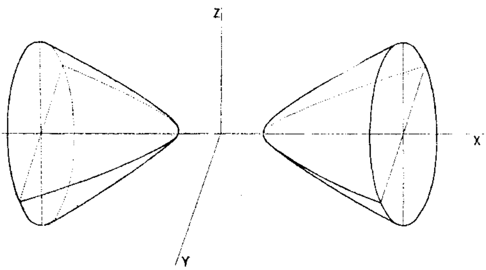

FIG. 4—HYPERBOLOID OF REVOLUTION OF TWO SHEETS

with semi-axes a and c and focal distance

$$\sqrt{a^2 + c^2} = \sqrt{a_1 - a_3}$$

The section by the YZ plane ($x = 0$) is the imaginary ellipses $\frac{y^2}{b^2} + \frac{z^2}{c^2} = -1$. With semi-axes b and c and focal distance

$$\sqrt{-(b^2 - c^2)} = \sqrt{a_2 - a_3}$$

If in equation (**7**) we let $b = c$ the section by any plane parallel to the plane $x = 0$ is a circle. Hence the surface is in this case formed by the revolution of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

about its tranverse axis. The form of the surface is shown in Fig. 4.

Third. Let all of the constants a_1, a_2 and a_3 be positive.

Thus, let

$$a_1 = a^2$$
$$a_2 = b^2$$
$$a_3 = c^2$$

Also let us assume that the relative magnitude of the constants are as indicated by the equation

$$a > b > c$$

Upon substituting these values in equation (**5**) we obtain

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \qquad (8)$$

which is seen to be the equation of an ellipsoid.

The sections by the co-ordinate planes and their focal distances are given below.

The section by the X-Y plane ($z = 0$) is the ellipse

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with semi-axes a and b and focal distance

$$f = \sqrt{a^2 - b^2} = \sqrt{a_1 - a_2}$$

The section by the Z-X plane ($y = 0$) is the ellipse

$\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1$ with semi-axes a and c and focal distance

$$f = \sqrt{a^2 - c^2} = \sqrt{a_1 - a_3}$$

The section by the Y-X plane ($x = 0$) is the ellipse

$\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ with semi-axes b and c and focal distance

$$f = \sqrt{b^2 - c^2} = \sqrt{a_2 - a_3}$$

If in equation (**8**) we let $a = b$ the sections by the planes parallel to the X-Y plane will be circles whose centres lie along the Z axis and the surface is cut out by revolving the ellipse

$$\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1$$

about the Z axis. If a is the major axis and c the minor axis of this ellipse as has been assumed and stated in the assumption

$a > c$, then the figure is cut out by revolving an ellipse about its minor axis. Such a figure is called an oblate spheroid. If the rotation occurs around the major axis the figure is called a prolade spheroid.

If $a = b = c$ the ellipsoid reduces to a sphere.

The form of the oblate spheroid is shown in Fig. 5.

Confocal Quadric Surfaces

In all three cases investigated above it will be observed that the squares of the focal distances of the principal sections (sections cut out on the X-Y, Y-Z, Z-X planes) are the square roots of the differences of the three constants a_1, a_2 and a_3. Therefore, if we add a constant to each of the three constants a_1, a_2 and a_3 we obtain a surface whose principal sections have

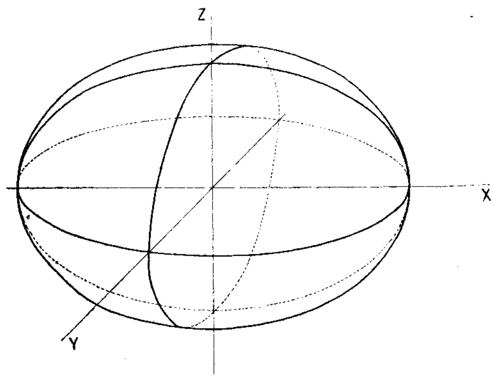

Fig. 5—Oblate Spheroid
Axis of revolution about Z axis.

the same foci as before or what is called a surface confocal with respect to the original.

For example, if we take the equation of an ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \qquad (9)$$

the focal distance of the principal section made by the X-Y plane has been seen to be

$$f = \sqrt{a^2 - b^2} = \sqrt{a_1 - a_2}$$

If we now add the constant quantity ρ to a_1 and a_2 we have as the focal distance of this principal section.

$$\sqrt{(a_1 - \rho) - (a_2 - \rho)} = \sqrt{a_1 - \rho - a_2 + \rho} = \sqrt{a_1 - a_2}$$

and thus the focal distance has remained the same. Similarly for the other principal sections, and we therefore see that

$$\frac{x^2}{a^2 + \rho} + \frac{y^2}{b^2 + \rho} + \frac{z^2}{c^2 + \rho} = 1 \qquad \textbf{(10)}$$

represents a surface confocal with the original for any real value positive or negative of ρ.

If as we have assumed throughout, a is greater than b and b is greater than c or,

$$a > b > c$$

then the character of the surface represented by equation (**10**) will be seen to be determined as follows:

(1) If $\rho > - c^2$ the surface is an ellipsoid.
(2) If $- c^2 > \rho > - b^2$ the surface is a hyperboloid of one sheet.
(3) If $- b^2 > \rho > - a^2$ the surface is a hyperboloid of two sheets.
(4) If $- a^2 > \rho$ the surface is imaginary.

Suppose we now wish to pass through a given point in space x, y, z a quadric surface confocal with the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \qquad \textbf{(11)}$$

Where $$a > b > c.$$

We have seen that its equation will be

$$\frac{x^2}{a^2 + \rho} + \frac{y^2}{b^2 + \rho} + \frac{z^2}{c^2 + \rho} = 1 \qquad \textbf{(12)}$$

from which the value of ρ can be determined when the values of x, y, z of the desired point are given.

It will be seen that this equation (**12**) is a cubic equation in ρ and therefore will have three roots say ρ_1, ρ_2, ρ_3 and if it is found that these roots are all real there will be three confocal surfaces which will pass through any desired point x, y, z in space. Since an equation of odd degree always has at least one real root we see that the other two roots will be real or both imaginary since imaginary roots always enter an equation in pairs.

In order to investigate the roots of our equation (**12**) we can write it when cleared of fractions as follows:

$$(a^2+\rho)(b^2+\rho)(c^2+\rho)-x^2(b^2+\rho)(c^2+\rho)-y^2(a^2+\rho)(c^2+\rho) - z^2(a^2+\rho)(b^2+\rho)=f(\rho)=0 \quad (\mathbf{13})$$

This is our cubic equation in ρ and a, b, c, x, y, z are the constants.

We have just seen that the character of the confocal surface depends upon the magnitude of ρ with respect to the three constants a, b and c. To recapitulate

(1) If $\rho > -c^2$ the surface is an ellipsoid.
(2) If $-c^2 > \rho > -b^2$ the surface is a hyperboloid of one sheet.
(3) If $-b^2 > \rho > -a^2$ the surface is a hyperboloid of two sheets.
(4) If $-a^2 > \rho$ the surface is imaginary.

We will now substitute for ρ in equation (**13**) the values

(1) $\rho = +\infty$
(2) $\rho = -c^2$
(3) $\rho = -b^2$
(4) $\rho = -a^2$

which will be seen to be the extreme values as outlined above. It should be observed that $\rho = -\infty$ would result in an imaginary surface and therefore is not considered.

Upon making these substitutions for ρ in equation (**13**) and remembering that $a > b > c$ we will obtain the following as regards to changes of sign of our cubic equation in ρ which is represented for brevity by $f(\rho)$.

When $\qquad \rho = +\infty$
sign $\qquad f(\rho) = +\infty$

and it is observed that $f(\rho)$ is a positive quantity for all values of x, y, z, a, b, c constant under the assumed relation $a > b > c$.

When $\qquad \rho = -c^2$
$\qquad f(\rho) = -z^2(a^2-c^2)(b^2-c^2)$
sign $\qquad f(\rho) = (-)(+)\quad(+) = (-)$

in this case we observe that $f(\rho)$ is negative and therefore $f(\rho)$ has changed from a positive value in the previous case to a negative value in this case. Therefore, $f(\rho)$ has passed through zero somewhere between $\rho = +\infty$ and $\rho = -c^2$, that is, there is a real root between these two values. Call this root λ and observe that it occurs in the region which specified that the confocal surface will be an ellipsoid.

When $\rho = -b^2$

$f(\rho) = -y^2 (c^2 - b^2)(a^2 - b^2)$

sign $f(\rho) = (-)(-) \quad (+) = (+)$

Here again there has been a change in sign and therefore there is another real root somewhere between $\rho = -c^2$ and $\rho = -b^2$. Call this root μ and observe that it occurs in the region which specifies that the confocal surface defined by it will be a hyperboloid of one sheet.

When $\rho = -a^2$

$f(\rho) = -x^2 (b^2 - a^2)(c^2 - a^2)$

sign $f(\rho) = (-) \quad (-) \quad (-) \quad = (-)$

and we see that there is another change in sign and therefore another real root in the interval between $\rho = -b^2$ and $\rho = -a^2$. Call this root ν and observe that it lies in the interval which specifies that the confocal shall be a hyperboloid of two sheets.

We have thus accounted for the three roots of the cubic and found their general location also that they are all real.[34] Thus we have seen that through any given point in space, say x', y', z' there will pass one surface of each of the three kinds and therefore we may represent the given point in space either by its rectangular co-ordinates x', y', z' or by the three values λ, μ, ν which are the three values of ρ as determined by the cubic equation in

$$\frac{x^2}{a^2 + \rho} + \frac{y^2}{b^2 + \rho} + \frac{z^2}{c^2 + \rho} = 1$$

in which the values of x, y, z are taken as x' y' z'.

If we write for the sake of brevity.

$$F(\lambda, x, y, z) = \frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} + \frac{z^2}{c^2 + \lambda} - 1 = 0$$

$$F(\lambda, x, y, z) = 0 \qquad \textbf{(14)}$$

We see that equation (**14**) defines λ as a function of x, y, z which

34. A complete treatment of the cubic equations will be found in "Todhunter's Theory of Equations". Page 109.

is a point in space and therefore λ is said to be defined as a point function. Similarly

$$F(\mu, x, y, z) = 0 \qquad (15)$$

$$F(\nu, x, y, z) = 0 \qquad (16)$$

define μ and ν as point functions.

If we have given the values of λ, μ, ν in these three equations we determine completely the ellipsoid λ = constant, and the hyperboloid of one sheet μ = constant, and the hyperboloid of two sheets ν = constant, and hence their point of intersection x, y, z and its seven symmetrical points in the other quadrants. It can be further shown that the three surfaces defined by

$$\lambda = \text{constant}$$
$$\mu = \text{constant}$$
$$\nu = \text{constant}$$

are mutually perpendicular at every point of intersection.[35] λ, μ and ν are called the *ellipsoidal* or *elliptic co-ordinates* of the point. They form a system of Orthogonal Curvilinear Co-ordinates.[36]

Normal Oblate Spheroidal Co-ordinates

When any two semi-axes of our standard ellipsoid become equal the above described system of curvilinear co-ordinates breaks down. For in that case the equation

$$\frac{x^2}{a^2 + \rho} + \frac{y^2}{b^2 + \rho} + \frac{z^2}{c^2 + \rho} = 1$$

reduces to a quadratic equation in ρ and therefore has only two roots which we may call λ and μ. The surfaces λ = constant and μ = constant are now confocal ellipsoids and hyperboloids of revolution. Obviously a third family of surfaces is required before the position of a point in space can be fixed by their intersection. Such a family of surfaces, orthogonal to the two present families, is supplied by the system of diametral planes through the axis of revolution of the standard spheroid. The two cases in which the standard ellipsoid is a prolate spheroid and an oblate spheroid require separate treatment. We are

35. A good discussion will be found in "Solid Geometry", Charles Smith, Macmillan and Co., Page 147.

36. The above discussion has mainly been taken from A. G. Webster, "Electricity and Magnetism", pages 27-31.

here principally interested in the later case which will now be described.

Let us assume as the standard oblate spheroid

$$\frac{x^2 + y^2}{a^2} + \frac{z^2}{c^2} = 1 \qquad \textbf{(17)}$$

where $a > c$ that is, we have taken the case where the figure is rotated about the z axis.

The equation of the confocal family of surfaces is then

$$\frac{x^2 + y^2}{a^2 + \rho} + \frac{z^2}{c^2 + \rho} = 1 \qquad \textbf{(18)}$$

which is seen to be a quadratic ρ regarding x, y, z, a, c as constants.

As we have seen before the character of the surface will depend upon the magnitude of ρ with respect to the constants a and c. Remembering $a > c$

(1) If $\rho > -c^2$ the surface is an oblate spheroid.
(2) If $-c^2 > \rho > -a^2$ the surface is a hyperboloid of revolution of one sheet.
(3) If $-a^2 > \rho$ the surface is imaginary.

In order to investigate the roots of (**18**) we may write it

$$(a^2 + \rho)(c^2 + \rho) - (x^2 + y^2)(c^2 + \rho) - z^2 (a^2 + \rho) = f(\rho) = 0 \qquad \textbf{(19)}$$

and observe the changes of sign when the following values are substituted for ρ in equation (**19**).

(1) $\rho = +\infty$
(2) $\rho = -c^2$
(3) $\rho = -a^2$

When $\rho = +\infty$

sign $f(\rho) = +\infty$

and it is observed that $f(\rho)$ is a positive quantity for all values of x, y, z, a, c consistent with the assumed relation $a > c$.

When $\rho = -c^2$

$f(\rho) = -z^2 (a^2 - c^2)$

sign $f(\rho) = (-) \quad (+) \quad = (-)$

in this case we see that $f(\rho)$ is negative and has therefore changed from a positive value in the previous case to a negative value in this case. Therefore, $f(\rho)$ has passed through zero somewhere between $\rho = +\infty$ and $\rho = -c^2$, that is, there is a real root between these two values. Call this root λ and observe that it occurs in the region which specifies that the confocal surface will be an oblate spheroid.

When $\rho = -a^2$

$$f(\rho) = -(x^2 + y^2)(c^2 - a^2)$$

sign $f(\rho) = (-) \quad (+) \quad (-) \quad = (+)$

Here again there has been a change in sign and therefore there is another real root somewhere between $\rho = -c^2$ and $\rho = -a^2$ Call this root μ and observe that it is in the region which specifies that the confocal surface will be an hyperboloid of revolution of one sheet. We have thus accounted for the two roots of our quadratic and also found their approximate location; also that they are real.

We may now write

$$F(\lambda, x, y, z) = \frac{x^2 + y^2}{a^2 + \lambda} + \frac{z^2}{c^2 + \lambda} - 1 = 0$$

or,

$$F(\lambda, x, y, z) = 0 \qquad (20)$$

Similarly,

$$F(\mu, x, y, z) = 0 \qquad (21)$$

Now, as we have seen before, in order to determine completely a point in space we must have another equation which defines a surface which will intersect the two present surfaces $\lambda =$ constant, and, $\mu =$ constant, at all real points. As has already been said such a system of surfaces is that of the diametral planes through the axis of revolution of our standard oblate spheroid.

In this case, we have taken the z-axis as the axis of revolution. We, therefore, wish the equation of the plane which passes through the intersection of the planes $x = 0$ and $y = 0$. The equation of this plane may be written.[37]

$$y - \nu x = 0$$

where ν may have any value.

37. See—Charles Smith "Solid Geometry," Macmillan and Co., page11.

Thus we may write

$$F(\nu, x, y, z) = y - \nu x = 0 \tag{22}$$

If we are given the values of λ, μ, ν these three equations determine completely the oblate spheroid, $\lambda =$ constant, the hyperboloid of revolution of one sheet, $\mu =$ constant, and the diametral plane through the axis of revolution, $\nu =$ constant. Hence their point of intersection x, y, z and its seven symmetrical points in the other quadrants is completely determined. It can further be shown that the three surfaces defined by

$$\begin{aligned} \lambda &= \text{constant} \\ \mu &= \text{constant} \\ \nu &= \text{constant} \end{aligned}$$

are mutually perpendicular at every point of intersection. We may thus take λ, μ, ν as a set of normal curvilinear co-ordinates.

In order to simplify the work which will follow we will make certain simplifications in the form of the equations of the three surfaces, the intersection of which represent our three co-ordinates of a point. Rewriting equations (**20**), (**21**), (**22**) of the three orthogonal surfaces,

$$F(\lambda, x, y, z) = \frac{x^2}{a^2 + \lambda} + \frac{y^2}{a^2 + \lambda} + \frac{z^2}{c^2 + \lambda} - 1 = 0 \tag{23}$$

$$F(\mu, x, y, z) = \frac{x^2}{a^2 + \mu} + \frac{y^2}{a^2 + \mu} + \frac{z^2}{c^2 + \mu} - 1 = 0 \tag{24}$$

$$F(\nu, x, y, z) = y - \nu x = 0 \tag{25}$$

Where

$$\begin{gathered} a > c \\ \lambda > -c^2 \\ -c^2 > \mu > -a^2 \end{gathered}$$

ν may have any real value positive or negative.

Let us now make the following substitutions in equations (**23**) and (**24**).

Let

$$a^2 - c^2 = E^2 \tag{26}$$

and

$$a^2 + \lambda = E^2 D \tag{27}$$

subtracting we obtain

$$c^2 + \lambda = E^2 (D - 1) \tag{28}$$

also let

$$a^2 + \mu = E^2 F \tag{29}$$

subtracting (**26**) from (**29**) we obtain

$$c^2 + \mu = E^2 (F - 1) \tag{30}$$

If we now substitute (**27**) and (**28**) in (**23**), and (**29**) and (**30**) in (**24**) we have

$$\frac{x^2}{E^2 D} + \frac{y^2}{E^2 D} + \frac{z^2}{E^2 (D - 1)} - 1 = 0 \tag{31}$$

$$\frac{x^2}{E^2 F} + \frac{y^2}{E^2 F} + \frac{z^2}{E^2 (F - 1)} - 1 = 0 \tag{32}$$

We may now make the further substitutions,

Putting

$$E^2 = c'^2$$
$$E^2 D = \lambda'^2$$
$$E^2 F = \mu'^2$$

We then may write (**31**) and (**32**)

$$\frac{x^2}{\lambda'^2} + \frac{y^2}{\lambda'^2} + \frac{z^2}{\lambda'^2 - c'^2} - 1 = 0 \tag{33}$$

$$\frac{x^2}{\mu'^2} + \frac{y^2}{\mu'^2} + \frac{z^2}{\mu'^2 - c'^2} - 1 = 0 \tag{34}$$

If in these equations λ'^2 is greater than c'^2 and c'^2 is greater than μ'^2, that is

$$\lambda'^2 > c'^2 > \mu'^2$$

then equation (**33**) is the equation of a confocal family of oblate spheroids in a simplified form and equation (**34**) the confocal family of hyperboloids of revolution. The Z-axis is the axis of revolution and $2\,c'$ is the focal distance. In other words, if we take an ellipse and a hyperbola having the same foci (focal distance $2\,c'$) and revolve them about the minor axis of the ellipse (in this case the z-axis) we shall get a pair of surfaces which are mutually perpendicular; also a plane through the axis of revolution will cut both the spheroid and hyperboloid orthogonally. We may now drop the primes and write the equations of the three surfaces:

$$F_1(x, y, z, \lambda) = \frac{x^2}{\lambda^2} + \frac{y^2}{\lambda^2} + \frac{z^2}{\lambda^2 - c^2} - 1 = 0 \qquad \textbf{(35)}$$

$$F_2(x, y, z, \mu) = \frac{x^2}{\mu^2} + \frac{y^2}{\mu^2} + \frac{z^2}{\mu^2 - c^2} - 1 = 0 \qquad \textbf{(36)}$$

$$F_3(x, y, z, \nu) = y - \nu x = 0 \qquad \textbf{(37)}$$

where $\lambda^2 > c^2 > \mu^2$, $2c$ being the distance between foci.

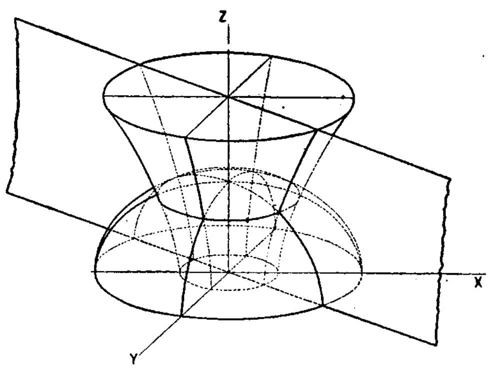

FIG. 6—INTERSECTION OF AN OBLATE SPHEROID, CONFOCAL HYPERBOLOID AND A PLANE THROUGH THE AXIS OF ROTATION

For all values of λ, μ and ν consistent with the inequality written above the surfaces (**35**), (**36**) and (**37**) intersect in real points and cut orthogonally. We can, therefore, represent any given point in space say x, y, z by the intersection of the three surfaces provided we assign the proper values to λ, μ and ν. Therefore, λ, μ and ν are a set of orthogonal curvilinear coordinates.

The form of the three surfaces is shown in Fig. 6.

LAPLACE'S EQUATION IN NORMAL OBLATE SPHEROIDAL CO-ORDINATES

If we solve equations (**35**), (**36**) and (**37**) simultaneously we will obtain the values of x, y and z in terms of our co-ordinates λ, μ and ν.

To do this we may divide equation (**35**) through by μ^2 and equation (**36**) through by λ^2 and then subtract. We obtain upon simplification.

$$z^2 = \frac{(\lambda^2 - c^2)\,(c^2 - \mu^2)}{c^2} \tag{38}$$

Substituting this value of z^2 in equation (**35**) we obtain

$$x^2 + y^2 = \frac{\lambda^2\,\mu^2}{c^2}$$

and upon substituting the value of y from equation (**37**) we obtain

$$x^2 = \frac{\lambda^2\,\mu^2}{c^2\,(1 + \nu^2)} \tag{39}$$

similarly we obtain

$$y^2 = \frac{\lambda^2\,\mu^2\,\nu^2}{c^2\,(1 + \nu^2)} \tag{40}$$

Equations (**38**), (**39**) and (**40**) enable us to express the position of a point in space x, y, z when the values of λ, μ and ν are given.

Let $x = x_0$, $y = y_0$, $z = z_0$ be the rectangular co-ordinates of the point in space determined by the intersection of the three surfaces $\lambda = \lambda_0$, $\mu = \mu_0$, $\nu = \nu_0$.

The rectangular co-ordinates of the point

$$\lambda = \lambda_0 + d\,\lambda$$
$$\mu = \mu_0$$
$$\nu = \nu_0$$

will be

$$x = x_0 + \frac{\partial\, x}{\partial\, \lambda}\, d\,\lambda$$

$$y = y_0 + \frac{\partial\, y}{\partial\, \lambda}\, d\,\lambda$$

$$z = z_0 + \frac{\partial\, z}{\partial\, \lambda}\, d\,\lambda$$

That is, it will be the original position plus the rate of change of x with respect to a change in λ multiplied by the increment in λ. This will be readily seen from Fig. 7.

The square of the distance between the points λ_0 and $\lambda_0 + d\,\lambda$ will be

$$dn_1^2 = \left(\frac{\partial\, x}{\partial\, \lambda}\right)^2 d\,\lambda^2 + \left(\frac{\partial\, y}{\partial\, \lambda}\right)^2 d\,\lambda^2 + \left(\frac{\partial\, z}{\partial\, \lambda}\right)^2 d\,\lambda^2$$

$$dn_1^2 = \left[\left(\frac{\partial\, x}{\partial\, \lambda}\right)^2 + \left(\frac{\partial\, y}{\partial\, \lambda}\right)^2 + \left(\frac{\partial\, z}{\partial\, \lambda}\right)^2\right] d\,\lambda^2 \tag{41}$$

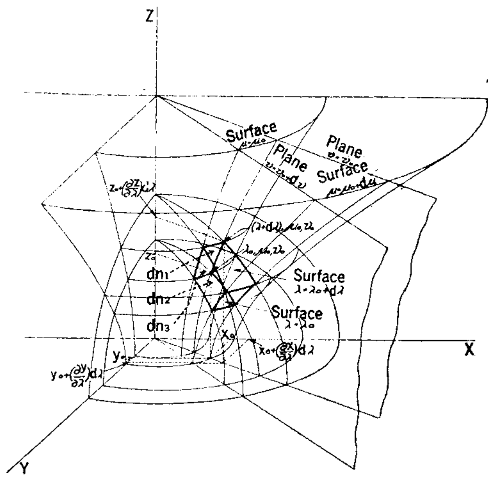

FIG. 7—ELEMENTARY SOLID CUT OUT BY THE INTERSECTION OF A PAIR OF CONFOCAL HYPERBOLOIDS OF REVOLUTION, A PAIR OF ELLIPSOIDS OF REVOLUTION AND A PAIR OF PLANES THROUGH THE AXIS OF REVOLUTION (Z AXIS)

In the limit the elementary curvilinear rectangular parallelopiped is simply a rectangular parallelopiped, that is, the effect of the curvative will introduce an effect which will be an infinitesimal of higher order than $d\,\lambda$ and dn_1 and, therefore, may be neglected. From this consideration it will be observed that dn_1 is the differential unit of length normal to the surface $\lambda = \lambda_0$. Thus, a considerable simplification has resulted from the fact that our system of curvilinear co-ordinates are orthogonal.

In like manner, we can write down the expressions for the differential unit normals to the other two surfaces

$\mu = \mu_0$ and $\nu = \nu_0$.

Thus,

$$dn_2^2 = \left[\left(\frac{\partial x}{\partial \mu}\right)^2 + \left(\frac{\partial y}{\partial \mu}\right)^2 + \left(\frac{\partial z}{\partial \mu}\right)^2\right] d\,\mu^2 \tag{42}$$

which is the elementary normal to the surface $\mu = \mu_0$
Similarly

$$dn_3^2 = \left[\left(\frac{\partial x}{\partial \nu}\right)^2 + \left(\frac{\partial y}{\partial \nu}\right)^2 + \left(\frac{\partial z}{\partial \nu}\right)^2\right] d\,\nu^2 \tag{43}$$

is the elementary normal to the surface $\nu = \nu_0$

If we now let

$$\frac{1}{h_1^2} = \left(\frac{\partial x}{\partial \lambda}\right)^2 + \left(\frac{\partial y}{\partial \lambda}\right)^2 + \left(\frac{\partial z}{\partial \lambda}\right)^2 \tag{44}$$

$$\frac{1}{h_2^2} = \left(\frac{\partial x}{\partial \mu}\right)^2 + \left(\frac{\partial y}{\partial \mu}\right)^2 + \left(\frac{\partial z}{\partial \mu}\right)^2 \tag{45}$$

$$\frac{1}{h_3^2} = \left(\frac{\partial x}{\partial \nu}\right)^2 + \left(\frac{\partial y}{\partial \nu}\right)^2 + \left(\frac{\partial z}{\partial \nu}\right)^2 \tag{46}$$

we can write the three equations for the differential normals in the more compact form.

$$dn_1 = \frac{d\,\lambda}{h_1} \tag{47}$$

$$dn_2 = \frac{d\,\mu}{h_2} \tag{48}$$

$$dn_3 = \frac{d\,\nu}{h_3} \tag{49}$$

The elementary surfaces which form the sides of the differential volume may now be written (see Fig. 7)

$$d\,S_1 = dn_2\,dn_3 = \left(\frac{d\,\mu\,d\,\nu}{h_2\,h_3}\right) \tag{50}$$

for the surface $\lambda = \lambda_0$

$$d\,S_2 = dn_1\,dn_3 = \frac{d\,\lambda\,d\,\nu}{h_1\,h_3} \tag{51}$$

for the surface $\mu = \mu_0$

$$d\,S_3 = dn_1\,dn_2 = \frac{d\,\lambda\,d\,\mu}{h_1\,h_2} \tag{52}$$

for the surface $\nu = \nu_0$

We may now write the differential volume $d\,\tau$ of the rectangular parallelopiped

$$d\,\tau = dn_1\,dn_2\,dn_3 = \frac{d\,\lambda\,d\,\mu\,d\,\nu}{h_1\,h_2\,h_3} \tag{53}$$

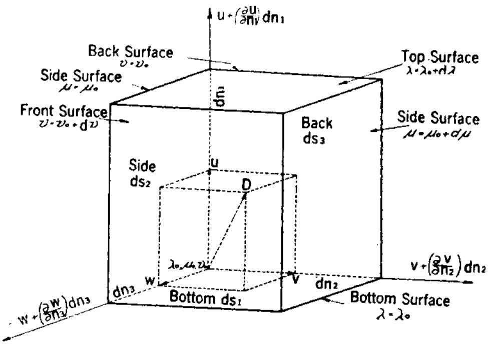

FIG. 8

We can now proceed to write down Laplace's equation in our present system of orthogonal curvilinear co-ordinates. We will here consider the case of a homogeneous and isotropic medium. The clearest method of obtaining Laplace's equation is to apply the well known laws of flow of an incompressible fluid to the infinitesimal unit of volume bounded by the intersection of the six surfaces,

$$\lambda = \lambda_0 \quad \mu = \mu_0 \quad \nu = \nu_0$$

$$\lambda = \lambda_0 + d\,\lambda \quad \mu = \mu_0 + d\,\mu \quad \nu = \nu_0 + d\,\nu$$

Now let (see Fig. 8)

u equal the component of the flow density or flux density,

D, at the point λ_0, μ_0, ν_0, parallel to dn_1

Similarly

$$v = \text{component of } D \text{ parallel to } dn_2.$$

$$w = \text{component of } D \text{ parallel to } dn_3.$$

Now the total amount of flux (or flow) leaving the infinitesimal parallelopiped is the sum of the six amounts leaving it through the six faces; each of these is expressed as; flow density × surface.

Therefore, they are,

$$\left.\begin{aligned} & -u\,d\,S_1 + \left(u + \frac{\partial u}{\partial n_1}\,dn_1\right)\left(d\,S_1 + \frac{\partial\,(d\,S_1)}{\partial n_1}\,dn_1\right) \\ & -v\,d\,S_2 + \left(v + \frac{\partial v}{\partial n_2}\,dn_2\right)\left(d\,S_2 + \frac{\partial\,(d\,S_2)}{\partial n_1}\,dn_2\right) \\ & -w\,d\,S_3 + \left(w + \frac{\partial w}{\partial n_3}\,dn_3\right)\left(d\,S_3 + \frac{\partial\,(d\,S_3)}{\partial n_3}\,d\,n_3\right) \end{aligned}\right\} \qquad (\mathbf{54})$$

In order to see the exact meaning of these three equations we will put the first one in word form. The first term says that the total flux (or flow) entering the lower surface of the infinitesimal parallelopiped is equal to the component of the flux density in the direction perpendicular to the surface × the area of the lower surface. Since we desire to express the outward flow we designate this inward flow by the minus sign. The second term expresses the total flux outward through the top surface. For this case the component of the flux density will be the original component plus the rate of increase of the component as we move towards the top of the cube multiplied by the height of the cube. This gives the component of the flux density at the top of the cube and if we multiply this by the area of the top we obtain the total outward flow at the top surface.

It will be observed that the area of the top surface is the area of the bottom plus the rate of increase of the area as we go from the bottom towards the top multiplied by the distance.

Multiplying out the second term of the first line of equation (**54**) we obtain:

$$u\,d\,S_1 + \frac{\partial u}{\partial n_1}\,d\,n_1\,d\,S_1 + u\,\frac{\partial\,(d\,S_1)}{\partial n_1}\,d\,n_1 + \frac{\partial u}{\partial n_1}\,\frac{\partial\,(d\,S_1)}{\partial n_1}\,d\,n_1^2 \qquad (\mathbf{55})$$

the fourth term is of higher order and therefore may be neglected

Factoring out the $d\,n_1$, we can write (**55**) as:

$$\left[\frac{\partial u}{\partial n_1}\,d\,S_1 + u\,\frac{\partial\,(d\,S_1)}{\partial n_1}\right] d\,n_1 \qquad \textbf{(56)}$$

It will now be evident that this is merely the differential of the product of the two functions u and $d\,S_1$. (I am indebted to Professor J. N. Vedder, who called my attention to this relation while discussing the problem with him.) That is

$$d\,[u\,d\,S_1] = u\,d\,(d\,S_1) + d\,S_1\,d\,u$$

We can therefore write (**56**)

$$\frac{\partial}{\partial n_1}\,[u\,d\,S_1]\,d\,n_1$$

Applying the same reasoning to the other similar equations and adding them all together, we obtain the following equation, which represents the total amount of flux leaving the infinitesimal volume.

$$\frac{\partial}{\partial n_1}\,(u\,d\,S_1)\,d\,n_1 + \frac{\partial}{\partial n_2}\,(v\,d\,S_2)\,d\,n_2 + \frac{\partial}{\partial n_3}(w\,d\,S_3)\,d\,n_3 \qquad \textbf{(57)}$$

If we substitute the values of $d\,n_1$, $d\,n_2$ and $d\,n_3$ as given by equations (**47**), (**48**) and (**49**); as well as $d\,S_1$, $d\,S_2$, $d\,S_3$, as given by equations (**50**), (**51**) and (**52**) in equation (**57**) we obtain

$$h_1\,\frac{\partial}{\partial\lambda}\left(u\,\frac{d\,\mu\,d\,\nu}{h_2\,h_3}\right)\frac{d\,\lambda}{h_1} + h_2\,\frac{\partial}{\partial\mu}\left(v\,\frac{d\,\lambda\,d\,\nu}{h_1\,h_3}\right)\frac{d\,\mu}{h_2}$$

$$+\ h_3\,\frac{\partial}{\partial\nu}\left(w\,\frac{d\,\lambda\,d\,\mu}{h_1\,h_2}\right)\frac{d\,\nu}{h_3} \qquad \textbf{(58)}$$

Simplifying and dividing by (**53**) to express the equation for unit of volume, we get:

$$\left[\frac{\partial}{\partial\lambda}\left(\frac{u}{h_2\,h_3}\right) + \frac{\partial}{\partial\mu}\left(\frac{v}{h_1\,h_3}\right) + \frac{\partial}{\partial\nu}\left(\frac{w}{h_1\,h_2}\right)\right] h_1\,h_2\,h_3 \qquad \textbf{(59)}$$

Now for any point in space which is neither a source nor a sink of flux, we see that the same quantity of flux must leave as enters any closed surface in order that there may be no accumulation within. Therefore, we may set this equation equal to zero. This now constitutes the familiar equation of continuity in the λ, μ, ν system of coordinates.

To obtain the same physical statement in terms of potential we merely have to substitute the relation between flux density and potential gradient as expressed by the Fourier-Ohm law namely,

$$\left.\begin{aligned} u &= -\kappa \frac{\partial V}{\partial n_1} \\ v &= -\kappa \frac{\partial V}{\partial n_2} \\ w &= -\kappa \frac{\partial V}{\partial n_3} \end{aligned}\right\} \qquad (60)$$

In other words these equations state that the flow density in a given direction is equal to the potential gradient in that direction multiplied by the specific conductivity. The minus sign indicating that the potential is decreasing in the direction of the flow. We will here take the specific conductivity to be unity and equal in all directions, that is, the material has unit specific resistance and is homogeneous and isotropic.

We may, therefore, rewrite equation (**56**) for this case.

$$\left.\begin{aligned} u &= -\frac{\partial V}{\partial n_1} \\ v &= -\frac{\partial V}{\partial n_2} \\ w &= -\frac{\partial V}{\partial n_3} \end{aligned}\right\} \qquad (61)$$

Substituting the values of dn_1, dn_2 and dn_3 as given by equations (**47**), (**48**) and (**49**) in equation (**61**) we obtain,

$$\left.\begin{aligned} u &= -h_1 \frac{\partial V}{\partial \lambda} \\ v &= -h_2 \frac{\partial V}{\partial \mu} \\ w &= -h_3 \frac{\partial V}{\partial \nu} \end{aligned}\right\} \qquad (62)$$

Therefore, if we substitute the values given in equation (**62**) in equation (**59**), we obtain after changing signs:

$$\frac{\partial}{\partial \lambda}\left(\frac{h_1}{h_2\,h_3}\;\frac{\partial V}{\partial \lambda}\right)+\frac{\partial}{\partial \mu}\left(\frac{h_2}{h_1\,h_3}\;\frac{\partial \nu}{\partial \mu}\right)$$

$$+\frac{\partial}{\partial \nu}\left(\frac{h_3}{h_1\,h_2}\;\frac{\partial v}{\partial \nu}\right)h_1\,h_2\,h_3=0 \qquad (\mathbf{63})$$

which is Laplace's equation in orthogonal curvilinear coordinates. This simple method of deducing the above equation was first given by Sir William Thomson (see Mathematical and Physical Papers Vol. I, p. 25).

We may now proceed to obtain the values of h_1, h_2 and h_3 for our case of normal oblate spheroidal co-ordinates from equations (**38**), (**39**) and (**40**) when combined with equations (**44**), (**45**) and (**46**).

Thus differentiating equations (**38**), (**39**) and (**40**) with respect to λ we obtain

$$\left.\begin{aligned}\frac{\partial x}{\partial \lambda}&=\frac{\mu}{c\sqrt{1+\nu^2}}\\ \frac{\partial y}{\partial \lambda}&=\frac{\mu\,\nu}{c\sqrt{1+\nu^2}}\\ \frac{\partial z}{\partial \lambda}&=\frac{\lambda}{c}\sqrt{\frac{c^2-\mu^2}{\lambda^2-c^2}}\end{aligned}\right\} \qquad (\mathbf{64})$$

Substituting these values in equation (**44**) we obtain

$$\frac{1}{h_1^2}=\left(\frac{\mu}{c\sqrt{1+\nu^2}}\right)^2+\left(\frac{\mu\,\nu}{c\sqrt{1+\nu^2}}\right)^2+\left(\frac{\lambda}{c}\sqrt{\frac{c^2-\mu^2}{\lambda^2-c^2}}\right)^2$$

Simplifying we get

$$\frac{1}{h_1^2}=\frac{\lambda^2-\mu^2}{\lambda^2-c^2} \qquad (\mathbf{65})$$

Similarly we obtain

$$\left.\begin{aligned} \frac{\partial x}{\partial \mu} &= \frac{\lambda}{c\sqrt{1+\nu^2}} \\ \frac{\partial y}{\partial \mu} &= \frac{\lambda \nu}{c\sqrt{1+\nu^2}} \\ \frac{\partial z}{\partial \mu} &= -\frac{\mu}{c}\sqrt{\frac{\lambda^2-c^2}{c^2-\mu^2}} \end{aligned}\right\} \tag{66}$$

From which

$$\frac{1}{h_2^{\,2}} = \frac{\lambda^2-\mu^2}{c^2-\mu^2} \tag{67}$$

Again

$$\left.\begin{aligned} \frac{\partial x}{\partial \nu} &= -\frac{\lambda\mu\nu}{c(1+\nu^2)^{3/2}} \\ \frac{\partial y}{\partial \nu} &= \frac{\lambda\mu}{c(1+\nu^2)^{3/2}} \\ \frac{\partial z}{\partial \nu} &= 0 \end{aligned}\right\} \tag{68}$$

and

$$\frac{1}{h_3^{\,2}} = \frac{\lambda^2\mu^2}{c^2(1+\nu^2)^2} \tag{69}$$

If we now substitute these values of $\frac{1}{h_1}$ $\frac{1}{h_2}$, $\frac{1}{h_3}$, in equation (**63**) we get

$$\frac{\mu}{c(1+\nu^2)\sqrt{c^2-\mu^2}}\frac{\partial}{\partial\lambda}\left[\lambda\sqrt{\lambda^2-c^2}\frac{\partial V}{\partial\lambda}\right]+$$

$$\frac{\lambda}{c(1+\nu^2)\sqrt{\lambda^2-c^2}}\frac{\partial}{\partial\mu}\left[\mu\sqrt{c^2-\mu^2}\frac{\partial V}{\partial\mu}\right]+$$

$$\frac{c(\lambda^2-\mu^2)}{\lambda\mu\sqrt{(\lambda^2-c^2)(c^2-\mu^2)}}\frac{\partial}{\partial\nu}\left[(1+\nu^2)\frac{\partial V}{\partial\nu}\right]=0 \tag{70}$$

which is Laplace's equation in our spheroidal coordinates λ, μ and ν.

If now in place of λ, μ and ν we can find some function of λ, some function of μ, and some function of ν which, therefore, will represent the same set of orthogonal surfaces, and if we can choose these functions which we shall designate by α, β and γ, which of course are also functions of x, y and z, so that they are solutions of Laplace's equation when expressed in rectangular co-ordinates, and, therefore, satisfy it when substituted therein. That is, we would have,

$$\left.\begin{aligned} \frac{\partial^2 \alpha}{\partial x^2} + \frac{\partial^2 \alpha}{\partial y^2} + \frac{\partial^2 \alpha}{\partial z^2} &= 0 \\ \frac{\partial^2 \beta}{\partial x^2} + \frac{\partial^2 \beta}{\partial y^2} + \frac{\partial^2 \beta}{\partial z^2} &= 0 \\ \frac{\partial^2 \gamma}{\partial x^2} + \frac{\partial^2 \gamma}{\partial y^2} + \frac{\partial^2 \gamma}{\partial z^2} &= 0 \end{aligned}\right\}$$

under these circumstances equation (**70**) will be reduced to a more simple and symmetrical form. It must be remembered that the existence of these functions α, β, γ, is merely tentative. The criterion for their existence will be given later.

Equation (**70**) is Laplace's equation expressed in terms of λ, μ and ν.

Assuming now that in equation (**70**) V is a function of λ only; in that case

$$\frac{\partial V}{\partial \mu} = 0 \text{ and } \frac{\partial V}{\partial \nu} = 0$$

and equation (**70**) would then reduce to

$$\frac{\partial}{\partial \lambda}\left[\lambda \sqrt{\lambda^2 - c^2}\, \frac{\partial V}{\partial \lambda}\right] = 0 \qquad \textbf{(71)}$$

Integrating with respect to λ we get

$$\lambda \sqrt{\lambda^2 - c^2}\, \frac{d V}{d \lambda} = c_1 \qquad \textbf{(72)}$$

whence

$$d\,V = \frac{c_1\,d\,\lambda}{\lambda\sqrt{\lambda^2 - c^2}} \tag{73}$$

and integrating again

$$V = \frac{c_1}{c}\sec^{-1}\frac{\lambda}{c} + K_1 \tag{74}$$

Here V is a function of λ which satisfies Laplace's equation. Call this value of V which, is a function of λ and which satisfies Laplace's equation, α leaving the constant c_1 undertermined thus

$$d\,\alpha = \frac{c_1\,d\,\lambda}{\lambda\sqrt{\lambda^2 - c^2}} \tag{75}$$

and

$$\alpha = \frac{c_1}{c}\sec^{-1}\frac{\lambda}{c} + K_1 \tag{76}$$

The constant K_1 is zero since,

$$\alpha = 0$$

when $\lambda = c$

and $\sec^{-1}1 = 0$

As above assumed now that V in equation (**70**) is a function of μ only; in this case $\frac{\partial V}{\partial \lambda} = 0$ and $\frac{\partial V}{\partial \nu} = 0$ and

$$\frac{\partial}{\partial \mu}\left[\mu\sqrt{c^2 - \mu^2}\,\frac{\partial V}{\partial \mu}\right] = 0 \tag{77}$$

Integrating

$$\mu\sqrt{c^2 - \mu^2}\,\frac{d\,V}{d\,\mu} = c_2 \tag{78}$$

$$d\,V = \frac{c_2\,d\,\mu}{\mu\sqrt{c^2 - \mu^2}} \tag{79}$$

$$V = -\frac{c_2}{c}\,\mathrm{sech}^{-1}\frac{\mu}{c} + K_2 \tag{80}$$

Call this value of V which is a function of μ only, and which satisfies Laplace's equation, β, leaving the constant c_2 undetermined we have

$$d\,\beta = \frac{c_2\,d\,\mu}{\mu\sqrt{c^2 - \mu^2}} \tag{81}$$

$$\beta = -\frac{c_2}{c}\,\text{sech}^{-1}\frac{\mu}{c} + K_2 \tag{82}$$

The constant of integrating K_2 is zero since

$$\beta = 0$$

when $\mu = c$

and $\text{sech}^{-1} 1 = 0$

Similarly

$$\frac{\partial}{\partial\,\nu}\left[(1+\nu^2)\,\frac{\partial\,V}{\partial\,\nu}\right] = 0 \tag{83}$$

$$(1+\nu^2)\,\frac{d\,V}{d\,\nu} = c_3 \tag{84}$$

$$d\,V = \frac{c_3\,d\,\nu}{1+\nu^2} \tag{85}$$

$$V = c_3 \tan^{-1}\nu + K_3 \tag{86}$$

If, as before, we call this value of V, which is a function of ν only and which satisfies Laplace's equation, γ, leaving the constant c_3 undetermined we have

$$d\,\gamma = \frac{c_3\,d\,\nu}{1+\nu^2} \tag{87}$$

$$\gamma = c_3 \tan^{-1}\nu + K_3 \tag{88}$$

The constant of integration K_3 vanishes since

$$\gamma = 0$$

when $\nu = 0$

and $\tan^{-1} 0 = 0$

If we now substitute these values in equation (**70**) we can express Laplace's equation in terms of our new functions α, β and γ and thereby obtain a simpler equation. To do this let us take equation (**70**) term by term.

The first term is

$$\frac{\mu}{c\,(1+\nu^2)\,\sqrt{c^2-\mu^2}}\,\frac{\partial}{\partial\,\lambda}\left[\lambda\,\sqrt{\lambda^2-c^2}\,\frac{\partial\,V}{\partial\,\lambda}\right]$$

expanding the indicated differentiation we get

$$\frac{\mu}{c\,(1+\nu^2)\,\sqrt{c^2-\mu^2}}\left[\frac{\partial}{\partial\,\lambda}\left(\lambda\,\sqrt{\lambda^2-c^2}\,\frac{\partial\,V}{\partial\,\lambda}\right.\right.$$
$$\left.\left.+\,\lambda\,\sqrt{\lambda^2-c^2}\,\frac{\partial^2\,V}{\partial\,\lambda^2}\right)\right] \qquad \textbf{(89)}$$

From equation (**71**) the first term within the bracket is equal to zero.

If we now assign the following values to the constants, c_1, c_2 and c_3

$$c_1 = -\,c_2 = c \text{ and } c_3 = 1$$

•We have from equation (**73**)

$$\frac{d\,V}{d\,\lambda} = \frac{c}{\lambda\,\sqrt{\lambda^2-c^2}}$$

$$\frac{d^2\,V}{d\,\lambda^2} = \frac{d}{d\,\lambda}\left(\frac{c}{\lambda\,\sqrt{\lambda^2-c^2}}\right)$$

$$\frac{d^2\,V}{d\,\lambda^2} = \frac{-\,c\,[\,(\lambda^2-c^2)+\lambda^2]}{\lambda^2\,(\lambda^2-c^2)\,\sqrt{\lambda^2-c^2}} \qquad \textbf{(90)}$$

Now from equation (**75**) we have

$$\frac{d\,\lambda}{d\,\alpha} = \frac{\lambda\,\sqrt{\lambda^2-c^2}}{c}$$

$$\left(\frac{d\,\lambda}{d\,\alpha}\right)^2 = \frac{\lambda^2\,(\lambda^2-c^2)}{c^2} \qquad \textbf{(91)}$$

Combining (**90**) and (**91**) we obtain

$$\frac{d^2\ V}{d\ \alpha^2} = \frac{d^2\ V}{d\ \lambda^2}\frac{d\ \lambda^2}{d\ \alpha^2} = -\frac{(\lambda^2 - c^2) + \lambda^2}{c\sqrt{\lambda^2 - c^2}} \tag{92}$$

Now comparing (**90**) and (**92**) it will be seen that

$$\frac{d^2\ V}{d\ \lambda^2} = \frac{c^2}{\lambda^2 (\lambda^2 - c^2)}\frac{d^2\ V}{d\ \alpha^2} \tag{93}$$

If we now substitute this value of $\frac{d^2\ V}{d\ \lambda^2}$ in the second term of equation (**89**) we get

$$\frac{\mu}{c\,(1 + \nu^2)\sqrt{c^2 - \mu^2}} \left\{\lambda \sqrt{\lambda^2 - c^2}\left[\frac{c^2}{\lambda^2(\lambda^2 - c^2)}\frac{\partial^2\ V}{\partial\ \alpha^2}\right]\right\}$$

or simplifying the first term of equation (**70**) when expressed in terms of $\frac{\partial^2\ V}{\partial\ \alpha^2}$ we get

$$\frac{c}{\lambda\,(1 + \nu^2)\sqrt{c^2 - \mu^2}\sqrt{\lambda^2 - c^2}}\frac{\partial^2\ V}{\partial\ \alpha^2} \tag{94}$$

If we now perform the same operation upon the second term of equation (**70**) which is

$$\frac{\lambda}{c\,(1 + \nu^2)\sqrt{\lambda^2 - c^2}}\frac{\partial}{\partial\ \mu}\left[\mu\sqrt{c^2 - \mu^2}\frac{\partial\ V}{\partial\ \mu}\right] \tag{95}$$

expanding the indicated differentiation we have

$$\frac{\lambda}{c\,(1 + \nu^2)\sqrt{\lambda^2 - c^2}}\left[\frac{\partial}{\partial\ \mu}\left(\mu\sqrt{c^2 - \mu^2}\right)\frac{\partial\ V}{\partial\ \mu} + \mu\sqrt{c^2 - \mu^2}\frac{\partial^2\ V}{\partial\ \mu^2}\right] \tag{96}$$

Equation (**77**) tells us that the first term within the bracket is zero.

Now from equation (**79**) putting $c_2 = -c$ we have

$$\frac{d\,V}{d\,\mu} = \frac{-c}{\mu\,\sqrt{c^2-\mu^2}}$$

and then

$$\frac{d^2\,V}{d\,\mu^2} = \frac{c\,[\,(c^2-\mu^2)-\mu^2]}{\mu^2\,(c^2-\mu^2)\,\sqrt{c^2-\mu^2}} \tag{97}$$

From equation (**81**) we have

$$\frac{d\,\mu}{d\,\beta} = \frac{\mu\,\sqrt{c^2-\mu^2}}{-c}$$

$$\frac{d\,\mu^2}{d\,\beta^2} = \frac{\mu^2(c^2-\mu^2)}{c^2} \tag{98}$$

From (**97**) and (**98**) we obtain

$$\frac{d^2\,V}{d\,\beta^2} = \frac{d^2\,V}{d\,\mu^2}\frac{d\,\mu^2}{d\,\beta^2} = \frac{(c^2-\mu^2)-\mu^2}{c\,\sqrt{c^2-\mu^2}} \tag{99}$$

Now from (**97**) and (**99**) it will be seen that the following relation exists

$$\frac{d^2\,V}{d\,\mu^2} = \frac{c^2}{\mu^2\,(c^2-\mu^2)}\frac{d^2\,V}{d\,\beta^2} \tag{100}$$

Hence substituting this value in equation (**96**) we get for the second term of equation (**70**) when expressed in terms of $\frac{\partial^2\,V}{\partial\,\beta^2}$ the following.

$$\frac{\lambda\,c}{\mu\,(1+\nu^2)\,\sqrt{c^2-\mu^2}\,\sqrt{\lambda^2-c^2}}\frac{\partial^2\,V}{\partial\,\beta^2} \tag{101}$$

Now as before expand the third term of equation (**70**)

$$\frac{c\,(\lambda^2-\mu^2)}{\lambda\,\mu\,\sqrt{\lambda^2-c^2}\,\sqrt{c^2-\mu^2}}\left[\frac{\partial}{\partial\,\nu}\,(1+\nu^2)\,\frac{\partial\,V}{\partial\,\nu} + (1+\nu^2)\,\frac{\partial^2\,V}{\partial\,\nu^2}\right] \tag{102}$$

and observe that the first term within the brackets vanishes by equation (**83**).

Now from equation (**85**) putting $c_3 = 1$, we get,

$$\frac{d\,V}{d\,\nu} = \frac{1}{1 + \nu^2}$$

$$\frac{d^2\,V}{d\,\nu^2} = \frac{-\,2\,\nu}{(1 + \nu^2)^2} \tag{103}$$

From equation (**87**)

$$\frac{d\,\nu}{d\,\gamma} = 1 + \nu^2$$

$$\frac{d\,\nu^2}{d\,\gamma^2} = (1 + \nu^2)^2 \tag{104}$$

From (**103**) and (**104**)

$$\frac{d^2\,V}{d\,\gamma^2} = \frac{d^2\,V}{d\,\nu^2}\,\frac{d\,\nu^2}{d\,\gamma^2} = -\,2\,\nu \tag{105}$$

Now from (**103**) and (**105**) we get

$$\frac{d^2\,V}{d\,\nu^2} = \frac{1}{(1 + \nu^2)^2}\,\frac{d^2\,V}{d\,\gamma^2} \tag{106}$$

Hence substituting this value in equation (**102**) we get for the third term of equation (**70**) when expressed in terms of $\frac{\partial^2 V}{\partial\,\gamma^2}$ the following.

$$\frac{c\,(\lambda^2 - \mu^2)}{\lambda\,\mu\,(1 + \nu^2)\,\sqrt{c^2 - \mu^2}\,\sqrt{\lambda^2 - c^2}}\,\frac{\partial^2\,V}{\partial\,\gamma^2} \tag{107}$$

If we now combine the three terms as given by equations (**94**), (**101**) and (**107**) we get

$$\frac{1}{\lambda^2}\,\frac{\partial^2\,V}{\partial\,\alpha^2} + \frac{1}{\mu^2}\,\frac{\partial^2\,V}{\partial\,\beta^2} + \frac{\lambda^2 - \mu^2}{\lambda^2 - \mu^2}\,\frac{\partial^2\,V}{\partial\,\gamma^2} = 0 \tag{108}$$

or since from equation (**76**), (**82**) and (**88**) and remembering that we put $c_1 = -c_2 = c_3 = 1$ we have

$$\left.\begin{aligned} \lambda &= c \sec \alpha \\ \mu &= c \operatorname{sech} \beta \\ \nu &= \tan \gamma \end{aligned}\right\} \quad (\mathbf{109})$$

we can now write equation (**108**)

$$\cos^2 \alpha \frac{\partial^2 V}{\partial \alpha^2} + \cosh^2 \beta \frac{\partial^2 V}{\partial \beta^2} + (\cosh^2 \beta - \cos^2 \alpha) \frac{\partial^2 V}{\partial \gamma^2} = 0 \quad (\mathbf{110})$$

which is Laplace's equation expressed in terms of what have been called normal oblate spheroidal co-ordinates.

When using this equation (**110**) it is to be noted that the point whose co-ordinates are (α, β, γ) is the point of intersection of an oblate spheroid whose semi-axes are

$c \sec \alpha$ (major axis)
$c \tan \alpha$ (minor axis and the axis of revolution.)

and a hyperboloid of revolution of one sheet whose semi-axes are

$c \operatorname{sech} \beta$ (transverse axis)
$c \tanh \beta$ (conjugate axis, the axis of revolution.)

and a plane containing the axis of revolution of the system and making the angle γ with a fixed plane.

If the axis of revolution is the z axis and the fixed plane of reference is the XZ plane; the rectangular co-ordinates of α, β, γ, will be obtained by substituting the values of λ, μ and ν given by equation (**109**) into equations (**38**), (**39**) and (**40**). This substitution gives

$$\left.\begin{aligned} x &= c \sec \alpha \operatorname{sech} \beta \cos \gamma \\ y &= c \sec \alpha \operatorname{sech} \beta \sin \gamma \\ z &= c \tan \alpha \tanh \beta \end{aligned}\right\} \quad (\mathbf{111})$$

If now we let α range from 0 to $\frac{\pi}{2}$, β from $-\infty$ to $+\infty$, and γ from 0 to 2π, we shall be able to represent all points in space;

and if we agree that negative values of β shall belong to points below a plane through the origin and perpendicular to the axis of revolution and positive values of β to points above that plane. Then not only shall we have no ambiguity, but also the rectangular co-ordinates of any point in space as given by equations (**111**) will have their proper signs. (See Fig. 9)

The above transformation of Laplace's equation, as expressed in terms of the orthogonal curvilinear co-ordinates λ, μ and ν which represent an orthogonal system of surfaces, to the simplified form in which it was expressed in terms of the new co-ordinates α, β, γ was made possible by the assumption that certain functions α, β, γ of λ, μ and ν exist. Thus we determined the value of α by solving equation (**70**) on the assumption that V is a function of λ only; μ and ν not entering. This means that

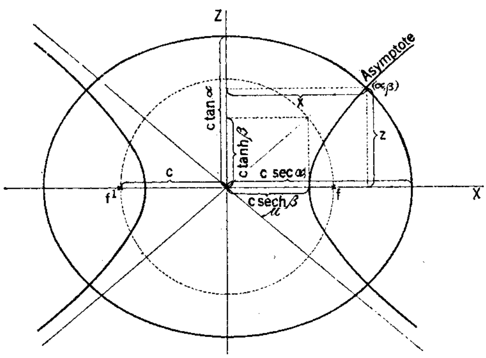

FIG. 9—CONFOCAL ELLIPSE AND HYPERBOLA

we assumed that the curvilinear co-ordinate λ corresponds to a possible equipotential or isothermal surface, for in that case there would be a function of λ which would satisfy Laplace's equation.

We may write it symbolically as follows:

Starting with Laplace's equation in terms of rectangular co-ordinates namely $\Delta^2 V = 0$ and remembering that $\lambda = f(x, y, z)$. Then if it happens that $V = f(\lambda)$ only, is a solution of Laplace's equation, then we may write it

$$V = f(\lambda) = \alpha$$

and upon substituting back in Laplace's equation we would get the condition that

$$\Delta^2 \alpha = 0$$

In other words, this means that α as found by this process is a value of the potential V corresponding to such a distribution that the surface obtained by giving particular values to λ are equipotential surfaces. The particular value of the function of λ, α is given in equation (**76**).

The condition that such a function α should exist, for a given system of surfaces, that is, that the distribution described above should be possible will be obtained analytically below.

We have seen that the required condition is merely that V in Laplace's equation may be a function of λ alone.

If V is a function of λ alone and we also remember that λ is a function of x, y and z, we may start from this information to write out Laplace's equation.

We have

$$V = f_1 (\lambda)$$

$$\lambda = f_2 (x, y, z)$$

Hence

$$\left.\begin{aligned} \frac{\partial V}{\partial x} &= \frac{d V}{d \lambda} \frac{\partial \lambda}{\partial x} \\ \frac{\partial V}{\partial y} &= \frac{d V}{d \lambda} \frac{\partial \lambda}{\partial y} \\ \frac{\partial V}{\partial z} &= \frac{d V}{d \lambda} \frac{\partial \lambda}{\partial z} \end{aligned}\right\}$$

Differentiating again we obtain

$$\left.\begin{aligned} \frac{\partial^2 V}{\partial x^2} &= \frac{d^2 V}{d \lambda^2} \frac{\partial \lambda^2}{\partial x^2} + \frac{d V}{d \lambda} \frac{\partial^2 \lambda}{\partial x^2} \\ \frac{\partial^2 V}{\partial y^2} &= \frac{d^2 V}{d \lambda^2} \frac{\partial \lambda^2}{\partial y^2} + \frac{d V}{d \lambda} \frac{\partial^2 \lambda}{\partial y^2} \\ \frac{\partial^2 V}{\partial z^2} &= \frac{d^2 V}{d \lambda^2} \frac{\partial \lambda^2}{\partial z^2} + \frac{d V}{d \lambda} \frac{\partial^2 \lambda}{\partial z^2} \end{aligned}\right\}$$

Adding these three equations and equating the result to zero in order to obtain Laplace's equation we get

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

also

$$\left[\left(\frac{\partial \lambda}{\partial x}\right)^2 + \left(\frac{\partial \lambda}{\partial y}\right)^2 + \left(\frac{\partial \lambda}{\partial z}\right)^2\right]\frac{d^2 V}{d \lambda^2} +$$

$$\left[\frac{\partial^2 \lambda}{\partial x^2} + \frac{\partial^2 \lambda}{\partial y^2} + \frac{\partial^2 \lambda}{\partial z^2}\right]\frac{d V}{d \lambda} = 0$$

We thus have obtained an expression for Laplace's equation based upon the assumption that the potential V shall be a function of λ alone. If we now write it

$$\frac{\dfrac{\partial^2 \lambda}{\partial x^2} + \dfrac{\partial^2 \lambda}{\partial y^2} + \dfrac{\partial^2 \lambda}{\partial z^2}}{\left(\dfrac{\partial \lambda}{\partial x}\right)^2 + \left(\dfrac{\partial \lambda}{\partial y}\right)^2 + \left(\dfrac{\partial \lambda}{\partial z}\right)^2} = - \frac{\dfrac{d^2 V}{d \lambda^2}}{\dfrac{d V}{d \lambda}}$$

This may be written in symbolic form

$$\frac{\Delta^2 \lambda}{h_1^2} = F_1 (\lambda)$$

where $F_1 (\lambda)$ may be any function of λ alone. Thus our required condition is that

$$\frac{\Delta^2 \lambda}{h_1^2} = F_1 (\lambda) \qquad \textbf{(112)}$$

and when this is fulfilled the original curvilinear co-ordinate λ corresponds to a possible equipotential or isothermal surface and in this case a function α exists. If similar relations are found to hold for β and γ the reduction of Laplace's equation to the so-called symmetrical form is possible.

Similarly we determine the values of β and γ on the assumption that V is a function of μ only in the case of β and a function of ν only in the case of γ. In that case it is evident that

$\Delta^2 \beta = 0$ and in the second case

$\Delta^2 \gamma = 0$

It is therefore evident that β is a value of V corresponding to such a distribution that the surfaces obtained by giving particular values to μ are equipotential surfaces; and that γ is a value of V corresponding to such a distribution that the surfaces obtained by giving particular value to ν are equipotential surfaces.

From what has been said above it is evident why Lamé, the inventor of this system of co-ordinates called the functions α, β and γ thermometric parameters.[38]

The analytical method of obtaining the criterion for the existence of these functions β and γ is exactly the same as that for the case of α which has been worked out. It is in the case of λ

$$\frac{\Delta^2 \lambda}{h_1^2} = F_1(\lambda)$$

and it is for μ

$$\left.\begin{aligned} \frac{\Delta^2 \mu}{h_2^2} &= F_2(\mu) \\ \text{and for } \nu \qquad & \\ \frac{\Delta^2 \nu}{h_3^2} &= F_3(\nu) \end{aligned}\right\} \qquad (\mathbf{113})$$

When these three conditions are fulfilled for λ, μ and ν, the original curvilinear co-ordinates λ, μ and ν correspond to possible equipotential or isothermal surfaces and thermometric parameters α, β and γ exist and the reduction of Laplace's equation to the so-called symmetrical form given by equation (**108**) is possible.

In all of the above work I have followed the development of the subject of orthogonal curvilinear co-ordinates as given in Byerly's Fourier Series and Spherical Harmonics pages 238 to 245 and merely tried to give an unexpurgated description of the process. That is, I have tried to fill in some of the steps in the process which are necessary in order that the average engineer may follow the development with ease.

The solution of the problems given in the following pages is

38. Numerous references to Lamé's work will be found in the following works.

Maxwell, Electricity and Magnetism, page 232; Webster, Electricity and Magnetism, page 22, 64,173; Byerly, Fourier's Series and Spherical Harmonics, page 244, 274; J. H. Jeans, Electricty and Magnetism, page 247; Goursat-Hedrick, Mathematical Analysis, Vol. I, page 80; I. Todhunter, The Functions of Laplace, Lamé and Bessel, page 210, due to Professor W. E. Byerly. I had been struggling with the problem for some time without success and finally took the liberty of writing Professor Byerly concerning my troubles. His reply of January 9th, 1915 to my letter exceeded my expectations for it gave the desired solution as given in the following pages. On February 17th, 1915, I had the further privilege of talking over the problems with him, at which time he cleared up the remaining troublesome points.

Potential at any Point in Space

Laplace's equation (**110**) expressed in the symmetrical form was seen to be

$$\cos^2\alpha \frac{\partial^2 V}{\partial \alpha^2} + \cosh^2\beta \frac{\partial^2 V}{\partial \beta^2} + (\cosh^2\beta - \cos^2\alpha)\frac{\partial^2 V}{\partial \gamma^2} = 0 \quad (\mathbf{114})$$

In the present problem we wish to determine the potential at any point in space between two confocal hyperboloids; one of which is chosen to represent the tank with a hole in it; the other representing the rod or connection which passes through the hole in the cover of the tank.

The potential V must satisfy the above equation and is a function of β only. This follows from the fact that the equipotential surfaces will be confocal hyperboloids of revolution.

Therefore, we have,

$$\frac{\partial^2 V}{\partial \alpha^2} = 0 \text{ and } \frac{\partial^2 V}{\partial \gamma^2} = 0$$

and Laplace's equation reduces in our case to

$$\cosh^2\beta \frac{\partial^2 V}{\partial \beta^2} = 0$$

This equation must be true for all values of β and V and since $\cosh^2\beta$ is not zero for all values of β, $\frac{\partial^2 V}{\partial \beta^2}$ must be equal to zero. Thus our equation will be

$$\frac{\partial^2 V}{\partial \beta^2} = 0 \quad (\mathbf{115})$$

The solution of this differential equation is seen to be obtained as follows:

Integrating,

$$\frac{dV}{d\beta} = A$$

$$dV = A\,d\beta$$

and

$$V = A\,\beta + C \tag{116}$$

which is the solution for the potential at any point in space. The values of the constants of integration A and C will be found from the conditions of the problem.

Thus if $\beta = \beta_0$ for the hyperboloid selected as the tank with hole in it, the potential is

$$V = 0$$

Assuming the tank to be grounded,

Hence $$0 = A\,\beta_0 + C$$

Also when $\beta = \beta_1$ for the hyperboloid selected as the electrode which passes through the hole in the tank

$$V = V_1$$

Hence

$$V_1 = A\,\beta_1 + C$$

Solving these equations simultaneously we have

$$\left.\begin{aligned} C &= \frac{V_1}{1 - \dfrac{\beta_1}{\beta_0}} \\ A &= \frac{V_1}{\beta_1 - \beta_0} \end{aligned}\right\} \tag{117}$$

Substituting the values of A and C as thus determined into equation (**116**) we have.

$$V = \frac{V_1\,\beta}{\beta_1 - \beta_0} + \frac{V_1}{1 - \dfrac{\beta_1}{\beta_0}}$$

or

$$V = \frac{V_1}{\beta_1 - \beta_0}(\beta - \beta_0) \tag{118}$$

which is the complete expression for the potential at any point in space under the assumed terminal conditions.

From considerations of symmetry it is obvious that the potential has circular symmetry about the axis of revolution; and, in this case, the z axis has been chosen.

We may now determine the potential at any point in space in rectangular co-ordinates as follows. Suppose we desire the potential at the point x, z for the case in which the semifocal distance is c.

We have for the value of μ which is the semi-transverse axis of the hyperbola

$$\mu = \frac{1}{2}(P\,F_2 - P\,F_1)$$

(see Fig. 10)

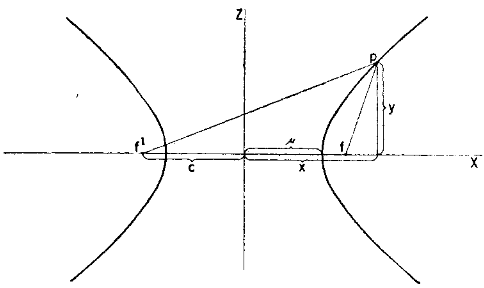

FIG. 10—HYPERBOLA

Hence,

$$\mu = \frac{1}{2}\left[\sqrt{(x+c)^2+z^2} - \sqrt{(x-c)^2+z^2}\right] \qquad \textbf{(119)}$$

Also from equation (**109**) we have,

$$\mu = c \operatorname{sech} \beta$$

$$\cosh \beta = \frac{c}{\mu}$$

$$\beta = \cosh^{-1}\frac{c}{\mu} \qquad \textbf{(120)}$$

Thus we can determine the potential at any point P in space under any desired conditions of c, β_1, β_0, V_1.

Potential Distribution on the X-Axis

For this case we have $z = 0$ in equation (**119**) and hence it reduces to

$$\mu = c \tag{121}$$

Now under these conditions μ and x are identical in value for points along the x axis between the origin and the focus. We can, therefore, use equation (**120**) directly without reference to equation (**119**) or (**121**).

Thus we can determine the values of β_1 and β_0 for any value of μ from equation (**120**) or rewriting it here

$$\beta = \cosh^{-1} \frac{c}{\mu} \tag{122}$$

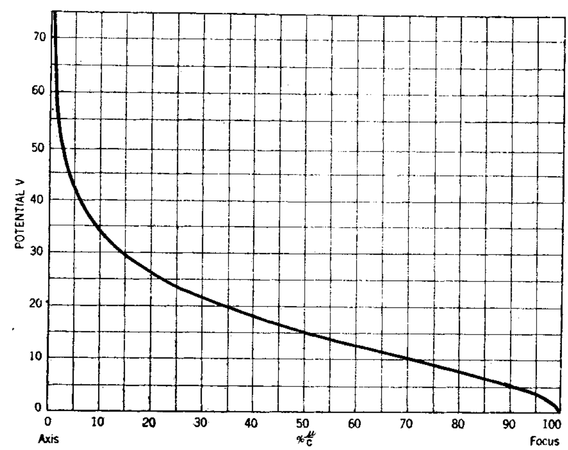

Fig. 11—Calculated Potential Distribution Along the X-Axis

If we select for the hyperboloid β_0 the limiting case that of the infinite plane with a hole in the centre $\beta_0 = 0$ for

$$\beta = \cosh^{-1} \frac{c}{\mu}$$

where

$$\mu = c$$

$$\beta = \beta_0 = \cosh^{-1} 1 = 0$$

We may then write equation (**118**) as follows:—

$$V = \left(\frac{V_1}{\beta_1}\right) \beta \tag{123}$$

The potential distribution for this case is shown in the curve, (see Fig. 11). From this curve we may find the proper spacing for the hyperbolas which when rotated about their z axis cut out the equipotential surfaces for unit differences of potential. All we have to do is to divide the potential into equal increments and note the corresponding value of μ or x. In this manner we can construct a diagram similar to Fig. 12. In the case before us the potential has been divided into 51 equal increments.

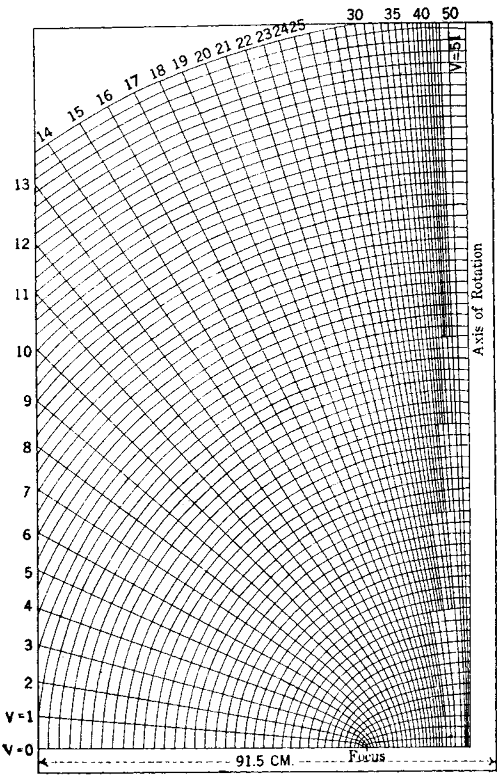

FIG. 12

POTENTIAL GRADIENT AT ANY POINT IN SPACE

The potential gradient is defined as the rate of change of potential in the direction of the greatest decrease in potential, that is, along the normal to the equipotential surface in the direction of decreasing potential. The sign will depend upon whether our unit normal is taken in the direction of increasing or decreasing potential. For the former case the expression for the potential gradient will be positive and for the latter negative. By reference to Figs. 7 and 8 and remembering that we have assumed the tank to be at zero potential it will be seen that we have selected the differential normal $d\,n_2$ in the direction of decreasing potential and, therefore, our expression for the potential gradient will be $-\frac{d\,V}{d\,n_2}$.

We have seen that the potential at any point in space is given by equation (**116**) and is

$$V = A\,\beta + C$$

Thus the potential gradient will be

$$-\frac{d\,V}{d\,n_2} = A\,\frac{d\,\beta}{d\,n_2} \qquad \textbf{(124)}$$

where $d\,n_2$ is the normal to the surface $\mu = \mu_0$, (a constant). and in the direction of decreasing potential.

In order to determine β we have the relation

$$\beta = \operatorname{sech}^{-1}\frac{\mu}{c}$$

Hence,

$$\frac{d\,\beta}{d\,\mu} = -\frac{c}{\mu\sqrt{c^2-\mu^2}} \qquad \textbf{(125)}$$

Now we have seen from equation (**48**) that

$$d\,n_2 = \frac{d\,\mu}{h_2} \qquad \textbf{(126)}$$

Equation (**67**) gives us the value of $\frac{1}{{h_2}^2}$

$$\frac{1}{{h_2}^2} = \frac{\lambda^2 - \mu^2}{c^2 - \mu^2}$$

$$\frac{1}{h_2} = \sqrt{\frac{\lambda^2-\mu^2}{c^2-\mu^2}} \qquad \textbf{(127)}$$

We may now substitute this value in equation (**126**) and we obtain

$$d\,n_2 = \sqrt{\frac{\lambda^2 - \mu^2}{c^2 - \mu^2}}\,d\,\mu \tag{128}$$

Thus,

$$\frac{d\,\mu}{d\,n_2} = \sqrt{\frac{c^2 - \mu^2}{\lambda^2 - \mu^2}} \tag{129}$$

Combining now equation (**125**) and (**129**) we have,

$$\frac{d\,V}{d\,n_2} = A\,\frac{d\,\beta}{d\,\mu}\,\frac{d\,\mu}{d\,n_2} \tag{130}$$

or,

$$\frac{d\,V}{d\,n_2} = -\frac{A\,c}{\mu\sqrt{c^2 - \mu^2}}\sqrt{\frac{c^2 - \mu^2}{\lambda^2 - \mu^2}} \tag{131}$$

and

$$-\frac{d\,V}{d\,n_2} = \frac{A\,c}{\mu\sqrt{\lambda^2 - \mu^2}} \tag{132}$$

The value of the potential gradient at any point in space.

The value of the constant A is that given in equation (**117**) and in order to determine the values of μ and λ in rectangular co-ordinates we have the relations given by equations (**109**) and (**111**). It will here be observed that the potential gradient is independent γ which is obvious from the symmetrical arrangement, that is, the potential gradient will be similar for any of the diametral planes γ = constant. Thus, we may write

$$\left.\begin{aligned} x &= c\sec\alpha\,\mathrm{sech}\,\beta \\ y &= c\tan\alpha\tanh\beta \end{aligned}\right\} \tag{133}$$

also,

$$\left.\begin{aligned} \lambda &= c\sec\alpha \\ \mu &= c\,\mathrm{sech}\,\beta \end{aligned}\right\} \tag{134}$$

These equations in combination with (**132**) enable us to calculate the potential gradient at any point in space in rectangular co-ordinates. (See Fig. 9.)

Curves of Constant Potential Gradient

The curves of constant potential gradient may now be obtained by putting

$$- \frac{d\,V}{d\,n_2} = G = \frac{A\,c}{\mu\,\sqrt{\lambda^2 - \mu^2}} \qquad \textbf{(135)}$$

Then

$$\mu\,\sqrt{\lambda^2 - \mu^2} = \frac{A\,c}{G} \qquad \textbf{(136)}$$

Thus for any given condition the right hand side of equation (**136**) is a constant. If we now substitute the values of λ and μ as given in equation (**134**) we may write

$$\sec\alpha = \pm \sqrt{\operatorname{sech}^2\beta + \frac{\left(\frac{A}{c\,G}\right)^2}{\operatorname{sech}^2\beta}}$$

or for purposes of calculation we may put for

$$\operatorname{sech}^2\beta = 1 - \tanh^2\beta$$

and we have

$$\sec\alpha = \pm \sqrt{(1 - \tanh^2\beta) + \frac{\left(\frac{A}{c\,G}\right)^2}{(1 - \tanh^2\beta)}} \qquad \textbf{(137)}$$

where the values in terms of x and y are

$$\left.\begin{aligned} x &= c \sec\alpha \operatorname{sech}\beta \\ y &= c \tan\alpha \tanh\beta \end{aligned}\right\}$$

The curves of constant potential gradient as calculated from these equations are shown in Fig. 13.

Potential Gradient Along the X Axis

The value of the potential gradient at any point along the x axis between the origin and the focus may be obtained from equation (**132**) as follows:

From equation (**111**) we have

$$z = c \tan\alpha \tanh\beta$$

Now when $z = 0$

$$c \tan \alpha \tanh \beta = 0$$

and since $c \tanh \beta$ is not zero we have

$$\tan \alpha = 0$$

$$\alpha = 0$$

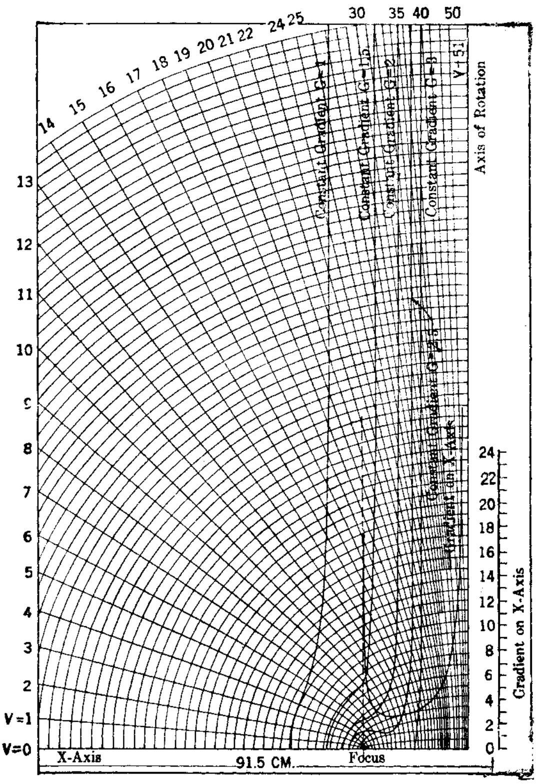

FIG. 13

Substituting this value of α in equation (**134**)

$$\lambda = c$$

and further upon substituting this value of λ in equation (**132**) we have

$$-\frac{d\,V}{d\,n_2} = \frac{A\,c}{\mu\sqrt{c^2 - \mu^2}} \tag{138}$$

The potential gradient along the x axis.

Another method of obtaining the same result is to observe that the potential gradient along the x axis is merely $-\frac{d\,V}{d\,\mu}$ since the hyperbolas cut the x axis normally. Thus as we have seen

$$V = A\,\beta + C$$

Differentiating with respect to μ

$$\frac{d\,V}{d\,\mu} = A\,\frac{d\,\beta}{d\,\mu}$$

Also,

$$\beta = \operatorname{sech}^{-1}\frac{\mu}{c}$$

Hence differentiating

$$\frac{d\,V}{d\,\mu} = A\,\frac{d\,\beta}{d\,\mu} = \frac{-A\,c}{\mu\sqrt{c^2-\mu^2}}$$

Thus,

$$-\frac{d\,V}{d\,\mu} = \frac{A\,c}{\mu\sqrt{c^2-\mu^2}} \qquad (\mathbf{139})$$

which is identical with equation (**138**) and is the potential gradient along the x axis between the origin and the focus. The potential gradient curve for this case is calculated and plotted in the curve Fig. 14.

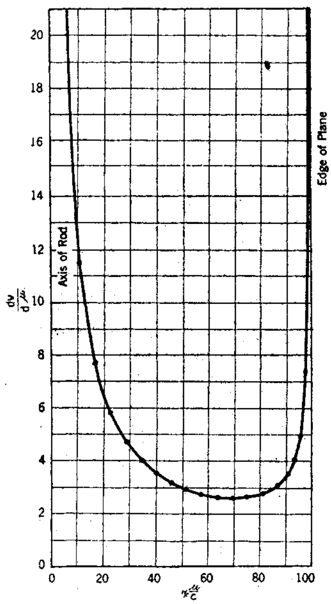

FIG. 14—CALCULATED POTENTIAL GRADIENT BETWEEN ROD AND EDGE OF PLANE

Testing this equation for the values $\mu = 0$ and $\mu = c$ it will be observed that the potential gradient is infinite at the edge of the hole in the plane or focus and at the origin or along the z axis. A minimum value of the gradient will exist somewhere between these two limits. Its value will be found in the usual manner as

$$\frac{d}{d\,\mu}\left(\frac{d\,V}{d\,\mu}\right) = 0$$

Thus differentiating equation (**139**) with respect to μ and equating the result to zero we have

$$\frac{d^2 V}{d \mu^2} = \frac{A c [(c^2 - \mu^2) - \mu^2]}{\mu^2 (b^2 - \mu^2)^{3/2}}$$

$$= \frac{A c [(c^2 - 2 \mu^2)]}{\mu^2 (b^2 - \mu^2)^{3/2}}$$

Equating to zero and observing that $A c$ is not zero we have

$$c^2 = 2 \mu^2$$

$$\mu = \frac{c}{\sqrt{2}} \tag{140}$$

Thus the minimum value of the potential gradient occurs at this point which is about 71 per cent of the distance from the axis towards the edge of the hole in the infinite plane.

The potential gradient curve enables us to determine the proper size of central electrode so as not to over stress the dielectric. Or by selecting the hyperboloid which we choose to represent our tank with hole we can select a central electrode which has the same value of the potential gradient or any other value which we may desire for other reasons. Knowing the characteristics of our dielectric used we can determine the puncture voltage of the apparatus, etc.

Solution for the Tubes of Flow or Flux

The flux or flow across any area may be defined as the surface integral of the normal component of the flux density, and as we have seen from equations (**56**) and (**57**) this in turn is equal to the normal component of the potential gradient.

Thus,

$$\left.\begin{aligned} &\iint -\frac{d V}{d n_2} \, d n_1 \, d n_3 \\ \text{or} \quad &\iint_s -\frac{d V}{d n_2} \, d S_2 \end{aligned}\right\} \tag{141}$$

We have from equation (**132**)

$$-\frac{d V}{d n_2} = \frac{A c}{\mu \sqrt{\lambda^2 - \mu^2}}$$

We also have for the values of $d\,n_1$ and $d\,n_3$ or $d\,S_2$, equation (**51**)

$$d\,S_2 = d\,n_1\,d\,n_3 = \frac{d\,\lambda\,d\,\nu}{h_1\,h_3} \tag{142}$$

and from equations (**65**) and (**69**)we have

$$\frac{1}{h_1} = \sqrt{\frac{\lambda^2 - \mu^2}{\lambda^2 - c^2}}$$

$$\frac{1}{h_3} = \frac{\lambda\,\mu}{c\,(1 + \nu^2)}$$

Substituting these values in (**142**) we get

$$d\,S_2 = \sqrt{\frac{\lambda^2 - \mu^2}{\lambda^2 - c^2}}\left(\frac{\lambda\,\mu}{c\,(1 + \nu^2)}\right) d\,\lambda\,d\,\nu$$

and combining in equation (**141**) and simplifying we get

$$\int\!\!\int_{S_2} - \frac{d\,V}{d\;n_2}\,d\,S_2 = \frac{A\,\lambda}{\sqrt{\lambda^2 - c^2}\,(1 + \nu^2)}\,d\,\lambda\,d\,\nu$$

$$= A\int\frac{d\,\nu}{(1 + \nu^2)}\int\frac{\lambda\,d\,\lambda}{\sqrt{\lambda^2 - c^2}}$$

Integrating the first part between the limits $\nu = 0$ and $\nu = \nu_0$ we have,

$$\Big[\tan^{-1}\nu + K\Big]_{\nu=0}^{\nu=\nu_0} = -\tan^{-1}\nu_0$$

and since ν_0 may have any value we can neglect it as far as the spacing of the surfaces of flow are concerned. Integrating the second part between the limits $\lambda = \lambda_0$ and $\lambda = \lambda_1$ we have

$$\int_{\lambda=\lambda_0}^{\lambda=\lambda_1}\frac{\lambda\,d\,\lambda}{\sqrt{\lambda^2 - c^2}} = \Big[\sqrt{\lambda^2 - c^2} + K\Big]_{\lambda=\lambda_0}^{\lambda=\lambda_1}$$

That is we have

$$\int_{S_2}\int - \frac{d\,V}{d\,n_2}\,d\,S_2 = A\,(\sqrt{\lambda_0^2 - c^2} - \sqrt{\lambda_1 - c^2}) \qquad \textbf{(143)}$$

If we now remember that from equation (**109**)

$$\lambda = c \sec \alpha$$

we may write

$$\int_{S_2}\int - \frac{d\,V}{d\,n_2}\,d\,S_2 = A\;c\;(\sqrt{\sec^2 \alpha_0 - 1} - \sqrt{\sec^2 \alpha_1 - 1}) \qquad \textbf{(144)}$$

Also, $\sec^2 \alpha - 1 = \tan^2 \alpha$

Substituting we get

$$\int_{S_2}\int - \frac{d\,V}{d\,n_2}\,d\,S_2 = A\,(c \tan \alpha_0 - c \tan \alpha_1) \qquad \textbf{(145)}$$

But it will be observed from Fig. 9 that $c \tan \alpha$ is the minor axis of an oblate spheroid, and, therefore, for equal tubes of flow the surfaces will have any constant spacing on the minor axis of the spheroid. (See Figs. 12 and 13.)

In the above work we have obtained the solution of the electrostatic problem in which any two confocal hyperboloids of revolution of one sheet and of the same family are maintained at definite potentials. The solution of this problem should be of interest to engineers as it furnishes us with a solution which may be applied as an approximation to many engineering problems. For example, it is a useful guide in studying the high-voltage bushing problem; it also gives us an interesting variety of possible electrode shapes for use in testing insulating materials, where it is very desirable to be able to calculate the gradients, etc., etc.

Another problem which should also be of considerable interest to engineers is the case when any two confocal hyperboloids of revolution of two sheets and of the same family are maintained at definite potentials. The solution of this problem may be applied as an approximation to another group of problems which are of frequent occurrence in engineering. For example, it may be applied to switch electrodes at various spacings; also as an approximation in vacuum-tube designs, such as X-ray tubes, kenotrons, etc. That is, it is an approximation to the group of problems of two elongated electrodes at various spacings, or such as an electrode and a plane (*i.e.*, two needle points or a needle and a plane).

The solution for this case is obtained in exactly the same manner as that described above except that in this case we select as our standard ellipsoid the prolate spheroid and express Laplace's equation in normal prolate spheroidal co-ordinates.

Two other problems namely, the distribution of the electrostatic field about the charged oblate or prolate spheroids are obtained in exactly the same manner.

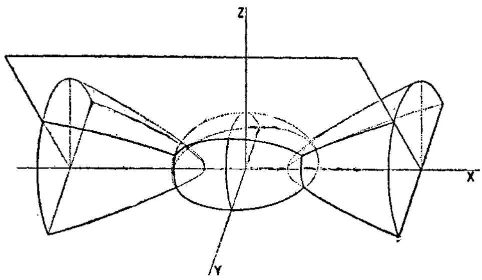

FIG. 15—INTERSECTION OF PROLATE SPHEROID, CONFOCAL HYPERBOLOID OF TWO SHEETS AND A PLANE THROUGH THE AXIS OF ROTATION

NORMAL PROLATE SPHEROIDAL CO-ORDINATES[39]

In this case we take for our standard ellipsoid the prolate spheroid, the axis of rotation is then the major axis and our plane must contain that axis. Taking the x axis as axis of rotation and the XY plane as plane of reference we have for the equation of our three orthogonal surfaces the following. (See Fig. 15).

$$\left.\begin{aligned}
\frac{x^2}{\lambda^2} + \frac{y^2}{\lambda^2 - b^2} + \frac{z^2}{\lambda^2 - b^2} &= 0 \\
\frac{x^2}{\mu^2} + \frac{y^2}{\mu^2 - b^2} + \frac{z^2}{\mu^2 - b^2} &= 0 \\
z - \nu\, y &= 0
\end{aligned}\right\} \quad (146)$$

39. See Byerly Fourier's Series and Spherical Harmonics, page 243, from which this discussion has been taken.

where $\lambda^2 > b^2 > \mu^2$

Here, b, is the semi-major axis of the prolate spheroid.

Solving the three equations simultaneously for the values of x, y, z as functions of λ, μ and ν as we did in the oblate spheroidal problem, and then differentiating these partially with respect to λ, μ and ν we will have functional equations as before, namely;

$$\frac{\partial x}{\partial \lambda} = f_1(\lambda, \mu, \nu)$$

$$\frac{\partial y}{\partial \lambda} = f_2(\lambda, \mu, \nu)$$

$$\frac{\partial z}{\partial \lambda} = f_3(\lambda, \mu, \nu)$$

and similar relations for x, y, z with respect to μ and ν. We also have as before

$$\left.\begin{aligned} \frac{1}{h_1^2} &= \left(\frac{\partial x}{\partial \lambda}\right)^2 + \left(\frac{\partial y}{\partial \lambda}\right)^2 + \left(\frac{\partial z}{\partial \lambda}\right)^2 \\ \frac{1}{h_2^2} &= \left(\frac{\partial x}{\partial \mu}\right)^2 + \left(\frac{\partial y}{\partial \mu}\right)^2 + \left(\frac{\partial z}{\partial \mu}\right)^2 \\ \frac{1}{h_3^2} &= \left(\frac{\partial x}{\partial \nu}\right)^2 + \left(\frac{\partial y}{\partial \nu}\right)^2 + \left(\frac{\partial z}{\partial \nu}\right)^2 \end{aligned}\right\}$$

and obtain the following values

$$\left.\begin{aligned} h_1^2 &= \frac{\lambda^2 - b^2}{\lambda^2 - \mu^2} \\ h_2^2 &= \frac{b^2 - \mu^2}{\lambda^2 - \mu^2} \\ h_3^2 &= \frac{b^2 (1 + \nu)^2}{(\lambda^2 - b^2)(b^2 - \mu^2)} \end{aligned}\right\} \qquad \textbf{(147)}$$

From this we have

$$\left.\begin{aligned} d\,n_1 &= \frac{d\,\lambda}{h_1} = \sqrt{\frac{\lambda^2-\mu^2}{\lambda^2-b^2}}\,d\,\lambda \\ d\,n_2 &= \frac{d\,\mu}{h_2} = \sqrt{\frac{\lambda^2-\mu^2}{b^2-\mu^2}}\,d\,\mu \\ d\,n_3 &= \frac{d\,\nu}{h_3} = \sqrt{\frac{(\lambda^2-b^2)\,(b^2-\mu^2)}{b^2\,(1+\nu^2)^2}}\,d\,\nu \end{aligned}\right\} \qquad (148)$$

and Laplace's equation in terms of our spheroidal co-ordinates λ, μ and ν becomes.

$$\frac{1}{b^2\,(1+\nu^2)}\frac{\partial}{\partial\,\lambda}\left[(\lambda^2-b^2)\,\frac{\partial\,V}{\partial\,\lambda}\right] + \frac{1}{b^2\,(1+\nu^2)} = \frac{\partial}{\partial\,\mu}$$

$$\left[(b^2-\mu^2)\,\frac{\partial\,V}{\partial\,\mu}\right] + \frac{\lambda^2-\mu^2}{(\lambda^2-b^2)\,(b^2-\mu^2)}\,\frac{\partial}{\partial\,\nu}$$

$$\left[(1+\nu^2)\,\frac{\partial\,V}{\partial\,\nu}\right] = 0 \qquad (149)$$

and reduces in exactly the same manner as the oblate spheroid problem to

$$\frac{1}{\lambda^2-b^2}\frac{\partial^2\,V}{\partial\,\alpha^2} + \frac{1}{b^2-\mu^2}\frac{\partial^2\,V}{\partial\,\beta^2} + \frac{\lambda^2-\mu^2}{(\lambda^2-b^2)\,(b^2-\mu^2)}\frac{\partial^2\,V}{\partial\,\gamma^2} = 0 \qquad (150)$$

Where

$$\left.\begin{aligned} d\,\alpha &= -\,\frac{b}{\lambda^2-b^2}\,d\,\lambda \\ d\,\beta &= \frac{b}{b^2-\mu^2}\,d\,\mu \\ d\,\gamma &= \frac{1}{1+\nu^2}\,d\,\nu \end{aligned}\right\} \qquad (151)$$

$$\left.\begin{aligned} \alpha &= \coth^{-1} \frac{\lambda}{b} \\ \beta &= \tanh^{-1} \frac{\mu}{b} \\ \gamma &= \tan^{-1} \nu \end{aligned}\right\} \qquad (152)$$

or,

$$\left.\begin{aligned} \lambda &= b \coth \alpha \\ \mu &= b \tanh \beta \\ \nu &= \tan \gamma \end{aligned}\right\} \qquad (153)$$

Substituting these values in equation (**150**) we obtain Laplace's equation in the form

$$\sinh^2 \alpha \frac{\partial^2 V}{\partial \alpha^2} + \cosh^2 \beta \frac{\partial^2 V}{\partial \beta^2} + (\sinh^2 \alpha + \cosh^2 \beta) \frac{\partial^2 V}{\partial \gamma^2} = 0 \qquad (154)$$

In using this equation it is to be noted that the point (α, β, γ) is the point of intersection of a prolate spheroid whose semi-axes are

$$\left.\begin{aligned} &b \coth \alpha \text{ major} \\ &b \operatorname{csch} \alpha \text{ minor} \end{aligned}\right\} \qquad (155)$$

a biparted hyperboloid of revolution whose semi-axes are

$$\left.\begin{aligned} &b \tanh \beta \text{ transverse axis} \\ &b \operatorname{sech} \beta \text{ conjugate axis} \end{aligned}\right\}$$

and a plane containing the axis of revolution and making the angle γ with a fixed plane. If the fixed plane of reference is that of XY the rectangular co-ordinates of any point in space (α, β, γ) are

$$\left.\begin{aligned} x &= b \coth \alpha \tanh \beta \\ y &= b \operatorname{csch} \alpha \operatorname{sech} \beta \cos \gamma \\ z &= b \operatorname{csch} \alpha \operatorname{sech} \beta \sin \gamma \end{aligned}\right\} \qquad (156)$$

and α may range from $+\infty$ to 0, β from $-\infty$ to $+\infty$, and γ from 0 to 2π. Negative values of β are to be taken for points lying to the left of a plane through the origin perpendicular to the axis of revolution; in this case, to the left of the YZ plane. (See Fig. 16).

Potential at any Point in Space

If we are given the potentials of any two confocal hyperboloids of two sheets, that is, for any two values of β we can obtain the potential at any point in space between them. For in that case we have seen exactly as in the oblate spheroid problem that V is a function of β only, hence in Laplace's equation

$$\frac{\partial^2 V}{\partial \alpha^2} = 0 \text{ and } \frac{\partial^2 V}{\partial \gamma^2} = 0$$

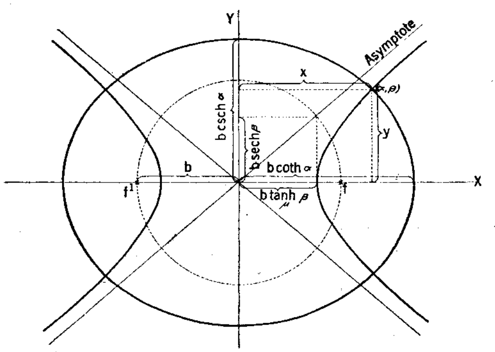

Fig. 16—Confocal Elipse and Hyperbola

Since $\cosh^2 \beta$ is not equal to zero for all values of β Laplace's equation reduces in our case to

$$\frac{\partial^2 V}{\partial \beta^2} = 0 \qquad \textbf{(157)}$$

The solution of which is seen to be

$$V = A\,\beta + C \qquad \textbf{(158)}$$

The constants A and C will be determined from the boundary conditions of the problem as before.
Assuming the case

$$\left.\begin{aligned} V &= 0 \text{ when } \beta = 0 \\ V &= V_1 \text{ when } \beta = \beta_1 \end{aligned}\right\} \qquad \textbf{(159)}$$

Substituting these values in equation (**158**) we get

$$\left.\begin{aligned} C &= 0 \\ A &= \frac{V_1}{\beta_1} \end{aligned}\right\}$$

Thus we may write our solution under these boundary conditions simply as

$$V = A\ \beta \tag{160}$$

or since $\beta = \tanh^{-1} \frac{\mu}{b}$

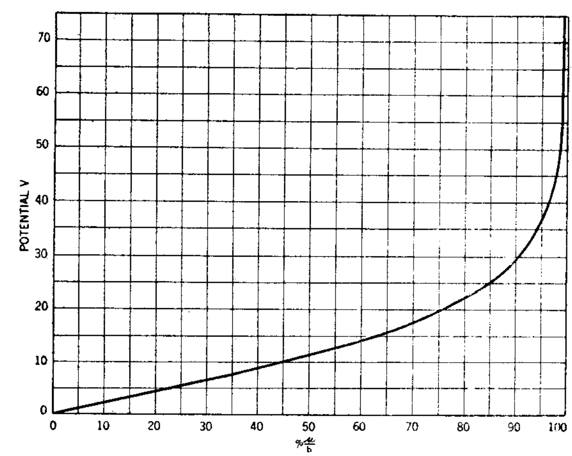

FIG. 17—CALCULATION OF POTENTIAL DISTRIBUTION ALONG THE X-AXIS

we may write it in terms of μ and b as follows.

$$V = \frac{V_1}{\tanh^{-1} \frac{\mu_1}{b}} \tanh^{-1} \frac{\mu}{b} \tag{161}$$

This equation enables us to calculate the potential at any point along the x axis between the foci and has been plotted in Fig. 17.

A diagram of the field has been constructed in the same manner as described above for the oblate problem. (See Fig. 18)

Potential Gradient at any Point in Space

The potential gradient will be

$$\frac{d\,V}{d\;n_2} = A\,\frac{d\,\beta}{d\;n_2} \tag{162}$$

where $d\,n_2$ is the elementary normal to the surface of the hyperboloid (say for $\mu = \mu_0$) under consideration. The sign $\dfrac{d\,V}{d\;n_2}$

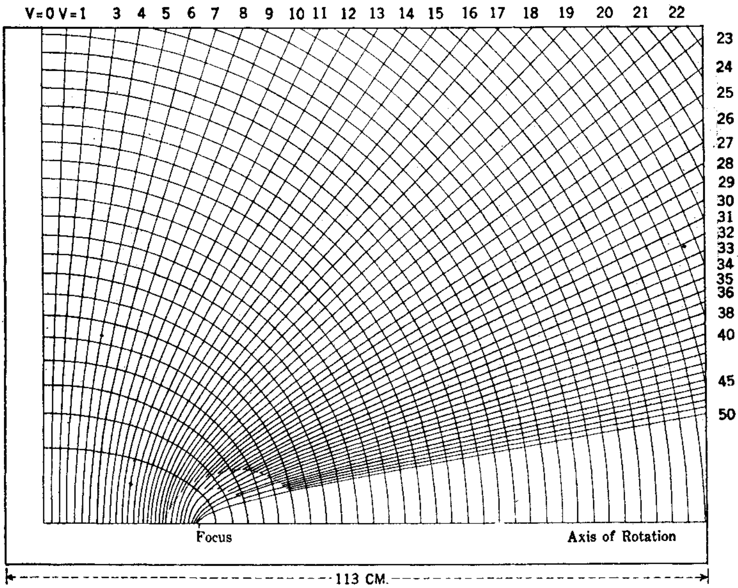

Fig. 18

which expresses the potential gradient is here taken as positive that is, $d\,n_2$ is assumed to be in the direction of increasing potential.

In order to determine β we have the relation

$$\beta = \tanh^{-1}\frac{\mu}{b}$$

Hence,

$$\frac{d\,\beta}{d\,\mu} = \frac{b}{b^2 - \mu^2} \tag{163}$$

We also have

$$d\,n_2 = \frac{d\,\mu}{h_2}$$

$$d\,n_2 = \sqrt{\frac{\lambda^2 - \mu^2}{b^2 - \mu^2}}\,d\,\mu$$

and

$$\frac{d\,\mu}{d\cdot n_2} = \sqrt{\frac{b^2 - \mu^2}{\lambda^2 - \mu^2}} \qquad \textbf{(164)}$$

Thus for the gradient we have from equations (**163**) and (**164**)

$$\frac{d\,V}{d\,n_2} = A\,\frac{d\,\beta}{d\,\mu}\,\frac{d\,\mu}{d\,n_2} = A\,\frac{b}{b^2 - \mu^2}\sqrt{\frac{b^2 - \mu^2}{\lambda^2 - \mu^2}}$$

Simplifying

$$\frac{d\,V}{d\,n_2} = \frac{A\,b}{\sqrt{(b^2 - \mu^2)\,(\lambda^2 - \mu^2)}} \qquad \textbf{(165)}$$

which is the potential gradient at any point in space. It is observed that the potential gradient is independent of γ as is obvious from the symmetry of the problem. We may, therefore, express the potential gradient at any point in space in rectangular co-ordinates with the help of the following relations (See Fig. 16).

$$\left.\begin{aligned} \lambda &= b \coth \alpha \\ \mu &= b \tanh \beta \end{aligned}\right\}$$

and

$$\left.\begin{aligned} x &= b \coth \alpha \tanh \beta \\ y &= b \operatorname{csch} \alpha \operatorname{sech} \beta \end{aligned}\right\}$$

Curves of Constant Potential Gradient

The curves of constant potential gradient may now be obtained by putting equation equal to the various constant values desired.

Thus,

$$\frac{d\,V}{d\,n_2} = G = \frac{A\,b}{\sqrt{(b^2 - \mu^2)\,(\lambda^2 - \mu^2)}}$$

$$\sqrt{(b^2 - \mu^2)\,(\lambda^2 - \mu^2)} = \frac{A\,b}{G} \qquad \textbf{(166)}$$

Thus for any given condition and value of potential gradient G the right hand member of this equation is a constant.

If we now substitute the values of λ and μ we may write it for the purpose of calculation.

$$\coth \alpha = \pm \sqrt{\tanh^2 \beta + \frac{\left(\frac{A}{G\,b}\right)^2}{\operatorname{sech}^2 \beta}}$$

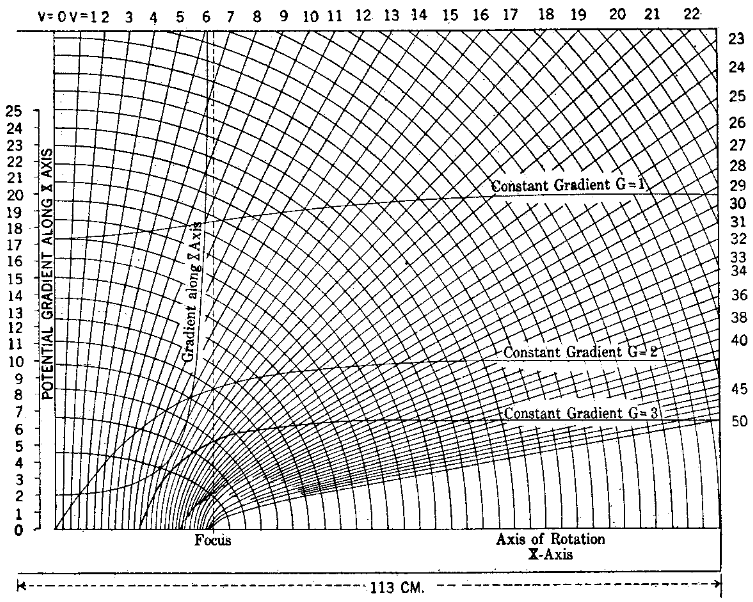

FIG. 19

or if tables of sech² β are not available we can write it

$$\coth \alpha = \pm \sqrt{\tanh^2 \beta + \frac{\left(\frac{A}{G\,b}\right)^2}{1-\tanh^2 \beta}} \qquad \textbf{(167)}$$

The values of x and y are

$$\left.\begin{aligned} x &= b \coth \alpha \tanh \beta \\ y &= b \operatorname{csch} \alpha \operatorname{sech} \beta \end{aligned}\right\}$$

The curves of constant potential gradient as calculated from these equations are shown in Fig. 19.

Potential Gradient Along the X Axis

The value of the potential gradient at any point along the x axis between the origin and the foci may be found directly from the general equation (**165**) or independently as $\frac{d\,V}{d\,\mu}$.

It is the value obtained when $y = 0$

$$y = 0 = b \text{ coch } \alpha \text{ sech } \alpha$$

and since b sech β is not zero we have

$$\text{csch } \alpha = 0$$

$$\frac{1}{\sinh \alpha} = 0$$

$$\alpha = \infty$$

Substituting this value in the equation

$$\lambda = b \coth \alpha$$

we have $\quad \coth \infty = 1$

and $\quad \lambda = b$

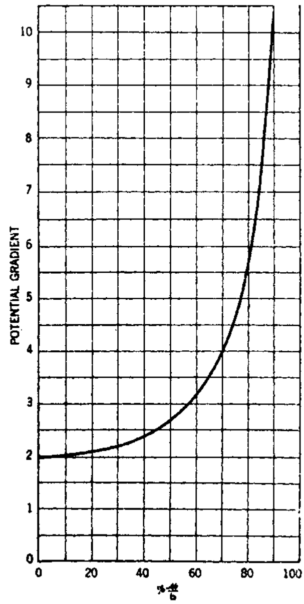

Fig. 20—Calculated Potential Gradient along the X-Axis

Hence substituting $\lambda = b$ in our equation (**165**) for the potential gradient we get

$$\frac{d\,V}{d\,n_2} = \frac{A\,b}{\sqrt{(b^2 - \mu^2)\,(b^2 - \mu^2)}} = \frac{A\,b}{b^2 - \mu^2} \tag{168}$$

for the value of the potential gradient along the x axis between the foci. The curve is shown in Fig. 20.

Solution for the Tubes of Flow of Flux

The total flux across any area may be defined as the surface integral of the normal component of the flux density; in the case of air dielectric, flux density and gradient are equivalent. We may, therefore, write it

$$\int\!\!\int \frac{d\,V}{d\,n_2}\,d\,n_1\,d\,n_3 = \int_{S_1}\!\!\int \frac{d\,V}{d\,n_2}\,d\,S_2$$

we have seen that

$$\frac{d\,V}{d\,n_2} = \frac{A\,b}{\sqrt{(b^2 - \mu^2)\,(\lambda^2 - \mu^2)}}$$

$$d\,n_1 = \sqrt{\frac{\lambda^2 - \mu^2}{\lambda^2 - b^2}}\,d\,\lambda$$

$$d\,n_3 = \sqrt{\frac{(\lambda^2 - b^2)\,(b^2 - \mu^2)}{b^2\,(1 + \nu^2)^2}}\,d\,\nu$$

Substituting these values and simplifying we get

$$\int\!\!\int \frac{A}{1 + \nu^2}\,d\,\lambda\,d\,\nu$$

As far as the spacing of the surfaces of flow is concerned we may neglect the integration with respect to ν by reason of the symmetry of the tubes of flow about the axis of rotation.
We thus have,

$$\int\!\!\int_{S_2} \frac{d\,V}{d\,n_2}\,d\,S_2 = A\,(\lambda_0 - \lambda_1)$$

But we have seen that

$$\lambda = b \coth \alpha$$

also that $b \coth \alpha$ is the major semi-axis of the prolate spheroids. Therefore, equal tubes of flux will be obtained by taking equal spaces along the major axis of the spheroids. (See Figs. 18 and 19).

Discussion on "An Experimental Method of Obtaining the Solution of Electrostatic Problems With Notes on High-Voltage Bushing Design," (Rice), New York, N. Y., November 9, 1917.

Ralph Mershon: There is one matter on which I shall be glad if Mr. Rice can give us some light. That is, as to what will happen in case the bushing has an arc-over backed up by considerable power; and as to what work, if any, has been done in devising means of protecting bushings from the effect of power arcs. This is a subject which is not perhaps immediately germane to Mr. Rice's paper, but which certainly is germane to the matter of bushings and insulators. It is a subject which has interested many of us for a long time.

Some time ago I conceived the idea that if we could build an insulator in the form of a framed structure the members of which were made up of units of massive porcelain we would have an insulator which it would be impossible to puncture and that if the members were so arranged that in case of a flash-over the resulting power arc could rise freely from them, we would have an insulator which would also be free from damage by power arcs.

After a great deal of difficulty I succeeded in having made out of porcelain the units for constructing the members of the structural insulator. These units were made up of solid porcelain, with comparatively small petticoats; they each had a net length of about nine inches. They were connected together end to end to form the structural member by cementing the ends into metal collars. The shape of the completed member was a good deal that of the outside of the bushing shown in Fig. 3 of Mr. Rice's paper. The manufacture of the units involved the production of massive porcelain absolutely free from flaws and having a high tensile strength. The units as constructed could be flashed over indefinitely under oil without puncture. They had a strength in tension of from 13,000 to 16,000 pounds, which meant, as I remember it, a unit stress of about 2000 pounds per square inch.

These insulators tested out very nicely with a testing transformer. It was absolutely impossible to puncture them, and the arc following a flash-over immediately rose from the surface of the insulator, as contemplated. But when the insulators were subject to a flash-over backed up by large power capacity the story was an entirely different one.

Through the kindness of the Pennsylvania Water & Power Company a portion of their generating plant was put at my disposal for power arc tests. A test was made on several of the porcelain units above described, assembled as they would be in one member of a structural insulator. This structural member was placed in a horizontal position and arcs started across its surface by means of fine wire fuses. In every case the arc instead of rising from the porcelain units as it had done in the case of testing transformers, and as it seemed to me it ought to, would not rise at all. It hugged in close against the petticoats and peeled them off, one by one. If the arc was allowed to continue long enough, it took pieces from the porcelain core, after having denuded it of the petticoats. The endeavor was made to protect the units from the power arc by arcing-rings, or similar devices, but without success. The power arc in every case stubbornly persisted in clinging close to the surface of the porcelain units. I have never been able to satisfactorily account for the peculiar behavior of the power arcs in this experiment, or to explain why the behavior should be so different in the case of heavy power arcs and the light arcs obtained from a source of limited capacity.

In the case of the bushing shown in this paper, which will presumably be used on transformers, switches, etc., to go out of doors, and which may arc over at times, no matter how carefully they may be designed or how great their factor of safety, I wonder what, if any, provision is contemplated to protect them from the effects of an arc backed up by a lot of power. If it is intended that they shall not arc over at all, we might recall some of the experiences we have had in the past with apparatus which was not going to do certain things.

C. O. Mailloux: The method described by the author, aside from its practical value as a means of graphical representation of the distribution of points of equal stress and of the course of the lines of action of force and energy between points having different degrees of stress, is of the highest theoretical interest.

The author has made a very important contribution to our working tools in physics, and to our methods of dealing with fields of force and their distribution in space. We find, in this paper, what is perhaps the first instance where engineers have been called upon to make personal acquaintance with the well known and time-honored "Equation of Laplace." I do not remember ever having seen, before, a paper not on a distinctly and purely physical subject in which that equation made its appearance. Anybody who has studied the potential function and who has made a passing acquaintance with Laplace's equation has acquired a great respect for it. He has learned that it is, indeed, a most wonderful means of analysis and a most potent tool for the physicist. It is agreeable to find that it will no longer be monopolized by the physicist, but that the electrical engineer is now going to make use of it. If, fifteen or twenty years ago, anyone had said that the electrical engineer would some day be able to handle the potential function and to use Laplace's equation as beautifully and as effectively as it is done in this paper, he would not have been believed.

It is especially in the study of fields of force in tri-dimensional space,—the most interesting, but also the most difficult field of physical research—that the equation of Laplace has proved a most wonderful instrument to the mathematical physicist. It has done this by virtue of certain remarkable properties which enable it to render valuable aid in the specification and the determination of the physical conditions obtaining in fields of force at the stage of action of the forces involved, at which a state of balance or equilibrium, either transitional or permanent, occurs. Its range of adaptability to fields of force problems is very great, being, indeed limited only by the mathematical knowledge and skill employed in its use. It can, theoretically, be applied to fields of force in which the lines of force, or the lines of flow of energy, follow paths of the most diversified character from the simplest, like straight parallel lines, to the most complex, like some of the cases discussed in the paper.

What interested me very much is the very ingenious development of methods by which the author, starting from the consideration of Laplace's equation in ordinary rectangular coordinates, develops and applies it to a very complex system of coordinates, designated as "Normal prolate spheroidal coordinates." That may be an awe-inspiring name to many of you, but it is a name well worth becoming familiar with. Even though it may take some time and study to learn all of the mathematics which precede it, it is well worth while. Laplace's equation has, as I just said, some remarkable properties; and one of its most remarkable properties, is that it enables physicists to reason about the phenomena incidental to the action of physical forces, and to reach absolutely logical, rigorous, scientific and true conclusions without the need of any postulates or any assumptions as to the intervening media, in other words, as to the nature and properties of the ether or of any medium in space through which force acts, and in which fields of force are produced by physical forces. It was, in my opinion, one of the great achievements of Laplace, that he was able to devise an instrument of thought, an instrument of mathematical analysis, the use of which was independent of any such complicated, worrisome and perplexing postulates. With Laplace's equation, used in each case in manner suitable for the purpose in view, it is possible to deal with the distribution of force under a great variety of circumstances, to determine exactly what is going to happen, and to obtain very interesting, useful and beautiful results.

The interesting feature of the mathematical part of this paper is that by the modification of the system of co-ordinates which the author has worked out, he is able to apply and utilize Laplace's equation in the study of the distribution, flow, and apportionment of forces under conditions where the lines of flow are no longer as you would find them in space which is free from all constraints or boundary conditions, but such as they are in certain special conditions, where the field is distorted by constraints and barriers, causing various reactions, as described in the paper and as shown in the diagrams.

The author's initial statement in the Appendix is almost contradicted by the fact that he himself has seemingly succeeded in finding solutions of Laplace's equation, which do fulfill at least some, if not all, of the arbitrary conditions involved in the problems. Further success and a nearer approach to the goal will result, presumably, from further modifications of co-ordinate systems still better adapted than those thus far developed and used for expressing the points of freedom, and the constraints necessary in the given case.

In conclusion, I consider this one of the most interesting mathematical papers that has been presented before the American Institute of Electrical Engineers in a long time. It is interesting, not only on account of the useful application made to an immediate practical purpose, but also because, it shows a new development and an extension of a method of using the equation of Laplace, and at the same time contains ideas which may be followed by others, and may lead to still further extensions of the method, so that, in presenting a new method of study of phenomena occurring in fields of force, the author has incidentally made a very interesting contribution to applied mathematics.

John B. Taylor: I am rather curious to know how you get such beautiful curves in a method where the devices introduced for observing the quantity you want to know, change the conditions under which you are working; that is, how is it possible to get this exploring point all through the medium without making apparent a change in conditions. It is trite to say when you put a voltmeter in the circuit you shall not draw so much current that the voltage is made different, or that an ammeter shall not have so much resistance that the current is changed. In acoustics we have difficulty in determining the form of vibration in the air, because the diaphragm usually changes the vibration at that particular point, and doubtless similar difficulties exist here, and I will be glad, if it is not covered in the paper, to hear a word in reply as to how that has been regarded.

A. M. Gray: I have had considerable success with the following photographic method of obtaining the dielectric flux lines in two-dimensional problems. To obtain the stress distribution, in slot insulation, for example, a model of part of the slot and conductors was made of metal as shown in Fig. 1, and placed on a photographic plate, the two pieces were connected to the high-voltage terminals of a transformer, the low-tension side of which was excited by direct current, and an impulse of high voltage was applied across the insulation by opening the direct-current circuit. No flash was visible but on developing the plate the flux lines showed up beautifully.

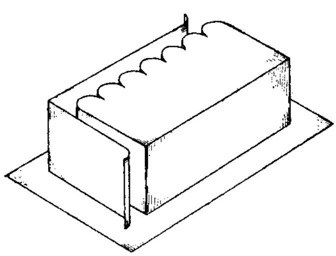

FIG. 1

To test whether or not this picture is a true representation of the stress distribution, two or three simpler cases were tried which could be solved mathematically, and the agreement was such as to justify the use of this experimental method for more complicated cases.

The application of the method to the three-dimensional problem of a rod passing through a plate would be more difficult. A small sector parallel to the lines of flux would have to be placed in the field and made of material which had the same permittivity as the film, the stress distribution due to a transient voltage, as well as that due to the contant potential from a static machine, could then be determined.

Charles L. Fortescue: In his text the author has several times referred to the paper by Mr. Farnsworth and myself on "Air as an Insulator in the Presence of Insulating Bodies of Higher Specific Inductive Capacity." The object of that paper was to show that dielectrics even though far from ideal do not substantially weaken the air path in contact with them, provided these dielectrics are introduced into the field in such a manner that no distortion is produced. Several errors crept into the paper, chiefly due to the limited time available for carrying out the work. Among those errors is that one pointed out by Mr. Rice in the present paper, namely, the use of circles for cross-sections of equipotential surfaces for the rod and ring problem. It was the original intention to work out the proper solution by the trial and error method, which Mr. Rice has described in his paper and the system of circles was taken as a first approximation, but the work proved so laborious that it was finally given up on account of lack of time.

While it would be extremely interesting from a theoretical point of view to obtain a solution of Laplace's equation which could be applied to any boundary conditions, it is more important from an engineering point of view to obtain a simple and accurate experimental method for determining the equipotential surfaces and flow lines for any system of insulated bodies. Mr. Rice's work is a valuable contribution in this direction.

The principles outlined in our original paper have been a great aid to me in determining the best form for insulating structures, but the lack of facilities for plotting the field form experimentally have been a great drawback. I hope to be able to set up a permanent outfit for carrying out such work in the future.

Referring to Mr. Rice's criterion for maximum efficiency in an insulator combining air and other materials, I wish to remark that it is the same as that given by us in our paper, although Mr. Weed among others, seems to find in the paper a general statement to the effect that any surface following a line of force was the strongest surface. What was actually intended, and I think brought out clearly in that paper, was the principle that the surface of the insulating body should conform to some system of flow lines and the electrodes should be so designed as to make the intensities along the surface as nearly uniform as possible. It appears probable that when substantial uniformity of stress is not present, it may be obtained by slightly deforming the insulation from the shape of the flow line.

Referring to Mr. Rice's calculation for the arcing over of the rubber test piece described in our paper, I regret to have to inform him that his conclusions do not agree with results of tests and since perhaps the original statement in our paper for the breakdown of this test piece led him astray, we owe him some explanation. The original test was made on a very jumpy circuit, as Mr. Chubb and I found to our great discomfort when calibrating the sphere gaps. The value quoted was the lowest value obtained, whereas, as I have pointed out in connection with the sphere-gap calibrations, on such a circuit, with apparatus on tests of such a character, the high values are the correct ones to use, as the low ratios are due to surges which the measuring devices do not record.

We found on this test piece that it did not make any appreciable difference in breakdown voltage if it were dirty or not. We let it lie on the floor for three months to accumulate a heavy coating of dust, and it tested just as high as when clean.

Later on, the electrodes were carefully re-cemented in such a way as to do away with the possibility of air pockets, etc., and when tested again under better conditions the breakdown value by ratio lay between 180 kv. and 190 kv. With proper corrections for crest value, this would represent an effective value of somewhere between 197,000 and 218,000 volts. Tests on this test piece were witnessed by a number of people, among them Prof. H. J. Ryan and Mr. Lieb.

I wish to take this opportunity to correct some wrong conceptions that have formed in connection with the condenser terminal. In some cases wrong descriptions have appeared in books on electric theory. I consider it rather regrettable that authors do not take the trouble to inform themselves as to the correctness of the information before presenting it. The condenser terminal is made with equal increments of length of the metallic cylinders for equal increments in their potential. Thus the adjacent pairs of cylinders have all the same capacity and decrease in length by equal steps as their diameter increases.

In my paper on "The Application of a Theorem of Electrostatics to Insulation Problems," I called attention to a theorem of electrostatics which is of much broader scope than that outlined by the first paper.* (Mr. Fortescue then read an extract from his paper referred to, followed by an extract from Maxwell's work.†

Engineers would do well to give this paragraph careful consideration, as it presents great possibilities in the design of

* A. I. E. E. Trans. 1913, Vol. XXXII, Part I, p. 907.

† 1904 Edition, Chapter VII, Art. 117.

insulating structures. Among the most noteworthy of the applications of this principle are the suspension type insulator and the condenser terminal. The principle of design of the latter may be presented as follows: Consider a cylinder passing through a zero potential plate whose thickness can be varied at will, and let us suppose the two ends of the cylinder to have conductors of a given form, for simplicity let us take infinite planes parallel to the plate, at each end. We shall first consider the region between the plates as having zero specific inductive capacity and let us consider an imaginary field of uniform strength mapped out in this region, conforming to that which would exist if the cylinder were not present. In order to present a surface around this cylinder having at all points the same potential as the assumed field, the dielectric must have the following characteristics:

(1) Radially it must be a perfect dielectric;

(2) Axially, it must be a perfect conductor.

With a dielectric of these characteristics, the bounding surface may be so formed that the equipotential surfaces of the same value in the region between the rod and bounding surfaces will coincide with those in the region between the two plates, external to the dielectric. If the external region has a finite specific inductive capacity there will be a displacement current necessary in an axial direction from the external region into the dielectric body. This will require a slight modification of the surface of the dielectric to take care of this additional displacement, which must eventually pass radially through the outer cylinders and plate. It is, of course, impracticable to obtain such a dielectric, although there are many ways in which it may be approximated, one of them being the use of a system of coaxial conductors cylindrically imbedded in the insulation. The external surfaces should, of course, be infinite, but in most cases they extend far enough to give satisfactory results. The dielectric surrounding the rod is not uniformly stressed, nor is it desirable that it should be; on the contrary the inner portion should be stressed more lightly than the outer, as the heat set up by the dielectric losses must be conducted out of the dielectric.

If the writer of the paper had made use of this theorem, he would naturally have been led to design his dielectric and system of conducting bodies so as to give the most favorable distribution in the external region, and he would have reached a different conclusion.

Some tests were made a few years ago on glass rods between plates by students at Worcester Polytechnic Institute, and, as I recall them, these tests were quite satisfactory in establishing the non-interference of the surface of high specific inductive capacity material on the breakdown value between plates.

I have recently obtained excellent results with porcelain, though not so good as with moulded or machined materials. The less satisfactory results with this material seem to be due to the irregular form of the contact surfaces between the conductors and the dielectric.

E. DeWitt Eby: With reference to the matter in the general statement for an ideal bushing, I wish to comment, perhaps take exception to Mr. Rice's statement, that in an ideal bushing all of the dielectric is stressed uniformly with respect to its strength, that is, that it will all begin to break down at the same time. This would mean there would be little energy consumption up to the time of rupture or breakdown. It would mean, secondarily, that under impulse conditions the arc-over would be very little different from that under low-frequency stresses. In actual operation or in practise, the low-frequency stresses to which the bushing is subjected are as a rule definitely related to the operating potential, whereas the impulse or lightning stresses are not necessarily related to the line potential, outside of their limitation by line insulation. Therefore, it appears to me that a better relation of the stresses would prevail, if instead of all the insulation breaking down uniformly or at the same time, a part of it should begin to break down, preferably so that it would not destroy or damage the bushing. For instance, the surrounding air should break down earlier than the rest of the structure. In that way energy consumption would take place which would precede the total breakdown, and increase the ratio between the impulse and the low-frequency arc-over.

I wish to emphasize, or add my approval, to the conditions for the design of bushing which Mr. Rice sets forth, that it should arc-over without puncture, at both low frequencies and impulses, and that it should do this without damage to the insulation; also that it should have a high wet arc-over with respect to its dry arc-over value, and, repeating what I said, that its impulse arc-over should be high with respect to its low-frequency arc-over.

I think it was Mr. Mershon who made inquiry as to whether any attempt had been made to control the performance of the bushing under power conditions, that is, with power behind the test circuit or the arc-over, and while not attempting to answer his question, I will simply remark that I know of a number of instances in which successful operation or performance of bushing has prevailed during arc-over with power behind the arc, without doing any damage to the bushing. I also know of one isolated case where the bushing was damaged by the power arc.

I wish to call attention briefly to the effect of the specific resistance of the testing water upon the wet arc-over voltage of a bushing or insulator, a thing which apparently has been overlooked a great deal in making rain or wet tests, and recording the data. The specific resistance of the water has a very great effect upon the wet arc-over voltage, so that, for instance, with such water as Mr. Rice probably used in his tests, the arc-over of the bushing was probably 75 per cent of the voltage it would have been had water been used which was of a higher resistance, such as is available in most natural water systems, that is mountain or reservoir water supplies. I assume that the tests which Mr. Rice has made, were made with water having the resistance of about 4000 ohms per cu. cm., whereas a resistance of 15,000 ohms per cu. cm. would have increased the results about one-third.

We are all aware of the effect of altitude upon the dry and wet arc-over voltage of bushings, both at low frequency and under impulse conditions, and these things have to be taken into account in the application of bushings in actual operation.

It is interesting also to know that there is a temperature effect, and that this effect of temperature is of the order that one degree centigrade difference in temperature is equivalent to about 100 ft. (30.4 m.) difference in altitude.

There is one other point I wish to bring to your attention, and that is, that there should be a relation between the arc-over voltage of the bushing and the voltage of the system to which it is applied. This can best be related to the protection of the system, that is, to the lightning arrester gap settings, since if the system is properly designed the bushing will have a high impulse ratio. It is necessary to consider principally the low-frequency arc over of the bushing, and this should be—theoretically anyway—at least twice the low-frequency arc-over of the lightning arrester gap which protects the system. This is because, as we are all aware, a reflected wave which is lower than the lightning arrester will discharge, will return with double its initial value, and the bushing should not arc-over at that voltage.

It is interesting to note that the factor of safety which the Institute has set down as a testing value for assembled apparatus such as lightning arresters, switches, etc., seems to be prompted by experience, so far as bushings are concerned; that is, that a test value of two and one-quarter times the operating line voltage gives satisfactory results in services, and very few arc-overs take place at that voltage.

F. W. Peek, Jr.: I will give a physical conception or picture, of the meaning of the dielectric flux diagrams in practise.

A bushing or insulator is made up of metal parts or electrodes, oil, necessary supporting solid dielectric, and the air in which it is immersed. When voltage is applied between electrodes, stresses are caused in the solid insulation and the air. If these stresses anywhere exceed the strength of the insulation, breakdown will occur. The strength of air is much less than porcelain. A bushing requiring the minimum amount of material would be one in which the stresses in the different insulations were uniform and in proportion to the respective strengths of the dielectrics. Such a bushing would break down everywhere at once. However, since the air part of the bushing automatically replaces itself when it punctures or flashes over, it is best to make the designs so that flash-over takes place before puncture of the porcelain. As the air is the weakest dielectric, it plays a great part in determining size. Mr. Rice's diagrams are maps of the stress everywhere around the electrodes; the importance of these diagrams in design is at once apparent.

The strength of air is 54 kv. per in. If the full strength of the air could be utilized, a bushing with a 500,000-volt arc-over need not be over 9.5 in. long. There are a number of good reasons why such bushings should, in practise, be perhaps six times that length:

(1) It is not practicable, and for certain reasons not desirable, to design for an abso utely uniform field.

(2) Even if the bushing is designed for a uniform field, the field is not uniform because of apparent conduction along the surface of the solid insulation, which greatly lowers the arc-over voltage. Mr. Rice's views on this are confirmed by experiments that I have made.* The effect takes place on clean porcelain rubber, or glass, and is not appreciably reduced if the surfaces are polished. I believe this is the reason why Mr. Fortescue discovered very little difference in surface effect over the range of surface condition that he investigated in his very interesting and important paper of several years ago. The effect is greatly reduced if the surfaces are oiled.

(3) The third reason that a bushing must be made long is that it must be used out of doors in rain, snow, and dirt. Petticoats must be used. Under such conditions a uniform distribution is not for a moment possible. The designs must include all of these practicable variables.

(4) When flash-over occurs, it is generally caused by lightning. The bushing or insulator should be designed for a "lightning arc-over voltage" much higher than the 60-cycle arc-over voltage, just as the lightning arrester gap should be designed for a low "lightning arc-over voltage."† I have seen apparatus designed in just the opposite way.

You have seen in Mr. Rice's paper many beautiful diagrams determined experimentally; mathematically, it is very difficult to draw any but the simplest diagrams. The meaning of these plots is, however, quite simple. They map out the stresses in the space surrounding the electrodes.

Look at Fig. 9. There is a metal plane, and a rod perpendicular to the plane, passing through a hole in the center of it. You will see certain lines starting at right angles from the plane, and ending perpendicularly, on the central rod. These are the lines of force. They are so drawn that the flux between any two lines is the same. In other words, it means that the capacity between any two lines is equal. You will see at right angles to these lines another set of lines or curves; these represent the equipotential surfaces. Each point of a given curve is at the same potential. They are so drawn that the potential differences

* "Dielectric Phenomena in High-Voltage Engineering", pp. 190 and 220-225.

† "The Effect of Transient Voltages on Dielectrics." F. W. Peek, Jr, A. I. E. E., TRANS., 1915, Vol. XXXIV, Part II, p. 1857.

"Lightning", F. W. Peek, Jr., *G. E. Review*, June, 1916.

between any two adjacent equipotential surfaces is equal. For instance, No. 1 starts at a plane at zero potential, and the next line has a potential of 2, the next 3, and so on up to 51, which is the potential of the rod. The potential difference between each adjacent surface is one. The capacities of all of the little cells, cut out by the lines of force and equipotential surfaces, are equal. If the diagram is revolved about the rod as axis, these cells cut out condensers of equal capacity in space. You will note that the equipotential surfaces are crowded together in places. The unit stress or gradient in volts per inch or per cm. at these points is high, since the voltage differences between two adjacent surfaces is the same as it is where the lines are much farther apart.

It can be seen that these fields may be easily changed, not only by changing the dielectric, but also by changing the metal parts. Obviously, insulation strength may often be increased by adding metal, and some times decreased by adding insulation.

Mr. Rice has been able to draw these wonderful diagrams by the use of the analogy between the electric and dielectric circuits. In the electric circuit there is voltage, and in the dielectric circuit voltage, in the dielectric circuit flux, in the electric circuit current, in the dielectric circuit permittivity and in the electric circuit conductivity, etc.

There is one point which Mr. Rice has made to which I wish to call attention. Mr. Rice states: "It seems to me a great pity that the mathematicians do not more frequently reduce their results to a readily utilizable form, and whenever possible sketch out, with examples, some of the applications which must occur to them while working on the subject." He also says: "Another difficulty, which I have frequently encountered, is the fact that the writer assume too great a familiarity with existing mathematical works on the part of his readers." I wish heartily to endorse these statements. Any one who has ever tried to use some of the mathematical solutions, and has had to give up in despair, because of the great time it required to puzzle out certain steps and the meaning of certain symbols, even in cases where the work is really quite simple, will appreciate this. It generally requires very little extra work to define terms and symbols. I think Mr. Rice has made a good example here of how mathematical work should be presented.

M. E. Tressler: Some very interesting and enlightening individual tests made to determine the cause for certain results, such as "the apparent decrease in disruptive voltage kilovolts per cm. with the increasing thickness of insulation"; "lowered arc-over voltage between terminals along the junction point of two dielectrics of different permittivity, in parallel"; etc.

Mr. Rice draws the conclusion that if the dielectric in parallel with the air is absolutely free from moisture and if perfect joints are made between the dielectric and the terminal, that the presence of insulation in parallel with air or oil in a uniform field would not result in a breakdown lower than that of the weakest material; this conclusion may be correct, but part of the tests would seem to prove that the dielectric strength of air was different when in contact with the test piece of solid dielectric and depending on the properties of this dielectric, as was suggested in the paper. For instance, the difference in arc-over voltage between an oiled hard-rubber surface and an oiled glass surface, Fig. 21, at 8 cm. spacing is 47 kv. In this particular case where a good comparison can be made we should be justified in assuming that the joint between the oiled glass and metal planes was equally as good as the joint between the oiled hard rubber and the metal planes, yet the arc-over voltage on the oiled hard rubber at the same arcing distance was about 65 per cent higher than on the oiled glass surface.

In examining the curves, Fig. 23, with grooved planes and comparing with those in Fig. 21, it will be noted that with the grooved planes or electrostatically shielded joints the arc-over voltages are much higher, but if the arc over occurs the full length of the dielectric cylinders to the bottom of the grooves, the arcing distance is greater than the distance between planes by the depth of the grooves, or if it occurs from the bend of the plane into the groove the air must be broken down here first due to non-uniform field and consequently a condition is obtained similar to the so-called joint effect.

One other reason for assuming that the introduction of a solid dielectric into the uniform field in air causes a change in the dielectric strength of air or else that the curves in Fig. 23 disprove the assumption that the joint effect is eliminated by electrostatic shielding of the ends of the solid dielectric, is that, if the arc-over curves on oiled glass and dry glass II, Fig. 23, are extrapolated a very short distance they cross the curve for arc-over between planes alone, indicating that arc-over voltage with the dielectric inserted is greater than the air alone between the planes, which is a result that would not be expected if the lowering of the arc-over voltages below certain spacings is caused by the joint-effect at the ends or moisture on the surface of the solid dielectric.

H. O. Stevens: The point I think we should all get in this paper is the fact that we have brought before us here in fairly simple mathematical manner a few fundamental principles. We have an indefinitely extended wire passing through a hole in an indefinitely extended plane. That is a fundamental theory worked out beautifully. In other words, we have an ideal there to work to. We can take that fundamental principle and add certain things to it, which may improve the design and work up a satisfactory bushing. Unfortunately, many of the so-called practical engineers when they start out to attack a problem do not have the time, or perhaps the proper mathematical and physical knowledge of the subject to study the problem in this manner, so the thing is done by a cut and try method. When we are designing a bushing, we put a certain amount of insulation on it, and bring it out through a transformer tank and the voltage is increased a little bit more, and we add more insulation, until ultimately we get in trouble. Then some one like Mr. Rice takes the problem and studies it, from a mathematical and scientific standpoint, and while oftentimes the results obtained do not correspond with what we would expect, yet as I said before, it gives us an ideal to work to, and that is the point we want to bear in mind in this paper. If we can work out the fundamental mathematical conception of a given problem, although our theory will not be strictly borne out in practise, yet if we apply to that our mathematical knowledge, the solution of the problem is made a great deal easier to us.

Sidney W. Farnsworth: Mr. Rice speaks very frankly of the difficulties which he has met and brings vividly to mind the early days of our work on this problem, in which we encountered similar difficulties—we would work to get the bushing completed, and then have it puncture on the inside and be obliged to tear it down, and we would feel that the work of first construction had been so difficult it would not warrant us in replacing it.

As engineers we are interested in the application we can make of the information which is given by the author, and the last speaker voiced, I think, the sentiments of those here—I would say that as a result of our work we were able to embody the principles in commercial apparatus and to save money for the company, and also to extend the upper limit of voltage range. Mr. Rice has worked at very low voltages, and perhaps it is not appreciated that the principles involved here permit of multiplication or expansion any number of times. For instance, if you just multiply his dimensions by two, you should get just twice the breakdown voltage, multiply by three, three times the breakdown voltage, etc. That is the principle upon which he has worked. Therefore, he has not solved the problem simply for the low voltages which are mentioned here, but for higher voltages as well.

There is one striking example of the consideration of the principles set forth in this paper, and those which we encounter in our work, and that is of the addition on a high-voltage condenser terminal of the hat, so-called. We had a terminal for a very high-voltage transformer which showed distress at something over 300,000 volts. It was equipped originally with a disk of some 18 in. diameter, with a rim of perhaps 0.75 in. (19 mm.) radius and 1.5 in. (38.1 mm.) diameter. The distress occurred on the terminal prior to distress anywhere else, and the terminal itself probably represented an outlay of some $600 or $700, and remembering the principles we encountered in the course of the design of the transformer and the work connected therewith, we had a wooden hat about six ft. (182.8 cm.) in diameter, and one ft. (30.4 cm.) thick, as I recall it, made, the hat being covered with tinfoil, and this was used as a temporary expedient. A thing that cost perhaps $15 or $20 and made almost over night. We put it on top of the terminal, and distress did not show in that terminal up to 571,000 volts; in other words, there had been an increase of about 150,000 volts in the usefulness of that terminal. By making a gain of 150,000 volts in this way we are doing well. It seems to me that is how we should make use of the principles which are so well set forth in this paper.

I would like to ask, why the charts were made with sheet metal and with wires. In our work we made them with rods of appreciable diameter and with sheets of appreciable thickness, with rounded edges. Fig. 9, giving the equipotential surfaces, shows only the location of the surfaces from 1 to about 30, up to about 24 with what an engineer would call a reasonable degree of accuracy of ease of reading. Now, that is only about half of the total number that should be there, that is, there is 50 volts between the rod and plate. I understand that having the plates and rod conform to line 30, and the plate conform to line 2, that the distribution of the field between the two does not change; that is, the principle is the same. But suppose we did do that, plates and rod conforming to line 30, and an irregular rod and irregular shaped piece conforming to line 2, then we could put that back into the path, and each of these do have the voltage distribution between these two surfaces, the surfaces which we intend to use, and in that way greatly increase the area of the disruption.

Selby Haar: I wish to ask Mr. Rice if he made any tests on his experimental insulators with high direct potentials?

B. A. Behrend: The parallel which exists between the electrostatic case, in which we are interested, and which Mr. Rice has treated with such skill, and the hydrodynamic case, is well known. The hydrodynamic case, so far as its consideration in two dimensions is concerned, has been beautifully worked out experimentally by Prof. Hele Shaw in some admirable papers which were published in the Royal Society Transactions. Now, then, in the latest work I have had an opportunity to consult in connection with hydrodynamics, the treatment is limited to two-dimensional problems altogether. I want to ask Mr. Rice, in replying to the many statements that have been made here to-night, to what extent a correct scientific solution has been given for three-dimensional problems. Prof. Horace Lamb states that three-dimensional cases have not been treated successfully, and a similar statement is made by Sir George Greenhill. The treatment of three-dimensional cases is, of course, fundamental in connection with the treatment of the subject laid before us by Mr. Rice. I trust that Mr. Rice, in his reply, will be kind enough to comment on this phase of the subject, and tell us how far it has been possible to obtain complete solutions of the three-dimensional cases.

Joseph B. Morrill (communicated after adjournment): I should like to discuss a few points that occurred to me in going through Mr. Rice's excellent paper.

First, in connection with the introduction of the diametral planes on page 1001 one might wonder where these planes come from and whether or not they are a new family. If, indeed, this were a new family we would have the impossible condition of *four* mutually orthogonal families, which our three-dimensional space will not permit. I shall endeavor to show that these planes are in reality a degeneration of the hyperboloids of two sheets.

Referring to the middle of page 1000, we find the root that of the cubic in the parameter, ρ, which gives us the hyperboloids of two sheets, lies between $-b^2$ and $-a^2$. Let us assume that the distance between $-b^2$ and $-a^2$ on the $\rho = 0$ axis is divided up into m equal parts each of which has a value or length p. We may assign any positive value to m. Since the root which gives us the hyperboloid of two sheets must lie between $-b^2$ and $-a^2$ we can call this root

$$\rho = -a^2 + n\,p$$

where n may have any positive value less than m. (It must be positive for $a > b$). Substituting this value of ρ in the equation

$$\frac{x^2}{a^2 + \rho} + \frac{y^2}{b^2 + \rho} + \frac{z^2}{c^2 + \rho} = 1$$

and remembering that $a > b > c$, we have

$$\frac{x^2}{n\,p} - \frac{y^2}{a^2 - b^2 - n\,p} - \frac{z^2}{a^2 - c^2 - n\,p} = 1$$

which is an hyperboloid of two sheets.

Since $$a^2 - b^2 = m\,p$$

we can rewrite the foregoing equation as follows:

$$\frac{x^2}{n\,p} - \frac{y^2}{(m-n)\,p} - \frac{z^2}{a^2 - c^2 - n\,p} = 1$$

Multiplying this equation through by p it becomes

$$\frac{x^2}{n} - \frac{y^2}{m - n} - \frac{p\,z^2}{a^2 - c^2 - n\,p} = p.$$

For the oblate spheriodal co-ordinates the z axis becomes an axis of revolution and a^2 becomes equal to b^2, which is the same as letting p approach zero. As p approaches zero, both m and n remain finite for as a^2 and b^2 come closer together we can still continue to divide the distance into m parts, each part smaller in the same ratio as the distance between a^2 and b^2 becomes smaller. If p approaches zero our last equation becomes

$$\frac{x^2}{n} - \frac{y^2}{m - n} = 0$$

or

$$y = \pm\sqrt{\frac{m - n}{n}}\,x$$

which gives us the diametral planes of equation (**22**). Since m and n are always positive and $m > n$ the radical

$$\pm \sqrt{\frac{m-n}{n}}$$

can take any value and we may substitute ν for it and obtain

$$y - \nu x = 0$$

where ν may have any value.

The same method may be employed to show that the third equation of (**146**) is a degenerate hyperboloid of one sheet.

A point in connection with the introduction of the thermometric parameters α, β and γ, occurs to me, which might possibly aid one who likes to have a picture of such a step in the analysis. In introducing the parameter β, for instance, it is assumed, and the assumption later justified analytically, that the potential, under certain conditions is a function of μ only. The conditions under which this is true is later shown to be that two of the hyperboloids be kept at definite potentials. Referring to Fig. 10, it is seen that μ is the intercept of the hyperboloids on the X Y plane, that is, μ determines the position of any hyperboloid. From a graphical standpoint, then, it is justifiable to assume that if two of the hyperboloids are maintained at a definite potential, made equipotential surfaces, the potential of the other hyperboloids will depend only on their position with respect to these two, in other words, tha t V is a function of μ only.

On page 1028 in solving the problem of the infinite rod through the hole in an infinite plane, the three following equations are given:

$$\frac{\partial^2 V}{\partial \alpha^2} = 0, \quad \frac{\partial^2 V}{\partial \gamma^2} = 0, \text{ and } \frac{\partial^2 V}{\partial \beta^2} = 0$$

the last one being equation (**115**) and is solved. One might well wonder why the other two equations are not solved if they are true equations. Of course the first two equations are *not* true for, if V is neither a function of α, nor γ, how can $\frac{\partial^2 V}{\partial \alpha^2}$ or $\frac{\partial^2 V}{\partial \gamma^2}$ be equal to zero unless it were purely an accident? Equation (**115**) is the natural consequence of equation (**114**) for neither the first nor the third term of (**114**) can have a meaning if we assume that V is a function of β *only*. Is not this point similar to the following case in algebra? If we have the equation $10\,X = 0$, we know that $X = 0$ and do not have to make the untrue statement that $10 = 1$ in order to get it. I believe this same mis-statement is made in Professor Byerly's, "Fouriers Series and Spherical Harmonics," to which Mr. Rice refers.

The two problems which Mr. Rice has solved by Laplace's equation in spheroidal coordinates can only be checked by experiment, as is done in his paper, and then only approximately, due to the impossibility of using either an infinite rod or an infinite plane. It might be interesting to work a simple problem using Laplace's equation in this form which could be checked by some other method. One such problem obviously is to calculate the electrostatic capacity of an isolated thin circular disk. Such a disk is obtained by allowing the oblate spheroids to degenerate into a circular disk of radius c (see Fig. 9). The oblate spheroids will be equipotential surfaces and the tubes of flux will be bounded by the hyperboloids of one sheet and the diametral planes. Under these conditions the potential V is a function of α only and equation (**115**) becomes

$$\cos^2 \alpha \frac{\partial^2 V}{\partial \alpha^2} = 0$$

and since this must be true for all values of α, we have

$$\frac{\partial^2 V}{\partial \alpha^2} = 0$$

Solving this differential equation we obtain

$$V = A \alpha + C$$

To evaluate the two constants of integration, A and C, we shall let the disk have a potential of V_0 and since the disk is isolated, $V=0$ at infinity. We have therefore the two boundary conditions.

$$V = V_0 \text{ when } \alpha = 0 \text{ and } V = 0 \text{ when } \alpha = \frac{\pi}{2}.$$

Substituting these conditions in the last equation we obtain

$$V = V_0 \left(1 - \frac{2\alpha}{\pi}\right)$$

The electrostatic flux, ψ, is given by the equation,

$$\psi = \iint - \frac{dV}{dn_1} \, dn_2 \, dn_3$$

The potential gradient is

$$\frac{dV}{dn_1} = - \frac{2 V_0}{\pi} \cdot \frac{d\alpha}{dn_1} = - \frac{2 V_0}{\pi} \cdot \frac{d\alpha}{d\lambda} \cdot \frac{d\lambda}{dn_1},$$

which is obtained by differentiating the equation for potential given above with respect to the normal, n_1.

Since

$$\lambda = c \sec \alpha,$$

$$\frac{d\alpha}{d\lambda} = \frac{c}{\lambda \sqrt{\lambda^2 - c^2}} \qquad \text{(Equation 75)}$$

From equation (**47**) and equation (**65**) we get

$$\frac{d\,\lambda}{d\,n_1} = h_1 = \frac{\sqrt{\lambda^2 - c^2}}{\sqrt{\lambda^2 - \mu^2}}$$

$$\therefore \frac{d\,V}{d\,n_1} = -\frac{2\,V_0}{\pi} \cdot \frac{c}{\lambda\sqrt{\lambda^2 - \mu^2}}$$

From equation (**48**) and equation (**67**) we obtain

$$d\,n_2 = \frac{d\,\mu}{h_2} = \frac{\sqrt{\lambda^2 - \mu^2}}{\sqrt{c^2 - \mu^2}}\,d\,\mu,$$

and from equation (**49**) and equation (**69**)

$$d\,n_3 = \frac{d\,\nu}{h_3} = \frac{\lambda\,\mu}{c\,(1 + \nu^2)}\,d\,\nu$$

Substituting these values for $\frac{d\,V}{d\,n_1}$, $d\,n_2$ and $d\,n_3$ in the equation for φ we obtain,

$$\varphi = \int\!\!\int \frac{2\,V_0}{\pi} \cdot \frac{\mu\,d\,\mu\,d\,\nu}{(1 + \nu^2)\sqrt{c^2 - \mu^2}}$$

$$= \frac{4\,V_0}{\pi}\int_{v=\tan 0}^{v=\tan 2\pi} \frac{d\,\nu}{1 + \nu^2}\int_{\mu=0}^{\mu=c} \frac{\mu\,d\,\mu}{\sqrt{c^2 - \mu^2}}$$

(Factor 2 is introduced to get flux above and below $X\ Y$ plane.)

$$= \frac{4\,V_0}{\pi}\Big[\text{arc tan}\,\nu\Big]_{v=\tan 0}^{v=\tan 2\pi}\Big[-\sqrt{c^2 - \mu^2}\Big]_{\mu=0}^{\mu=c}$$

$$= 8\,V_0\,c$$

Since $$\varphi = 4\,\pi\,Q$$

where Q is the total charge on the disk,

$$4\,\pi\,Q = 8\,V_0\,c$$

or

$$\frac{Q}{V_0} = \frac{2\,c}{\pi} = \text{Capacity of disk.}$$

This is the well known value for the capacity of an isolated circular disk of radius c and can be checked by other methods, for instance, see Berg's "Electrical Engineering, Advance Course" page 201.

Lastly, in connection with the integration for the tubes of flow where the integration for ν is neglected, would it not be more exact to say that the spacing obtained is for any diametral plane and hence independent of ν? That is, the last paragraph on page 1039 shoul dread "and since ν_0 may have any value we may, neglect it as far as the spacing *on any diametral plane* is concerned"; and the paragraph in the middle of page 1051 should read, "As far as the spacing of the surfaces of flow *on any diametral* plane is concerned we may..........."

Chester W. Rice: Major Mershon has asked what effect power arcs will have on bushings. I am sorry to say that I have never witnessed such tests but suppose that they will blow the bushings up, as they do almost everything else when they get started.

Mr. Taylor was interested to know why we did not introduce bad distortion into the diagrams by moving the pointer around in our electrolyte, as you would in an electrostatic problem if you tried to obtain the potential distribution by moving the pointer around in space. The beauty of the thing is that we have substituted an electric current for the electrostatic flux. This has two very important advantages, in the first place, we can insulate all but the end of our exploring pointer from the current whereas, there is no insulator for the electrostatic flux, in the second place, we substitute a large conduction current for a small displacement current which allows us to use a reasonable amount of power for operating our instrument without fear of distorting the field. Of course, the insulated pointer does displace a slight amount of liquid and therefore introduces a slight distortion. A method of eliminating this error is pictured in Fig. 5. The energy required to operate a quadrant electrometer is so small compared with the energy flowing through the circuit that distortion due to this cause is considered entirely negligible.

Prof. Gray asked about using the photographic method for three-dimensional problems. I think photographic methods and all of the allied methods, such as little fine particles of glass-wool or mica dust, are well adapted to show the general system of flow lines. I have made some experiments with little glass rods obtained by grinding up glass-wool, and in that way obtained a general picture through any plane, of the desired three-dimensional figure. A very considerable assistance is obtained by this method where one wishes to apply the cut and try method of obtaining a diagram for an electrostatic problem as it gives you something to start guessing with.

Mr. Fortescue has made a correction of the arc-over value of his bushing, which indicates that no surface effect exists in the material which he has used. He gets an arc-over of approximately 207 kilovolts effective, and the calculated arc-over, if my assumptions as to the size and general structure of his test piece are correct, comes out at 205 kilovolts effective. There is therefore no room left for surface effect if his tests are correct.

He also mentioned some further data which have apparently convinced him that there is no such thing as surface effect. For my part, I am confident that there is a large effect of this nature. The difference in opinion shows that further study of this phenomenon is very desirable.

I have never been able to see any very direct bearing between the calculations on the condenser bushing and the bushings as built. In the first place, the calculations assume that the bushing has very large hats on the top and bottom ends (infinite planes); in the second place, the structure is assumed to be symmetrical about the cover of the tank. Neither of these conditions is even approximated to in the bushings of this type which I have seen.

Mr. Farnsworth has asked why the charts were made using thin sheet metal to form the electrodes. This was done in the first place to facilitate the construction and in the second place to make the boundary conditions as simple as possible, so that in case a mathematical solution some day becomes possible the diagrams may be used as a check.

Mr. Haar has asked whether I have made any high-voltage direct-current tests. No, I am sorry to say that I have not had the opportunity.

Mr. Behrend has brought up the question of three-dimensional hydrodynamic problems. If we assume an incompressible and inviscid fluid, the electrodynamic method is of course available for obtaining the solution of any desired problem in hydrodynamics. For example; if we wish to obtain the solution of a torpedo shaped body at rest in an infinite current of fluid, all we have to do is to immerse a non-conducting torpedo shaped body in our tank between large parallel plane electrodes and investigate the potential distribution in the usual manner. This, of course, is also the solution of the analagous electrostatic problem, namely, a torpedo shaped body of zero permittivity in a uniform electrostatic field.

The solution of the type of problem, where a body is moving through a stationary perfect fluid can also be solved by this method. Taking the case of a sphere we could obtain the desired solution by immersing two pointed electrodes at close proximity in the tank and exploring the field in the usual manner. This would result in the well known solution of a sphere moving through an infinite stationary fluid. Of course it is understood that the solution of these problems does not give very interesting data for the naval constructor because actual fluids exert appreciable forces due to viscosity and radiation of compressional waves. A sphere set in motion in a stationary fluid would move on forever at the same speed also a stationary sphere having mass would remain at rest in a uniform stream of perfect fluid. In the ideal fluid theory all bodies are streamline bodies and eddys are non-existent.

Mr. J. B. Morrill's discussion has added some very interesting material, which I believe will be greatly appreciated by all those engineers and students who have occasion to study this subject. I believe that his analysis which shows that the diametral planes can be considered as a degeneration of the hyperboloids of two sheets is new and very illuminating. He is also entirely correct in pointing out the incorrectness of setting

$$\frac{d^2 V}{d \alpha^2} = 0, \qquad \frac{d^2 V}{d \gamma^2} = 0.$$

as a method of indicating that they are meaningless for the case under consideration and therefore may be struck out of the equation. I think that this slip is often made and therefore it is very well to draw attention to its incorrectness.

The solution of the limiting oblate spheroid is a very appropriate addition.

I am greatly indebted to Mr. Morrill for pointing out by letter an inconsistency in my original equations used in deducing Laplace's equation (pages 1006-1014). On getting into the matter, I found that the trouble arose from considering the top area of the infinitesimal volume equal to the bottom area whereas it is necessary to consider the rate of change of the bottom area as we go towards the top along the normal.

I have therefore taken the liberty of making this correction for publication in the Transactions.

I am further indebted to Mr. Morrill for the corrections of many typographical errors to which he has so kindly drawn my attention.